ALEA, Lat. Am. J. Probab. Math. Stat. **23**, 597–627 (2026)
DOI: 10.30757/ALEA.v23-23

Gamma invariance and Dirichlet proportion for random exchange processes on graphs

Jean-Christophe Breton and Kilian Lebreton

Univ Rennes, CNRS, IRMAR – UMR 6625, F-35000 Rennes, France
E-mail address: jean-christophe.breton@univ-rennes.fr
URL: https://jcbreton.pages.math.cnrs.fr/jcb/home.html

Université de Lorraine, CNRS, IECL – UMR 7502, F-54000 Nancy, France
E-mail address: kilian.lebreton@univ-lorraine.fr

Abstract. We investigate invariant measures on random weighted directed graphs representing random exchange processes. More precisely, we consider measures consisting of independent random variables on each vertex of the graph, and we assume that their expectations are not decomposable according to the graph structure. When in addition such a measure is invariant under a Markov dynamic on the graph, we show that the random variables constituting the considered measure are necessarily gamma-distributed and that the proportions from each vertex along its graph-neighbors are Dirichlet distributed. We illustrate and discuss our results on several examples, including the related original setting of Muratov and Zuyev (2017).

Introduction

We consider a collection of interacting individuals, each possessing some resources and exchanging, over time, a random proportion of their resources with that of some other individuals, defining a random exchange process. Such a model can be used to represent various situations: the wealth of economic agents in Muratov and Zuyev (2017); Düring et al. (2022), opinion dynamics in Sociophysics in Oestereich et al. (2022), interacting spiking neurons in De Masi et al. (2015), but also epidemiological models, or population dynamics. See also McKinlay (2014, Section 4.1) for related models and DeGroot and Rao (1963) for a very simple historical such toy-model.

In our setting, we represent a complex network of interacting individuals by a weighted directed graph where the vertices represent the individuals and (non-negative) random variables attached to each vertex represent their resources. We refer to van der Hofstad (2017) for the use of graphs to model complex networks. Then, the random exchange process consists in splitting each random variable according to the structure of the graph. In this article, we are interested in the invariance

Received by the editors May 7th, 2025; accepted March 25th, 2026; published under license CC BY 4.0.

2010 *Mathematics Subject Classification.* 60E05, 60J05, 60J10, 05C80, 05C81.

Key words and phrases. Random exchange process; gamma distribution; Dirichlet distribution; random stochastic graph; Markov dynamic; invariance in distribution.

The first author has been supported by the Centre Henri Lebesgue, program ANR-11-LABX-0020-0.

in distribution of the collection of these random variables under this random exchange process, together with related questions.

More precisely, given a weighted directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$, the vertices $i \in \mathcal{I}$ of the graph are interpreted as individuals and a (directed) edge $(i, j) \in \mathcal{F}$ between two individuals i and j indicates that individual i gives a part of its resource to individual j in a (random) proportion given by the non-negative weight $a_{i,j}$ of the edge. In this model, the weights of the edges starting from any vertex i must sum up to 1, so that the collection of all the proportions forms a (random) stochastic matrix $\mathbf{A} = (a_{i,j})_{i,j \in \mathcal{I}} \in \mathcal{M}_{\mathcal{I},\mathcal{I}}(\mathbb{R}_+)$ agreeing with the structure of the graph. We assume that all the resources are given by independent random variables X_i, $i \in \mathcal{I}$, so that the collection of all the resources is represented by a (random) measure on the graph given by $\mathbf{X} = (X_i)_{i \in \mathcal{I}}$, also assumed to be independent of the matrix $\mathbf{A}$. The random exchange process transforms an initial resource vector $\mathbf{X}$ into $\mathbf{X} \cdot \mathbf{A}$ where $(\mathbf{X} \cdot \mathbf{A})_j = \sum_{i \in \mathcal{I}} X_i a_{i,j}$ for any $j \in \mathcal{I}$. In this context, our main objective is to investigate under which conditions the collection of resources is invariant in distribution under the random exchange model, i.e. $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$. Roughly speaking, our main results prove that when such an invariance holds, then the random variables X_i must be gamma-distributed, see Theorem 3.3. To the best of our knowledge, this is the first such invariance result in such a general framework. Moreover proportions governed by Dirichlet distribution in each edge are also thoroughly discussed, see Theorem 3.5, Theorem 3.7 and related remarks.

This paper is organized as follows. Section 1 specifies the notations for graphs and measures on graphs, it also gives the main properties of such objects that will be considered all along the article. Section 2 gives some properties of gamma and Dirichlet distributions, and discusses Lukacs' theorem which specifies, in terms of gamma and Dirichlet distributions, when the shape and the size of a vector of independent non-negative random variables are independent, see Theorem 2.3 and Theorem 2.4. Our main results discussing gamma-distributed resources and Dirichlet distributions of proportions between individuals are given in Section 3 (Theorems 3.3, 3.5 and 3.7). Such Dirichlet proportions coincide with typical interesting distributions in related models, see McKinlay (2014) and references therein. These results are illustrated on several examples in Section 4, where our main conditions are also discussed. In particular, the setting of Muratov and Zuyev (2017) is recovered as a particular graph whose vertices are given by the set $\mathbb{Z}$; in this case, we recover the invariance result of Muratov and Zuyev (2017) as a special case of Theorem 3.3, and we prove a conjecture stated therein, see Section 4.2. Finally, the appendix presents some technical results.

1. Stochastic graphs and measures

A weighted directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{F})$ is a collection of vertices $\mathcal{I}$ equipped with a set $\mathcal{F}$ of edges given by ordered pairs of vertices (i, j), $i, j \in \mathcal{I}$, where each edge $(i, j) \in \mathcal{F}$ has a weight. In the sequel, we assume $\mathcal{I}$ is countable. When the weights are given by a stochastic matrix $\mathcal{A} = (\alpha_{i,j})_{(i,j) \in \mathcal{F}}$ on $\mathcal{I}$ (i.e. $\alpha_{i,j} \in [0, 1]$ and $\sum_{j \in \mathcal{I}} \alpha_{i,j} = 1$ for all $i \in \mathcal{I}$), we define a stochastic graph:

Definition 1.1 (Stochastic graph)**.** A weighted directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ is said to be a stochastic graph when the stochastic matrix $\mathcal{A}$ satisfies $\alpha_{i,j} > 0$ if and only if $(i, j) \in \mathcal{F}$.

We refer to West (1996) for a general reference on graph theory. In the sequel, we shall assume the degrees of the vertices are finite, more precisely, we consider graph satisfying:

$$\forall j \in \mathcal{I}, \quad \#\{i \in \mathcal{I} : (i, j) \in \mathcal{F}\} < +\infty \quad \text{and} \quad \forall i \in \mathcal{I}, \quad \#\{j \in \mathcal{I} : (i, j) \in \mathcal{F}\} < +\infty. \tag{H}$$

We shall also identify a measure $\mathbf{m}$ on $\mathcal{I}$ with the vector $(m_i)_{i \in \mathcal{I}}$ and we define its support by $\mathrm{Supp}(\mathbf{m}) = \{i \in \mathcal{I} : m_i \neq 0\}$. We set $(\mathbf{m} \cdot \mathcal{A})_j = \sum_{i \in \mathcal{I}} m_i \alpha_{i,j}$, and $\mathrm{Span}(E)$, resp. $\mathrm{Span}^+(E)$ for the set of linear combinations (resp. linear combinations with non-negative coefficients) of $\mathbf{x} \in E$.

Definition 1.2. Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph.

- A signed (resp. non-negative) measure $\mathbf{x} = (x_i)_{i\in\mathcal{I}}$ is said to be invariant when $\mathbf{x}\cdot\mathcal{A} = \mathbf{x}$ and we set $\mathrm{Ker}(\mathcal{A} - Id) = \{\mathbf{x} = (x_i)_{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}} : \mathbf{x}\cdot\mathcal{A} = \mathbf{x}\}$, resp. $\mathrm{Ker}(\mathcal{A} - Id)^+ = \mathrm{Ker}(\mathcal{A} - Id) \cap [0, +\infty)^{\mathcal{I}}$, for the sets of such measures.
- An invariant measure $\mathbf{x}$ is prime when

  $$\forall\, \mathbf{y}, \mathbf{z} \in \mathrm{Ker}(\mathcal{A} - Id)^+, \quad \mathbf{y} + \mathbf{z} = \mathbf{x} \implies \mathbf{y}, \mathbf{z} \in \mathrm{Span}^+(\mathbf{x}) = \{\lambda\mathbf{x} : \lambda \in [0, +\infty)\},$$

  this is equivalent to the following: $\forall \mathbf{y} \in \mathrm{Ker}(\mathcal{A} - Id)^+$ such that $\mathbf{y} \leqslant \mathbf{x}$ (i.e. $y_i \leqslant x_i\ \forall i \in \mathcal{I}$) then there exists $\lambda \in [0, 1]$ such that $\mathbf{y} = \lambda\mathbf{x}$;
- An invariant measure $\mathbf{x}$ is elementary when $\#\{j \in \mathcal{I} : (i, j) \in \mathcal{F}\} = 1$ for all $i \in \mathrm{Supp}(\mathbf{x})$.

Since $(i, j) \in \mathcal{F}$ is equivalent to $\alpha_{i,j} > 0$, the existence of an elementary $\mathbf{x}$ requires $\alpha_{i,j} = 1$ for all $i \in \mathrm{Supp}(\mathbf{x})$.

Roughly speaking, the notion of prime invariant measure is related to that of an extreme ray in Ito and Lourenço (2017) in order to regain a canonical basis of $\mathrm{Ker}(\mathcal{A} - Id)^+$, like in Davis (1954) when the set $\mathcal{I}$ is finite. For further details in this direction and for some properties on prime invariant measures, we refer to Section A.1 in the Appendix. In particular, by using such properties, we show that when there exists an invariant measure which is prime and elementary, then we have a complete description of the corresponding graph, at least on the support of this measure:

Proposition 1.3. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)^+$ be a non-negative elementary prime invariant measure. Then the subgraph*

$$\big(\mathrm{Supp}(\mathbf{x}), \mathcal{F} \cap \mathrm{Supp}(\mathbf{x})^2, \mathcal{A}_{|\mathrm{Supp}(\mathbf{x})}\big)$$

is in bijection with (up to a possible permutation of $\mathrm{Supp}(\mathbf{x})$*):*

- $(\mathbb{Z}/N\mathbb{Z}, \{(k, k+1) : k \in \mathbb{Z}/N\mathbb{Z}\})$ *when* $\#\mathrm{Supp}(\mathbf{x}) = N < +\infty$*;*
- $(\mathbb{Z}, \{(k, k+1) : k \in \mathbb{Z}\})$ *when* $\#\mathrm{Supp}(\mathbf{x}) = +\infty$.

In particular, $\mathbf{x} = c\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}$ *for some* $c > 0$.

Below, this result allows to deal only with generic cases when we consider elementary prime invariant measure, see Proposition 3.1. It is illustrated for instance in Example 4.7. We refer to page 622 in the appendix for a proof of this Proposition 1.3.

In the sequel, we consider a random stochastic matrix $\mathbf{A} = (a_{i,j})_{(i,j)\in\mathcal{F}}$, i.e. random variables $a_{i,j}$ for all $(i, j) \in \mathcal{F}$ such that $\mathbf{A}$ is almost surely a stochastic matrix. In particular by linearity, $\mathcal{A} := \mathbb{E}[\mathbf{A}] = (\mathbb{E}[a_{i,j}])_{(i,j)\in\mathcal{F}}$ is also a stochastic matrix.

Definition 1.4. We say that a weighted directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ with random weight matrix $\mathbf{A} = (a_{i,j})_{(i,j)\in\mathcal{F}}$ is a random stochastic graph if $\mathbf{A}$ is a random stochastic matrix such that moreover $\sigma(a_{i,k} : k \in \mathcal{I})$, $i \in \mathcal{I}$, are independent σ-algebras and $(\mathcal{I}, \mathcal{F}, \mathcal{A})$ with $\mathcal{A} = \mathbb{E}[\mathbf{A}]$ is a stochastic graph enjoying (H).

In the sequel, we investigate the following invariance property for the random measures $\mathbf{X} = (X_i)_{i\in\mathcal{I}} \in [0, +\infty)^{\mathcal{I}}$:

$$\text{the } X_i,\ i \in \mathcal{I}, \text{ are independent, } \mathbb{E}[|X_i|] < +\infty, \forall i \in \mathcal{I}, \text{ and } \mathbf{X}\cdot\mathbf{A} \sim \mathbf{X} \tag{Inv}$$

and we prove that when, in addition, the measure $\mathbb{E}[\mathbf{X}]$ is prime, then (Inv) requires the random variables X_i to follow gamma distributions, see Theorem 3.3. In Theorem 3.5 and Theorem 3.7, we also discuss Dirichlet distributions for the proportions governed by $\mathbf{A}$. Note that it follows directly from (Inv) that $\mathbb{E}[\mathbf{X}] := (\mathbb{E}[X_i])_{i\in\mathcal{I}}$ is an invariant measure for $\mathbb{E}[\mathbf{A}]$, i.e. $\mathbb{E}[\mathbf{X}] \in \mathrm{Ker}(\mathbb{E}[\mathbf{A}] - Id)^+$, since, by independence, $\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X}\cdot\mathbf{A}] = \mathbb{E}[\mathbf{X}]\cdot\mathbb{E}[\mathbf{A}]$.

2. Gamma distributions and related results

Since our forthcoming results are in terms of gamma distributions, we give in this section some properties of gamma and related distributions. In particular in Section 2.2, we state Lukacs' theorem on the independence of the shape and the size of a vector of independent non-negative random variables in terms of these distributions.

2.1. *Definitions and properties.*

Definition 2.1 (Gamma and Dirichlet distributions)**.** Let $n \in \mathbb{N}^*$ and $\alpha, \gamma, \beta_1, \dots, \beta_n$ be positive reals. Recall $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

- The gamma distribution $\Gamma(\alpha, \gamma)$, with parameters $\alpha > 0$, $\gamma > 0$, is the distribution on $\mathbb{R}_+^*$ given by the density:
$$\frac{\gamma^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\gamma x} \mathbf{1}_{\mathbb{R}_+^*}(x).$$
- The Dirichlet distribution $\textsc{Dir}(\beta_1, \dots, \beta_n)$ with n variables is the distribution on the simplex $\{(x_1, \dots, x_n) \in [0,1]^n : x_1 + \dots + x_n = 1\}$ given by the density:
$$\Gamma\left(\sum_{k=1}^n \beta_k\right) \prod_{k=1}^n \frac{x_k^{\beta_k - 1}}{\Gamma(\beta_k)} \mathbf{1}_{\{(x_1,\dots,x_n)\in[0,1]^n : x_1+\dots+x_n=1\}}.$$

Recall that $\Gamma(\alpha_1, \gamma) * \Gamma(\alpha_2, \gamma) = \Gamma(\alpha_1 + \alpha_2, \gamma)$ and for $X \sim \Gamma(\alpha, \gamma)$: $\mathbb{E}[X] = \alpha/\gamma$, $\mathrm{Var}(X) = \alpha/\gamma^2$, $\mathbb{E}[e^{zX}] = (1 - z/\gamma)^{-\alpha}$ for $z \in \mathbb{C}^- := \{z \in \mathbb{C}$ with real part $\Re(z) < 0\}$ and $\gamma X \sim \Gamma(\alpha, 1)$. When $(a_k)_{k \in [\![1,n]\!]} \sim \textsc{Dir}(\beta_1, \dots, \beta_n)$:

$$\mathbb{E}[a_k] = \frac{\beta_k}{\sum_{i=1}^n \beta_i}, \quad \mathrm{Var}(a_k) = \frac{\beta_k(\sum_{i=1}^n \beta_i - \beta_k)}{\left(\sum_{i=1}^n \beta_i\right)^2 (1 + \sum_{i=1}^n \beta_i)}.$$

Note that the marginals of a Dirichlet distribution are Beta distributions. A standard property of gamma and Dirichlet distributions is the following:

Lemma 2.2. *Let $(Y_k)_{k \in [\![1,n]\!]}$ be independent random variables such that $Y_k \sim \Gamma(\alpha_k, \gamma)$ with $\alpha_k > 0$ and the same scale parameter $\gamma > 0$. Then, setting $Y = Y_1 + \dots + Y_n$, we have:*

$$Y \perp\!\!\!\perp \left(\frac{Y_1}{Y}, \dots, \frac{Y_n}{Y}\right) \sim \textsc{Dir}(\alpha_1, \dots, \alpha_n). \tag{2.1}$$

For the sake of completeness, we give a proof of this property in the Appendix A.3. More generally, we refer to Ng et al. (2011) for a large review on Dirichlet and related distributions.

In order to deal, in the sequel, simultaneously with degenerate and non-degenerate cases, we include degenerate distributions into the families of both gamma and Dirichlet distributions. We set $\Gamma_\gamma(x) := \Gamma(\gamma x, \gamma)$ for $x > 0$ and $\gamma \in \overline{\mathbb{R}}_+^*$ and adopt the convention $\Gamma_\gamma(x) = \delta_x$ when $\gamma = +\infty$ or $x = 0$. This is justified by the observation that when $X \sim \Gamma(\gamma x, \gamma)$:

$$\mathbb{E}[X] = x \xrightarrow[x \to 0]{} 0 \quad \text{and} \quad \mathrm{Var}(X) = \frac{x}{\gamma} \to 0, \text{ when } x \to 0 \text{ or } \gamma \to +\infty.$$

Let $\gamma > 0$ and $(\alpha_k)_{k \in [\![1,n]\!]} \in [0,1]^n$ such that $\sum_{k=1}^n \alpha_k = 1$, we set

$$\textsc{Dir}_\gamma((\alpha_k)_{k \in [\![1,n]\!]}) := \textsc{Dir}((\gamma \alpha_k)_{k \in [\![1,n]\!]})$$

and adopt the convention that for $(a_k)_{k\in[\![1,n]\!]} \sim \mathrm{Dir}_\gamma((\alpha_k)_{k\in[\![1,n]\!]})$: $\mathcal{L}(a_k) = \delta_{\alpha_k}$ when $\gamma = +\infty$ and $\mathcal{L}(a_k) = \delta_0$ when $\alpha_k = 0$. This is justified by the observation that:

$$\mathbb{E}[a_k] = \alpha_k \xrightarrow[\alpha_k\to 0]{} 0,$$

$$\mathrm{Var}(a_k) = \frac{\gamma\alpha_k(\gamma\alpha_k+1)}{\gamma(\gamma+1)} - \alpha_k^2 = \frac{\alpha_k(1-\alpha_k)}{\gamma+1} \to 0, \quad \text{when } \alpha_k \to 0 \text{ or } \gamma \to +\infty.$$

We also extend the definition of Dirichlet distribution to an infinite vector $(a_k)_{k\geqslant 1}$ such that $\sum_{k\geqslant 1} a_k = 1$. For parameters $\beta_k \geqslant 0$, $k \geqslant 1$, such that $\sum_{k\geqslant 1}\beta_k < +\infty$, consider for any $n \geqslant 1$:

$$(a_1,\dots,a_n) \sim \mathrm{Dir}\left(\beta_1,\dots,\beta_{n-1},\sum_{k\geqslant n}\beta_k\right).$$

Due to the aggregation property of (standard) Dirichlet distributions [1], the definition above is consistent and Kolmogorov's extension theorem applies to define $(a_k)_{k\geqslant 1} \sim \mathrm{Dir}\big((\beta_k)_{k\geqslant 1}\big)$. Note that Lemma 2.2 still holds true with these extensions of gamma and Dirichlet distributions.

2.2. *Lukacs' Theorem.* Lukacs' theorem is one of the most classical results of characterization for probability distributions. With a slight generalization including degenerate random variables, it states:

Theorem 2.3 (Lukacs)**.** *Let U, V be two positive independent random variables, non-zero a.s. Then $U+V$ and U/V are independent if and only if there are $u,v>0$ and $\gamma\in\overline{\mathbb{R}}_+^*$ such that $U\sim\Gamma_\gamma(u)$, $V\sim\Gamma_\gamma(v)$.*

See also Theorem A.13 in the Appendix for the original result from Lukacs (1955). This result has been generalized in many directions. For instance for a multivariate version, see Bobecka and Wesołowski (2004). For an independence structure generalizing that of shape and size in Theorem 2.3, see Matsumoto and Yor (2000, Prop. 9.1), this independence property involves Kummer and gamma distributions, cf. Letac and Wesołowski (2000); Koudou and Wesołowski (2025). In the sequel, we shall use the following generalization of Theorem 2.3 designed for our setting:

Theorem 2.4. *Let $X\in[0,+\infty), X\neq 0$ a.s. and, independently, $\mathbf{a} = (a_i)_{i\in\mathcal{I}} \in [0,1]^{\mathcal{I}}$ be independent random variables under the constraint that $\sum_{i\in\mathcal{I}} a_i = 1$, $\mathbb{E}[a_i] = \alpha_i > 0$ where $\mathcal{I}\subset\mathbb{N}$ has cardinality at least 2. Then, the following statements are equivalent:*

(1) $(Xa_i)_{i\in\mathcal{I}}$ are independent;
(2) there exist $x>0$ and $\gamma\in\overline{\mathbb{R}}_+^*$ such that $Xa_i\sim\Gamma_\gamma(x\alpha_i)$ for all $i\in\mathcal{I}$ and $\mathbf{a}\sim\mathrm{Dir}_{\gamma x}((\alpha_i)_{i\in\mathcal{I}})$;
(3) there exist $x>0$ and $\gamma\in\overline{\mathbb{R}}_+^*$ such that $Xa_i\sim\Gamma_\gamma(x\alpha_i)$ for some $i\in\mathcal{I}$ and $\mathbf{a}\sim\mathrm{Dir}_{\gamma x}((\alpha_i)_{i\in\mathcal{I}})$;
(4) there exist $x>0$ and $\gamma\in\overline{\mathbb{R}}_+^*$ such that $X\sim\Gamma_\gamma(x)$ and $\mathbf{a}\sim\mathrm{Dir}_{\gamma x}((\alpha_i)_{i\in\mathcal{I}})$.

Proof: For any $j\in\mathcal{I}$, we set $\overline{a_j} = 1-a_j = \sum_{i\in\mathcal{I}\setminus\{j\}} a_i$.

1)⇒ 2) When the Xa_i, $i\in\mathcal{I}$, are independent, for any $j\in\mathcal{I}$, the random variables $U := Xa_j$ and $V := X(1-a_j)$ are independent and a.s. non-zero. Lukacs' theorem (Th. 2.3) applies to U,V: there are $x,\gamma>0$ such that $Xa_j\sim\Gamma_\gamma(x\alpha_j)$ and $X\sim\Gamma_\gamma(x)$. For any finite subset $J := \{j_1,\dots,j_n\}\subset\mathcal{I}$, setting $J^c = \mathcal{I}\setminus J$, the random variables $Xa_{j_1},\dots,Xa_{j_n}$ and $\sum_{j\in J^c} Xa_j$ are independent and gamma-distributed with the same scale parameter γ. Lemma 2.2 implies:

$$\left(a_{j_1},\dots,a_{j_n},\sum_{j\in J^c} a_j\right) \sim \mathrm{Dir}_{\gamma x}\left(\alpha_{j_1},\dots,\alpha_{j_n},\sum_{j\in J^c}\alpha_j\right)$$

[1] When $(a_1,\dots,a_n)\sim\mathrm{Dir}(\beta_1,\dots,\beta_n)$, then $(a_1,\dots,a_i+a_j,\dots,a_n)\sim\mathrm{Dir}(\beta_1,\dots,\beta_i+\beta_j,\dots\beta_n)$.

and we obtain $a \sim \mathrm{DIR}_{\gamma x}((\alpha_i)_{i\in\mathcal{I}})$ by Kolmogorov's extension theorem (see page 601 in Section 2.1).

The implication 2)⇒3) is straightforward.

The implication 3)⇒4) is proved by the method of moments which identifies the distribution $\Gamma_\gamma(x)$ by its moments: Since $X \perp\!\!\!\perp a$, we have $\mathbb{E}[(Xa_i)^n] = \mathbb{E}[X^n]\,\mathbb{E}[a_i^n]$ for any $n \in \mathbb{N}$, and using the explicit moments of $Xa_i \sim \Gamma_\gamma(x\alpha_i)$, $a_i \sim \mathrm{Beta}(\gamma x\alpha_i, \gamma x(1-\alpha_i))$, we have for any $n \in \mathbb{N}$:

$$\mathbb{E}[X^n] = \frac{\mathbb{E}[(Xa_i)^n]}{\mathbb{E}[a_i^n]} = \frac{\Gamma(\gamma x\alpha_i + n)}{\gamma^n\Gamma(\gamma x\alpha_i)}\,\frac{\Gamma(\gamma x+n)\Gamma(\gamma x\alpha_i)}{\Gamma(\gamma x\alpha_i+n)\Gamma(\gamma x)} = \frac{\Gamma(\gamma x+n)}{\gamma^n\Gamma(\gamma x)}.$$

Finally, we show 4)⇒1). When $\gamma = +\infty$, all the random variables are degenerate and in particular independent. When $\gamma < +\infty$, since it is enough to prove the independence for any finite family of $(Xa_i)_{i\in\mathcal{I}}$, we assume $X \sim \Gamma_\gamma(x) \perp\!\!\!\perp (a_j)_{j=1,\dots,n} \sim \mathrm{DIR}(\gamma x\alpha_j)_{j=1,\dots,n}$. Then, for any $u_j \in \mathbb{R}$, $j \in [\![1,n]\!]$, we have:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{j=1}^n Xa_ju_j\right)\right]\\
&= \int_{\{y\geqslant 0, s_1+\dots+s_n=1\}} e^{i\sum_{j=1}^n ys_ju_j}\frac{\gamma^{\gamma x}}{\Gamma(\gamma x)}y^{\gamma x-1}e^{-\gamma y}\frac{1}{B((x\gamma\alpha_j)_j)}\left(\prod_{j=1}^n s_j^{\gamma x\alpha_j-1}\right)\left(\prod_{j=1}^{n-1} ds_j\right)dy\\
&= \left(\prod_{j=1}^n \frac{\gamma^{\gamma x\alpha_j}}{\Gamma(\gamma x\alpha_j)}\right)\int_{\{y\geqslant 0, s_1+\dots+s_n=1\}} e^{i\sum_{j=1}^n ys_ju_j}\left(\prod_{j=1}^n y^{\gamma x\alpha_j-1}\right)y^{n-1}\left(\prod_{j=1}^n e^{-\gamma s_jy}\right)\times\\
&\qquad\qquad\times\left(\prod_{j=1}^n s_j^{\gamma x\alpha_j-1}\right)\left(\prod_{j=1}^{n-1} ds_j\right)dy
\end{aligned}$$

(since $B((x\gamma\alpha_j)_j) = \left(\prod_{j=1}^n\Gamma(x\gamma\alpha_j)\right)/\Gamma(\gamma x)$ and $\sum_{j=1}^n s_j = 1$)

$$= \left(\prod_{j=1}^n \frac{\gamma^{\gamma x\alpha_j}}{\Gamma(\gamma x\alpha_j)}\right)\int_{\{t_1,\dots,t_n\geqslant 0\}} e^{i\sum_{j=1}^n t_ju_j}\left(\prod_{j=1}^n e^{-\gamma t_j}\right)\left(\prod_{j=1}^n t_j^{\gamma x\alpha_j-1}\right)\left(\prod_{j=1}^n dt_j\right)$$

by using the change of variables $(s_1,\dots,s_{n-1},y)\mapsto(t_1,\dots,t_n)$ where $t_j = s_jy$ for $j\in[\![1,n]\!]$ with $s_n := 1-(s_1+\dots+s_{n-1})$, and whose Jacobian is

$$\begin{vmatrix} y & & & s_1\\ & \ddots & & \vdots\\ & & y & s_{n-1}\\ -y & \dots & -y & s_n\end{vmatrix} = \begin{vmatrix} y & & & s_1\\ & \ddots & & \vdots\\ & & y & s_{n-1}\\ 0 & \dots & 0 & 1\end{vmatrix} = y^{n-1}.$$

Thus by Fubini's theorem, we obtain for any $u_j\in\mathbb{R}$ and $j\in[\![1,n]\!]$:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\sum_{j=1}^n Xa_ju_j\right)\right] &= \prod_{j=1}^n\left(\frac{\gamma^{\gamma x\alpha_j}}{\Gamma(\gamma x\alpha_j)}\int_{\{t_j\geqslant 0\}} e^{it_ju_j}t_j^{\gamma x\alpha_j-1}e^{-\gamma t_j}\,dt_j\right)\\
&= \prod_{j=1}^n \mathbb{E}\left[\exp\left(iXt_ju_j\right)\right],
\end{aligned}$$

proving that the $(Xa_j)_{j=1,\dots,n}$ are independent. The independence of the whole family follows. □

Corollary 2.5. *Let $X = (X_i)_{i\in\mathcal{I}}$ be a family of independent random variables and $A = (a_{i,j})_{i,j\in\mathcal{I}}$ be a random matrix, independent of X and with independent rows such that*

$$\forall i \in \mathcal{I}, X_i \sim \Gamma_\gamma(x_i) = \Gamma(\gamma x_i, \gamma), \quad \text{and } (a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{D{\scriptstyle IR}}_{\gamma x_i}(\alpha_{i,j})_{j\in\mathcal{I}}. \tag{2.2}$$

Then the random variables $Y_j := \sum_{i\in\mathcal{I}} X_i a_{i,j}$, $j \in \mathcal{I}$, are independent with, for each $j \in \mathcal{I}$: $Y_j \sim \Gamma_\gamma\big(\sum_{i\in\mathcal{I}} x_i \alpha_{i,j}\big)$. Moreover, $(X_i a_{i,j})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ is an independent family.

Proof: For each $i \in \mathcal{I}$, condition (2.2) ensures 4/ in Theorem 2.4 for the gamma random variable X_i and the Dirichlet vector $(a_{i,j})_{j\in\mathcal{I}}$. Then Theorem 2.4 entails that the family $(X_i a_{i,j})_{j\in\mathcal{I}}$ is independent (statement 1/ in Th. 2.4) and $X_i a_{i,j} \sim \Gamma_\gamma(x_i\alpha_{i,j})$ (statement 3/ in Th. 2.4). Moreover since the X_i's and the rows of the matrix A are all independent, the $(X_i a_{i,j})_{j\in\mathcal{I}}$ are independent for different indices $i \in \mathcal{I}$. Then, $(X_i a_{i,j})_{i,j\in\mathcal{I}}$ is independent and $Y_j = \sum_{i\in\mathcal{I}} X_i a_{i,j} = \ast_{i\in\mathcal{I}} \Gamma_\gamma(x_i\alpha_{i,j}) = \Gamma_\gamma\big(\sum_{i\in\mathcal{I}} x_i\alpha_{i,j}\big)$, where $\ast$ stands for the convolution. □

2.3. *Cumulant generating function.*

Definition 2.6. The moment-generating function of a non-negative random variable X is defined by $\psi_X(z) = \mathbb{E}[e^{zX}]$ for all $z \in \mathbb{C}$ with $\Re(z) < 0$. The function $\phi_X(z) = \log(\psi_X(z))$ defines the cumulant generating function.

For example, when $X \in L^p(\Omega)$, the derivative of order p writes $\psi_X^{(p)}(t) = \mathbb{E}[X^p e^{tX}]$ for $t \in (-\infty, 0]$ and under appropriate integrability assumption:

$$\phi'_X = \frac{\psi'_X}{\psi_X}, \qquad \phi''_X + {\phi'_X}^2 = \frac{\psi''_X}{\psi_X}.$$

In particular, when $X \sim \Gamma_\gamma(x)$ with $\gamma \in (0, +\infty)$, we have:

$$\psi_X(t) = \left(1 - \frac{t}{\gamma}\right)^{-x\gamma}, \qquad \psi'_X(t) = x\left(1 - \frac{t}{\gamma}\right)^{-x\gamma-1}, \qquad \phi'_X(t) = \frac{x}{1 - \frac{t}{\gamma}}, \tag{2.3}$$

and we derive the differential equation:

$$\phi''_X(t) = \frac{x/\gamma}{(1 - \frac{t}{\gamma})^2} = \frac{1}{x\gamma}\left(\frac{x}{1 - \frac{t}{\gamma}}\right)^2 = \frac{1}{x\gamma}{\phi'_X}^2(t). \tag{2.4}$$

Equation (2.4) still holds true when $\gamma = +\infty$ since it reduces to $\phi''_X(t) = 0$ in this case, see below. Following Lukacs (1955), whose proof relies on the differential equation (2.4), we prove that such an equation characterizes the gamma distribution $\Gamma_\gamma(x)$ for any $\gamma \in \overline{\mathbb{R}}_+^*$. In order to include the case $\gamma = +\infty$ in (2.5) (i.e. $\rho = 0$ therein), we solve this equation in all cases in Lemma 2.7.

Lemma 2.7. *Let $\rho \geqslant 0$ and X be a non-negative random variable such that ϕ_X satisfies the differential equation:*

$$\phi''_X(t) \;=\; \rho\, {\phi'_X}^2(t) \quad \forall t \in (-\infty, 0]. \tag{2.5}$$

Then $X \sim \Gamma_\gamma(x)$ where $x = \mathbb{E}[X]$ and $\gamma = 1/(\rho x) \in \overline{\mathbb{R}}_+^*$.

Proof: First, with an initial condition $y(-1) = 1/\alpha$ in $t = -1$ for some $\alpha > 0$, we have uniqueness for (2.5) in $(-\infty, 0]$ since the Cauchy-Lipschitz conditions are satisfied. In the case $\rho > 0$, we have:

$$y'(t) = \rho y^2(t), \;\; y(-1) = \frac{\mathbb{E}[Xe^{-X}]}{\mathbb{E}[e^{-X}]} = \frac{1}{\alpha} > 0 \quad \Longrightarrow \quad y(t) = \frac{1}{\alpha - \rho(t+1)}, \tag{2.6}$$

which can be rewritten

$$\phi'_X(t) = \psi'_X(t)/\psi_X(t) = \frac{x}{1 - t/\gamma} \tag{2.7}$$

for a constant x related to γ and ρ by $\gamma\rho x = 1$. By uniqueness in the Cauchy-Lipschitz's theorem, and since $\psi(0) = 1$, we solve the differential equation (2.7), which gives $\psi_X(t) = \left(1 - \frac{t}{\gamma}\right)^{-x\gamma}$, proving $X \sim \Gamma_\gamma(x)$ with necessarily $x = \mathbb{E}[X]$.

In the case $\rho = 0$, $\phi'_X(t) = x \in \mathbb{R}$ and ψ_X is solution of $y'(t) = xy(t)$ with $y(0) = 1$. We derive $\psi_X(t) = e^{tx}$, proving $X = x$ which agrees with $X \sim \Gamma_\gamma(x)$ when $\gamma = +\infty$. □

Note that for an integrable non-negative random variable X, $\phi'_X(z)$ is well defined for any $z \in \mathbb{C}$ with $\Re(z) \leqslant 0$ and we shall also use the following property of ϕ_X:

Lemma 2.8. *Let X be an integrable non-negative random variable with cumulant generating function ϕ_X. Then ϕ'_X is non-negative on $(-\infty, 0]$ and non-decreasing to $\phi'_X(0) = \mathbb{E}[X]$.*

Proof: Since the function ϕ_X is C^1 on $(-\infty, 0]$ and C^2 on $(-\infty, 0)$, we have for $t < 0$:

$$\phi''_X(t) = \frac{\psi''_X(t)\psi_X(t) - \psi'_X(t)^2}{\psi_X(t)^2} = \frac{\mathbb{E}[X^2 e^{tX}]\,\mathbb{E}[e^{tX}] - \mathbb{E}[Xe^{tX}]^2}{\mathbb{E}[e^{tX}]^2}.$$

But $\mathbb{E}[(Xe^{tX/2})(e^{tX/2})]^2 \leqslant \mathbb{E}[X^2 e^{tX}]\,\mathbb{E}[e^{tX}]$ by the Cauchy-Schwarz inequality, and we conclude that $\phi''_X(t) \geqslant 0$ for $t < 0$ and $t \mapsto \phi'_X(t)$ is non-decreasing so that $\phi'_X(t) \leqslant \phi'_X(0) = \mathbb{E}[X]$ for $t \leqslant 0$. □

3. Prime invariant measures on a graph

3.1. *Main result.* In this section, we give our main result, generalizing Theorem 2 in Muratov and Zuyev (2017) for random stochastic graphs $(\mathcal{I}, \mathcal{F}, \mathbf{A})$. We use similar notations as in Section 1, in particular for a random stochastic matrix $\mathbf{A} = (a_{i,j})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ on a graph $(\mathcal{I}, \mathcal{F})$, we set $\mathbb{E}[\mathbf{A}] = (\mathbb{E}[a_{i,j}])_{(i,j)\in\mathcal{I}\times\mathcal{I}} =: (\alpha_{i,j})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ and we recall that the row of $\mathbf{A}$ are independent (see Def. 1.4). For a vector $\mathbf{X} = (X_i)_{i\in\mathcal{I}}$ with $\mathbb{E}[X_i] < +\infty$, we set $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_i])_{i\in\mathcal{I}} =: (x_i)_{i\in\mathcal{I}}$. We shall assume that the measure $\mathbb{E}[\mathbf{X}]$ is prime (cf. Def. 1.2). First, we deal with a trivial case when moreover $\mathbb{E}[\mathbf{X}]$ is elementary in the sense of Def. 1.2:

Proposition 3.1. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a random stochastic graph and $(X_i)_{i\in\mathcal{I}}$ be an independent family of non-negative random variable satisfying (Inv). We assume that $\mathbf{x} := (x_i)_{i\in\mathcal{I}} \in Ker(\mathcal{A} - Id)^+$ is a non-negative elementary prime invariant measure. Then $\mathbf{X} = (X_i \cdot \mathbf{1}_{\mathrm{Supp}(\mathbf{x})}(i))_{i\in\mathcal{I}}$ is a family of i.i.d. random variables on the support of $\mathbf{x}$. Conversely, if $\mathbf{X}$ consists of i.i.d. random variables on the support of $\mathbf{x}$ then we have $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$.*

Proof: Thanks to Proposition 1.3, it is enough to consider that the restriction of the graph $(\mathcal{I}, \mathcal{F})$ to $\mathrm{Supp}(\mathbf{x})$ is either $(\mathbb{Z}/N\mathbb{Z}, \{(k, k+1) : k \in \mathbb{Z}/N\mathbb{Z}\})$ or $(\mathbb{Z}, \{(k, k+1) : k \in \mathbb{Z}\})$. In this context, due to the elementarity of $\mathbf{x}$, we have $\alpha_{i,j} = \mathbb{E}[a_{i,j}] \in \{0, 1\}$ for all $i, j \in \mathrm{Supp}(\mathbf{x})$ and, in turn, the matrix $\mathbf{A}$ is deterministic with $a_{i,j} \in \{0, 1\}$. The conclusion follows easily from the structure of the graph considered. □

Moreover, for any non-negative invariant measure $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)^+$, we show we can always find a non-zero solution $\mathbf{X}$ to (Inv), such that $\mathbb{E}[\mathbf{X}] = \mathbf{x}$, when $\mathbf{A}$ is a Dirichlet type matrix.

Proposition 3.2. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a random stochastic graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)^+$ be any non-negative invariant measure. Let $\gamma \in \overline{\mathbb{R}}_+^*$ and consider independent random variables $(X_i)_{i\in\mathcal{I}}$ with expectation $\mathbb{E}[X_i] = x_i$ for all $i \in \mathcal{I}$, independent of $\mathbf{A}$ and both satisfying (2.2). Then $\mathbf{X} = (X_i)_{i\in\mathcal{I}}$ is solution to (Inv). Moreover, $(X_i a_{i,j})_{i,j\in\mathcal{I}\times\mathcal{I}}$ is an independent family.*

Proof: By assumption, we have $\sum_{i\in\mathcal{I}} x_i\alpha_{i,j} = x_j$ for all $j \in \mathcal{I}$, so that Corollary 2.5 entails $\sum_{i\in\mathcal{I}} X_i a_{i,j} \sim \Gamma_\gamma(x_j) \sim X_j$ for any $j \in \mathcal{I}$. By independence of the $\sum_{i\in\mathcal{I}} X_i a_{i,j}$, $j \in \mathcal{I}$, and of the X_j, $j \in J$, we also have $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$. □

The general case is considered in the next theorem which is our main result.

Theorem 3.3. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a random stochastic graph equipped with a random stochastic matrix $\mathbf{A}$ and satisfying (H), and let $\mathbf{X} = (X_i)_{i\in\mathcal{I}} \in [0, +\infty)^{\mathcal{I}}$ be an independent, non-negative, integrable random vector. We assume that $\mathbf{X}$ is independent of $\mathbf{A}$, invariant, i.e. $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$, and that $\mathbb{E}[\mathbf{X}]$ is prime and non-elementary. Then, there exists $\gamma \in \overline{\mathbb{R}}_+^*$ such that:*

$$\forall i \in \mathcal{I}, \quad X_i \sim \Gamma_\gamma(x_i) = \Gamma(\gamma x_i, \gamma).$$

Proof: The proof consists of 4 steps: in Step 1, we show that $(\phi'_{X_i}(t))_{i\in\mathcal{I}}$ is invariant; in Step 2, we obtain relations between the ϕ'_{X_i}'s and differential equations for each ϕ'_{X_i} in Step 3 that we solve in Step 4 to deduce ϕ_{X_i}.
First, we have immediately that $\mathbf{x} = \mathbb{E}[\mathbf{X}] \in \mathrm{Ker}(\mathbb{E}[\mathbf{A}] - Id)^+$ since, by independence and linearity, $\mathbb{E}[\mathbf{X}] \cdot \mathbb{E}[\mathbf{A}] = \mathbb{E}[\mathbf{X} \cdot \mathbf{A}] = \mathbb{E}[\mathbf{X}]$. In the sequel, we set $\widetilde{\mathbf{X}} := \mathbf{X} \cdot \mathbf{A}$.

Step 1: Under (H), we have

$$\phi'_{\mathbf{X}}(t) := (\phi'_{X_i}(t))_{i\in\mathcal{I}} = \left(\frac{\mathbb{E}[X_i e^{tX_i}]}{\mathbb{E}[e^{tX_i}]}\right)_{i\in\mathcal{I}} \in \mathrm{Ker}(\mathbb{E}[\mathbf{A}] - Id)^+ \quad \forall t \in (-\infty, 0)$$

viz.

$$\phi'_{\mathbf{X}} \cdot \mathbb{E}[\mathbf{A}] = \phi'_{\mathbf{X}}. \tag{3.1}$$

Indeed, we show that for any fixed $j_0 \in \mathcal{I}$ we have:

$$\phi'_{X_{j_0}} = \sum_{i\in\mathcal{I}} \alpha_{i,j_0} \phi'_{X_i}. \tag{3.2}$$

For that purpose, set:

$$\begin{aligned} J = J(j_0) &:= \{i \in \mathcal{I} : (i, j_0) \in \mathcal{F}\} \cup \{j_0\}, \\ K = K(J) &:= \{k \in \mathcal{I} : \exists i \in J,\ (i,k) \in \mathcal{F}\} \cup J, \end{aligned}$$

and observe that, under (H), J and K are determinitic with $\#J < +\infty$ and $\#K < +\infty$; moreover, since $(i,k) \in \mathcal{F} \iff \alpha_{i,k} > 0 \iff a_{i,k} \neq 0$ a.s., we have $\widetilde{X}_{j_0} = \sum_{i\in J} X_i a_{i,j_0}$, and $\sum_{j\in K} a_{i,j} = 1$ for all $i \in J$ by definition of J. We have

$$\begin{aligned} \sum_{j\in K} \widetilde{X}_j &= \sum_{j\in K}\sum_{i\in\mathcal{I}} X_i a_{i,j} = \sum_{i\in J}\sum_{j\in K} X_i a_{i,j} + \sum_{j\in K}\sum_{i\in\mathcal{I}\setminus J} X_i a_{i,j} \\ &= \sum_{i\in J} X_i + \sum_{j\in K}\sum_{i\in\mathcal{I}\setminus J} X_i a_{i,j}. \end{aligned} \tag{3.3}$$

Setting $R = \sum_{i\in\mathcal{I}\setminus J}\sum_{j\in K} X_i a_{i,j}$ and $\widetilde{R} = \sum_{j\in K\setminus J} \widetilde{X}_j$, (3.3) can be rewritten as

$$\sum_{j\in J} \widetilde{X}_j + \widetilde{R} = \sum_{i\in J} X_i + R. \tag{3.4}$$

Note that $\widetilde{R}$ and $(\widetilde{X}_j)_{j\in J}$ are independent by the independence of the rows of $\mathbf{A}$ and independence with $\mathbf{X}$. In turn, R, $(X_i)_{i\in J}$ and $((a_{i,j})_{j\in\mathcal{I}})_{i\in J}$ are also independent. Moreover, since $\sum_{j\in J} \widetilde{X}_j \sim \sum_{j\in J} X_j$ under (Inv), we have for any $t \in (-\infty, 0)$:

$$\psi_{\widetilde{R}}(t) \prod_{j\in J} \psi_{\widetilde{X}_j}(t) = \psi_{\widetilde{R}+\sum_{j\in J} X_j}(t) = \psi_{R+\sum_{i\in J} X'_i}(t) = \psi_R(t) \prod_{i\in J} \psi_{X_i}(t)$$

from which we derive $\psi_R(t) = \psi_{\widetilde{R}}(t)$ and thus $R \sim \widetilde{R}$. Next by using (3.4) and independence, we have:

$$\begin{aligned}
&\psi'_{\widetilde{X}_{j_0}}(t)\,\psi_{\widetilde{R}}(t) \prod_{l\in J\setminus\{j_0\}} \psi_{\widetilde{X}_l}(t) \\
&= \mathbb{E}\left[\widetilde{X}_{j_0} e^{t\left(\widetilde{R}+\sum_{l\in J}\widetilde{X}_l\right)}\right] = \mathbb{E}\left[\sum_{i\in J} X_i a_{i,j_0} e^{t\left(R+\sum_{i\in J} X_i\right)}\right] \\
&= \sum_{i\in J} \alpha_{i,j_0}\psi'_{X_i}(t)\,\psi_R(t) \prod_{l\in J\setminus\{i\}} \psi_{X_l}(t) = \left(\sum_{i\in J}\alpha_{i,j_0}\frac{\psi'_{X_i}(t)}{\psi_{X_i}(t)}\right)\psi_{\widetilde{R}}(t)\prod_{l\in J}\psi_{\widetilde{X}_l}(t)
\end{aligned}$$

since $\widetilde{R} \sim R$ and $\widetilde{X}_i \sim X_i$. Thus, we have:

$$\phi'_{\widetilde{X}_{j_0}}(t) = \frac{\psi'_{\widetilde{X}_{j_0}}(t)}{\psi_{\widetilde{X}_{j_0}}(t)} = \sum_{i\in J}\alpha_{i,j_0}\frac{\psi'_{X_i}(t)}{\psi_{X_i}(t)} = \sum_{i\in J}\alpha_{i,j_0}\phi'_{X_i}(t) = \sum_{i\in\mathcal{I}}\alpha_{i,j_0}\phi'_{X_i}(t),$$

where the last equality comes from the choice of J from (H). This proves (3.2) which implies (3.1) and achieves Step 1.

Step 2: Note that $\phi'_{\mathbf{X}}(t) \in \mathrm{Ker}(\mathbb{E}[\mathbf{A}] - Id)^+$ for $t < 0$ by Step 1 and that $\phi'_{\mathbf{X}}(t) \leqslant \mathbb{E}[\mathbf{X}]$ for all $t \leqslant 0$ (where, here, $\leqslant$ is to be understood coordinatewisely) since for all $i \in \mathcal{I}$, by Lemma 2.8:

$$0 \leqslant \phi'_{X_i}(t) = \frac{\mathbb{E}[X_i e^{X_i t}]}{\mathbb{E}[e^{X_i t}]} \leqslant \phi'_{X_i}(0) = x_i.$$

Since $\mathbb{E}[\mathbf{X}]$ is prime (Def. 1.2), for all $t < 0$, there is some $\lambda(t) \in [0,1]$ such that $\lambda(t)\,\mathbb{E}[\mathbf{X}] = \phi'_{\mathbf{X}}(t)$. Thus, we have $x_j\phi'_{X_i}(t) = x_i\phi'_{X_j}(t)$ for all $t \leqslant 0$ and all $(i,j) \in \mathcal{I}\times\mathcal{I}$. By holomorphy, we extend the equality on $\mathbb{C}^-$ (the set of complex numbers with non-positive real part):

$$x_j\phi'_{X_i}(z) = x_i\phi'_{X_j}(z) \quad \text{for all } z \in \mathbb{C}^- \text{ and } (i,j) \in \mathcal{I}\times\mathcal{I}. \tag{3.5}$$

Step 3: We establish the following differential equations on $\mathbb{C}^-$: for all $i \in \mathcal{I}$, there are $\mu_i, \lambda_i \in [0,+\infty)$ such that

$$\mu_i\phi'^2_{X_i} = \lambda_i\phi''_{X_i}. \tag{3.6}$$

First, observe that (3.6) is immediately satisfied for $j_0 \notin \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ (i.e. $x_{j_0} = 0$), since in this case (3.5) entails $\phi'_{X_{j_0}} = 0$ and thus $X_{j_0} = x_{j_0} = 0$. Next, we consider a fixed $j_0 \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$. With the same notation as in Step 1, by using successively the notation $\widetilde{\mathbf{X}} = \mathbf{X}\cdot\mathbf{A}$, the identity (3.4) and the independence $R \perp\!\!\!\perp (X_i)_{i\in J}$, we have for $t < 0$:

$$\begin{aligned}
&\psi''_{\widetilde{X}_{j_0}}(t)\,\psi_{\widetilde{R}}(t)\prod_{l\in J\setminus\{j_0\}}\psi_{\widetilde{X}_l}(t) = \mathbb{E}\left[\widetilde{X}^2_{j_0} e^{t\left(\widetilde{R}+\sum_{l\in J}\widetilde{X}_l\right)}\right] \\
&= \mathbb{E}\left[\left(\sum_{i\in J} a_{i,j_0}X_i\right)^2 e^{t\left(R+\sum_{i\in J}X_i\right)}\right] = \mathbb{E}\left[\left(\sum_{i\in J} a_{i,j_0}X_i\right)^2 e^{t\sum_{i\in J}X_i}\right]\mathbb{E}\left[e^{tR}\right] \\
&= \sum_{i\in J}\mathbb{E}\left[a^2_{i,j_0}X_i^2 e^{t\sum_{k\in J}X_k}\right]\psi_R(t) + \sum_{\substack{i,j\in J\\ i\neq j}}\mathbb{E}\left[a_{i,j_0}a_{j,j_0}X_iX_j e^{t\sum_{k\in J}X_k}\right]\psi_R(t) \\
&= \left(\sum_{i\in J}\mathbb{E}\left[a^2_{i,j_0}\right]\frac{\psi''_{X_i}(t)}{\psi_{X_i}(t)} + \sum_{\substack{i,j\in J\\ i\neq j}}\alpha_{i,j_0}\frac{\psi'_{X_i}(t)}{\psi_{X_i}(t)}\alpha_{j,j_0}\frac{\psi'_{X_j}(t)}{\psi_{X_j}(t)}\right)\psi_R(t)\prod_{k\in J}\psi_{X_k}(t).
\end{aligned}$$

Setting $\nu_{i,j_0} := \mathbb{E}\left[a_{i,j_0}^2\right]$, we conclude that:

$$\frac{\psi''_{X_{j_0}}}{\psi_{X_{j_0}}} = \sum_{i\in\mathcal{I}} \nu_{i,j_0}\left(\phi''_{X_i} + \phi'^2_{X_i}\right) + \sum_{\substack{i,j\in J\\ i\neq j}} \alpha_{i,j_0}\alpha_{j,j_0}\phi'_{X_i}\phi'_{X_j}. \tag{3.7}$$

But using the invariance (3.1) of $\phi'_{\mathbf{X}}$, we also have:

$$\begin{aligned}\frac{\psi''_{X_{j_0}}}{\psi_{X_{j_0}}} &= \phi''_{X_{j_0}} + \phi'^2_{X_{j_0}} = \phi''_{X_{j_0}} + \left(\sum_{i\in\mathcal{I}} \alpha_{i,j_0}\phi'_{X_i}\right)^2 \\ &= \sum_{i\in\mathcal{I}} \alpha_{i,j_0}\phi''_{X_i} + \sum_{i\in\mathcal{I}} \alpha^2_{i,j_0}\phi'^2_{X_i} + \sum_{\substack{i,j\in J\\ i\neq j}} \alpha_{i,j_0}\alpha_{j,j_0}\phi'_{X_i}\phi'_{X_j}.\end{aligned} \tag{3.8}$$

Since $x_{j_0} \neq 0$ for $j_0 \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$, comparing (3.7) and (3.8), and using (3.5), we get:

$$\begin{aligned}\left(\sum_{i\in\mathcal{I}}(\alpha_{i,j_0} - \nu_{i,j_0})\frac{x_i}{x_{j_0}}\right)\phi''_{X_{j_0}} &= \sum_{i\in\mathcal{I}}(\alpha_{i,j_0} - \nu_{i,j_0})\phi''_{X_i} \\ &= \sum_{i\in\mathcal{I}}(\nu_{i,j_0} - \alpha^2_{i,j_0})\phi'^2_{X_i} = \left(\sum_{i\in\mathcal{I}}(\nu_{i,j_0} - \alpha^2_{i,j_0})\frac{x_i^2}{x_{j_0}^2}\right)\phi'^2_{X_{j_0}},\end{aligned}$$

proving equation (3.6) on $\mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ with the constants

$$\begin{aligned}\mu_{j_0} &= \sum_{i\in\mathcal{I}}(\nu_{i,j_0} - \alpha^2_{i,j_0})\frac{x_i^2}{x_{j_0}^2} = \sum_{i\in\mathcal{I}}\left(\mathbb{E}[a^2_{i,j_0}] - \mathbb{E}[a_{i,j_0}]^2\right)\frac{x_i^2}{x_{j_0}^2} \geqslant 0 \\ \lambda_{j_0} &= \sum_{i\in\mathcal{I}}(\alpha_{i,j_0} - \nu_{i,j_0})\frac{x_i}{x_{j_0}} = \sum_{i\in\mathcal{I}}\mathbb{E}[a_{i,j_0}(1 - a_{i,j_0})]\frac{x_i}{x_{j_0}} \geqslant 0.\end{aligned}$$

Step 4: Since the measure $\mathbb{E}[\mathbf{X}]$ is not elementary, there is $i \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ with $0 < \alpha_{i,j} < 1$ for some j, necessarily in $\mathrm{Supp}(\mathbb{E}[\mathbf{X}])$, since $x_j = \sum_{k\in\mathcal{I}} x_k\alpha_{k,j} \geqslant x_i\alpha_{i,j} > 0$. Thus, $\alpha^2_{i,j} < \alpha_{i,j}$ and at least one of the two inequalities $\alpha^2_{i,j} \leqslant \nu_{i,j} \leqslant \alpha_{i,j}$ is strict so that either $\lambda_j > 0$ or $\mu_j > 0$. If $\lambda_j = 0$ then $\phi'^2_{X_j} = 0$ from (3.6); this requires $X_j = 0$ which is in contradiction with $x_j \neq 0$ since $j \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$. As a consequence, we can assume there is $j \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ such that $\lambda_j > 0$ and for this index the differential equation (3.6) can be written as $\rho_j\phi'^2_{X_j} = \phi''_{X_j}$ for some parameter $\rho_j \geqslant 0$. We conclude with Lemma 2.7 that for all $i \in \mathcal{I}$, there exists c_i such that

$$\phi'_{X_i}(t) = c_i\phi'_{X_{j_0}}(t) = \frac{c_i x_{j_0}}{1 - \frac{t}{\gamma}} = \frac{x_i}{1 - \frac{t}{\gamma}}.$$

□

Remark 3.4. When the measure $\mathbb{E}[\mathbf{X}]$ is elementary we have $\alpha_{i,j} \in \{0,1\}$ for all $i \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ so that $\alpha_{i,j} = \nu_{i,j} = \alpha^2_{i,j}$ for all $i \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ and we derive $\lambda_j = \mu_j = 0\ \forall j \in \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ in Step 3 above. In this case, the differential equation (3.6) reduces to 0 and the strategy of the previous proof is not applicable.

3.2. *Related results.* In this section, we discuss results related to our main result, Theorem 3.3. First in the following result, $\mathbf{X}$ is not assumed to be invariant, in contrast to Theorem 3.3, and the point in Theorem 3.5 is to investigate what can be said beyond invariance.

Theorem 3.5. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a random stochastic graph equipped with a random stochastic matrix $\mathbf{A}$, and $\mathbf{X} = (X_i)_{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$ be an independent, non-negative, integrable random vector. We assume that $\mathbf{X}$ is independent of $\mathbf{A}$ and $\mathbb{E}[\mathbf{X}]$ is prime and non-elementary. Set $\widetilde{\mathbf{X}} := \mathbf{X} \cdot \mathbf{A}$ and further assume that $\widetilde{X}_j \sim X_j$ for each $j \in \mathcal{I}$. Then:*

(1) When (H) is in force, the following statements are equivalent:

(a) The family $(\widetilde{X}_j)_{j\in\mathcal{I}}$ consists of independent random variables;

(b) For some $\gamma \in \overline{\mathbb{R}}_+^*$, and all finite subset $J \subset \mathcal{I}$, we have:

$$\widetilde{X}_j \sim \Gamma_\gamma(x_j) \quad \forall j \in J \qquad \perp\!\!\!\perp \qquad \left(\frac{\widetilde{X}_j}{\sum_{i\in J} \widetilde{X}_i}\right)_{j\in J} \sim \mathrm{D{\scriptstyle IR}}_{\gamma \sum_{i\in J} x_i}\left(\frac{x_j}{\sum_{i\in J} x_i}\right)_{j\in J};$$

(2) The following statements are equivalent (i.e. without assuming (H)):

(a) The family $(X_i a_{i,j})_{i,j\in\mathcal{I}}$ consists of independent random variables;

(b) For some $\gamma \in \overline{\mathbb{R}}_+^*$, and all $i \in \mathcal{I}$, we have:

$$X_i \sim \Gamma_\gamma(x_i) \quad \text{and} \quad (a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{D{\scriptstyle IR}}_{\gamma x_i}(\alpha_{i,j})_{j\in\mathcal{I}}.$$

Proof: (1a)⇒(1b): Under (1a), we have $\widetilde{\mathbf{X}} \sim \mathbf{X}$ and thus $\mathbf{X}\cdot\mathbf{A} \sim \mathbf{X}$, i.e. $\mathbf{X}$ is invariant. Theorem 3.3 applies and gives $\widetilde{X}_i \sim X_i \sim \Gamma_\gamma(x_i)$. The Dirichlet distribution and the independence in (1b) follows from Lemma 2.2 applied to the $\widetilde{X}_j \sim \Gamma_\gamma(x_j)$, $j \in J$.

(1b)⇒(1a) derives from Theorem 2.4 applied to the gamma-distributed random variable $\sum_{j\in J} \widetilde{X}_j$ and the independent Dirichlet-distributed random vector $\big(\widetilde{X}_j / \sum_{i\in J} \widetilde{X}_i\big)_{j\in J}$.

(2a)⇒(2b): Applying Theorem 2.4 for each $i \in \mathcal{I}$ to $(X_i a_{i,j})_{j\in J}$, we have, for all $(i,j) \in \mathcal{F}$, $X_i a_{i,j} \sim \Gamma_{\gamma_i}(x_i \alpha_{i,j})$ and $(a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{D{\scriptstyle IR}}_{\gamma_i x_i}(\alpha_{i,j})_{j\in\mathcal{I}}$, which is the conclusion except that, at this point, the index γ_i depends, a priori, on $i \in \mathcal{I}$. Using Lemma A.15 for any edge $(i,j) \in \mathcal{F}$, we deduce $\gamma_i = \gamma_j$. Finally, since $\mathcal{I} = \mathrm{Supp}(\mathbf{X})$ is connected, all the γ_i's must coincide and we have a global scale parameter $\gamma \in \overline{\mathbb{R}}_+^*$ for the gamma distribution, i.e. $\gamma_i = \gamma$ for all $i \in \mathcal{I}$.

(2b)⇒(2a): Applying Theorem 2.4 for each $i \in \mathcal{I}$ to X_i and $(a_{i,j})_{j\in\mathcal{I}}$ gives the independence of the family $\{X_i a_{i,j} : j \in \mathcal{I}\}$. This independence extends to $(X_i a_{i,j})_{i,j\in\mathcal{I}}$ since the X_i, $(a_{i,j})_{j\in\mathcal{I}}$ are independent for different index i. □

Remark 3.6 (On Theorem 3.5).

- In (1b) in Theorem 3.5, $\widetilde{X}_j$ cannot be replaced by X_j.
- In Theorem 3.5, the statements (1a) and (2b) are not equivalent, see Example 4.5.
- Roughly speaking, part (1) of Theorem 3.5 is global while part (2) is local describing the proportion in each edge (i,j) of the graph. Note that the Dirichlet proportion obtained in (2b) is typical of such random stochastic matrix setting, see McKinlay (2014, Th. 4) and references therein.

Theorem 3.7. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a random stochastic connected graph equipped with a random stochastic matrix $\mathbf{A} = (a_{i,j})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ and, independently, $\mathbf{X} = (X_i)_{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$ be a non-negative random vector. We assume that $\#\{j \in \mathcal{I} : (i,j) \in \mathcal{F}\} \geqslant 2$ for all $i \in \mathcal{I}$, $\mathcal{I} = \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ (i.e. $\mathbb{E}[X_i] > 0$ $\forall i \in \mathcal{I}$) and $(X_i a_{i,j})_{i,j\in\mathcal{I}^2}$ is a collection of independent random variables. Then, the following statements are equivalent:*

(1) $\mathbf{X}$ is invariant, i.e. $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$;

(2) $\exists \gamma \in \overline{\mathbb{R}}_+^*$ such that $\forall i \in \mathcal{I}$: $X_i \sim \Gamma_\gamma(x_i)$ and $(a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{D{\scriptstyle IR}}_{\gamma x_i}(\alpha_{i,j})_{j\in\mathcal{I}}$ and $\mathbb{E}[\mathbf{X}] = (x_i)_{i\in\mathcal{I}}$ is an invariant measure.

Moreover in this case, if $\mathbf{Y} = (Y_i)_{i\in\mathcal{I}}$ is a non-negative random vector independent of $\mathbf{A}$ such that $\mathbf{Y} \cdot \mathbf{A} \sim \mathbf{Y}$ and $(Y_i a_{i,j})_{i,j\in\mathcal{I}^2}$ are independent, then $\mathbf{Y}$ is integrable and

- *If $\gamma < +\infty$ in 2), then $\mathcal{L}(\mathbf{Y}) = \mathcal{L}(\lambda\,\mathbf{X})$ for some $\lambda > 0$;*
- *If $\gamma = +\infty$ in 2), then $\mathbf{Y} = \mathbb{E}[\mathbf{Y}]$ a.s. and $\mathbb{E}[\mathbf{Y}]$ is an invariant measure.*

Conversely, if $\mathbf{Y}$ has one the two previous forms then it is a solution.

Proof: Since the converse part 2)⇒1) is given by Proposition 3.2, it is enough to prove the direct implication 1)⇒2) and we thus assume 1). Since $\#\{j \in \mathcal{I} : (i,j) \in \mathcal{F}\} \geqslant 2$ for all $i \in \mathcal{I}$, we can use Theorem 2.4 at all vertex $i \in \mathcal{I}$ and obtain:

$$\forall i \in \mathcal{I}, \exists \gamma_i \in \overline{\mathbb{R}}_+^* \text{ such that } X_i \sim \Gamma_{\gamma_i}(x_i) \text{ and } (a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{Dir}_{\gamma x_i}(\alpha_{i,j})_{j\in\mathcal{I}}.$$

As an immediate consequence, $\mathbf{X}$ is integrable and $\mathbb{E}[\mathbf{X}]$ is an invariant measure. We conclude that all the γ_i's must be equal like above in the proof of (2b)⇒(2a) for Theorem 3.5.

For the second part, if $\mathbf{Y} = 0$ a.s. then the result is immediate. We can thus assume that $\mathrm{Supp}(\mathbf{Y}) \neq \emptyset$. Just like for $\mathbf{X}$ above, we have that $Y_i \sim \Gamma_{\gamma'_i}(y_i)$ for all $i \in \mathrm{Supp}(\mathbf{Y})$ and $(a_{i,j})_{j\in\mathcal{I}} \sim \mathrm{Dir}_{\gamma x_i}(\alpha_{i,j})_{j\in\mathcal{I}} = \mathrm{Dir}_{\gamma'_i y_i}(\alpha_{i,j})_{j\in\mathcal{I}}$. Then, $\mathbf{Y}$ is integrable and $\mathbb{E}[\mathbf{Y}]$ is an invariant measure and we have $\gamma x_i = \gamma'_i y_i$.
If $\gamma = +\infty$, then necessarily $\gamma'_i = +\infty$ for $i \in \mathrm{Supp}(\mathbf{Y})$, and $\mathbf{X}, \mathbf{A}, \mathbf{Y}$ are all degenerate, i.e. $\mathbf{Y} = \mathbb{E}[\mathbf{Y}]$ a.s.
If $\gamma < +\infty$, Corollary 2.5 entails $Y_i a_{i,j} \sim \Gamma_{\gamma'_i}(x_i\alpha_{i,j})$ for all $(i,j) \in \mathcal{F}$. We show $\mathrm{Supp}(\mathbf{Y}) = \mathrm{Supp}(\mathbf{X})$: indeed, for any fixed $j \in \mathrm{Supp}(\mathbf{Y})$, we have:

$$\sum_{i\in\mathrm{Supp}(\mathbf{X})} x_i\alpha_{i,j} = x_j = \frac{\gamma'_j}{\gamma} y_j = \frac{\gamma'_j}{\gamma} \sum_{i\in\mathrm{Supp}(\mathbf{Y})} y_i\alpha_{i,j} = \sum_{i\in\mathrm{Supp}(\mathbf{Y})} x_i\alpha_{i,j}, \tag{3.9}$$

where in the last equality we used $\gamma'_j y_i = \gamma x_i$ stemming from $\gamma'_i = \gamma'_j$ derived with Lemma A.15 for $(i,j) \in \mathcal{F}$. Since $\mathrm{Supp}(\mathbf{Y}) \subset \mathrm{Supp}(\mathbf{X})$ and $x_i, \alpha_{i,j} > 0$ for $(i,j) \in \mathcal{F}$ and $i \in \mathrm{Supp}(\mathbf{X})$, the equality (3.9) requires the equality $\mathrm{Supp}(\mathbf{Y}) = \mathrm{Supp}(\mathbf{X}) = \mathcal{I}$. Since $\mathcal{I}$ is connected, one more application of Lemma A.15 yields $\gamma'_i = \gamma'_j =: \gamma'$ for all $i,j \in \mathcal{I}$ and $\mathbf{Y} \sim (\gamma/\gamma')\mathbf{X}$. □

Remark 3.8 (On Theorem 3.7).

1) Note that in the second part of Theorem 3.7, when $\gamma = +\infty$, the random variables Y_i's and the $a_{i,j}$'s are a.s. constant, justifying the condition on $(Y_i a_{i,j})_{i,j\in\mathcal{I}^2}$ is indeed fulfilled.
2) The equivalence (1)⟺(2) still holds true when instead of assuming $\#\{j \in \mathcal{I} : (i,j) \in \mathcal{F}\} \geqslant 2$ for all $i \in \mathcal{I}$, with $\mathcal{I} = \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$, we assume $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_i])_{i\in\mathcal{I}}$ is a non-elementary, prime invariant measure. Indeed in this setting, the implication (1)⇒(2) is due to (2a)⇒(2b) in Theorem 3.5. While (2)⇒(1) comes from Proposition 3.2. However, the second part of Theorem 3.7 does not hold in this setting as illustrated by Example 4.7 and Example 4.8.

4. Examples and applications

In this section, we specify our results in the context of finite graphs (Examples 4.1 and 4.2) and of bistochastic graphs (Example 4.4). Several examples also discuss the conditions of our main results (Examples 4.5, 4.6, 4.7, 4.8).

4.1. *Finite graphs.* For finite graphs, the condition (H) is clearly superfluous.

Example 4.1 (Irreducible finite graphs). Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ be a finite random stochastic graph such that the stochastic matrix $\mathcal{A} := \mathbb{E}[\mathbf{A}]$ is irreducible and let $\mathbf{X} = (X_i)_{i\in\mathcal{I}}$ be an independent non-negative random vector, satisfying $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$. Since the (finite) matrix $\mathcal{A}$ is irreducible, the Perron-Frobenius theorem ensures that $\dim(\mathrm{Ker}(\mathcal{A} - Id)) = 1$ and the corresponding eigenvector is positive, thus $\mathrm{Ker}(\mathcal{A} - Id)^+ \neq \{0\}$. Since $\mathbb{E}[\mathbf{X}] \in \mathrm{Ker}(\mathcal{A} - Id)$, it is necessarily prime. Then:

- when $\mathbb{E}[\mathbf{X}]$ is elementary, Proposition 3.1 applies and $\mathbf{X}$ consists of i.i.d. random variables X_i;

- when $\mathbb{E}[\mathbf{X}]$ is non-elementary, Theorem 3.3 applies and there exists some $\gamma \in \overline{\mathbb{R}}_+^*$ such that, for all $i \in \mathcal{I}$, $X_i \sim \Gamma_\gamma(\mathbb{E}[X_i])$.

Example 4.2 (General finite graph). For a finite random stochastic graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$, we can specify completely the solutions of (Inv) as follows.

First, we can ignore the vertices $i \in \mathcal{I}$ such that $\forall \mathbf{x} \in \operatorname{Ker}(\mathcal{A} - Id),\ x_i = 0$. Moreover by Proposition A.7, we can assume that the matrix $\mathcal{A}$ consists of diagonal irreducible blocks, corresponding to its positive recurrence classes. There is a family $\mathcal{P}$ of prime invariant measures generating $\operatorname{Ker}(\mathcal{A} - Id)^+$ in the sense that any $\mathbf{x} \in \operatorname{Ker}(\mathcal{A} - Id)^+$ can be uniquely written $\mathbf{x} = \sum_{\mathbf{p} \in \mathcal{P}} \lambda_\mathbf{p}\, \mathbf{p}$ for $\lambda_\mathbf{p} \geqslant 0$ such that $\sum_{\mathbf{p} \in \mathcal{P}} \lambda_\mathbf{p} p_i < +\infty$ for all $i \in \mathcal{I}$. On the one hand, note that by Proposition A.7 two prime invariant measures are either proportional or with disjoint supports, and on the other hand if $\mathbf{x}, \mathbf{y} \in \operatorname{Ker}(\mathcal{A} - Id)^+$ are such that $\operatorname{Supp}(\mathbf{x}) \cap \operatorname{Supp}(\mathbf{y}) = \emptyset$ and $\mathbf{X}$, $\mathbf{Y}$ are two independent solutions of (Inv) such that $\mathbb{E}[\mathbf{X}] = \mathbf{x}$ and $\mathbb{E}[\mathbf{Y}] = \mathbf{y}$, then $\mathbf{X} + \mathbf{Y}$ is also a solution of (Inv), since both the $(X_i)_{i \in \mathcal{I}}$ and the $(Y_i)_{i \in \mathcal{I}}$ are independent. As a consequence, it is enough to take one solution $\mathbf{X}(\mathbf{p})$ for each irreducible block following Example 4.1. Finally, we have:

$$\{\mathbf{X} \text{ solution of (Inv)}\} = \left\{ \sum_{p \in \mathcal{P}} \lambda_\mathbf{p} \mathbf{X}(\mathbf{p}) : \begin{array}{l} \mathbf{X}(\mathbf{p}), \mathbf{p} \in \mathcal{P}, \text{ are independent solutions of (Inv)} \\ \text{with } \mathbb{E}[\mathbf{X}(\mathbf{p})] = \mathbf{p} \text{ and } \lambda_\mathbf{p} \geqslant 0 \end{array} \right\}.$$

Remark 4.3. Note that the problem (Inv) may not have solution $\mathbf{X}(\mathbf{p})$, the existence of such a solution depends on the distribution of the matrix $\mathbf{A}$.

4.2. *Graph $\mathbb{Z}^d$ and random exchange process for economic agents.* In Muratov and Zuyev (2017), a random exchange process is considered for the wealth of economic agents, represented by the lengths τ_k of the intervals $[T_{k-1}, T_k]$ delimited by two successive points of a renewal process $(T_k)_{k \in \mathbb{Z}}$ on the real line. In this setting, each agent gives a random proportion of its wealth to all the other agents while, simultaneously, receiving independent proportions of the wealth of all the other agents. Considering the graph $\mathbb{Z}$ supporting the random variables $X_k = \tau_k$ in each $k \in \mathbb{Z}$, Theorem 3.3 reduces to Theorem 3 and Corollary 1 in Muratov and Zuyev (2017), i.e. when there is invariance, then necessarily the wealth random variables X_k's are gamma-distributed and, for all k, $(a_{k,l})_{l \in \mathbb{Z}}$ follows a Dirichlet distribution. See also Section 2 in Muratov and Zuyev (2017) and Theorem 1 therein for a simpler model where each agent exchanges wealth only with its right-neighbor.

Furthermore, Muratov and Zuyev (2017) conjectures that on $\mathcal{I} = \mathbb{Z}^d$, if we assume that $\mathbf{X} = (X_i)_{i \in \mathbb{Z}^d}$ is i.i.d and $((a_{i,i+j})_{j \in \mathbb{Z}^d})_{i \in \mathbb{Z}^d}$ is also i.i.d, then under the hypothesis (H), $\mathbf{X}$ consists of gamma-distributed random variables and $\mathbf{A}$ is Dirichlet-distributed (for further details in this direction, see Corollary 2.5). In the following Example 4.4, we prove the gamma distribution part of this conjecture in a more general context. However, in this general setting, $\mathbf{A}$ is not necessarily Dirichlet-distributed, see Example 4.5 below.

Example 4.4 (Bistochastic graphs). Let $(G, +)$ be a finitely generated abelian group and let $J \subset G$ be a finite set of generators (i.e. the sub-group $\langle J \rangle$ generated by J is G). Consider the following graph:

$$\mathcal{G} = (\mathcal{I}, \mathcal{F}, A), \quad \text{where } \mathcal{I} = G \text{ and } \mathcal{F} = \{(g, g + j) : g \in G, j \in J\},$$

and assume that $\mathcal{A} := \mathbb{E}[\mathbf{A}] = (\alpha_{g,g'})_{g,g' \in G}$ satisfies $\alpha_{g,g+j} = \alpha_{0,j} > 0\ \forall (g,j) \in G \times J$. By Proposition A.12, $\mathbb{1} := (1)_{i \in \mathcal{I}}$ is a prime invariant measure of $\mathcal{A}$ and every bounded $\mathbf{x} \in \operatorname{Ker}(\mathcal{A} - Id)$ is constant.

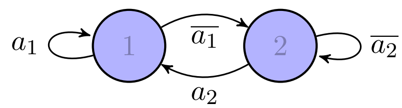
If $\#J = 1$, then $\mathbf{x}$ is an elementary invariant measure like in Proposition 3.1.
If $\#J \geqslant 2$, when $(a_{g,g+j})_{(g,j) \in G \times J}$ are i.i.d. random variables in $g \in G$ such that $\mathbb{E}[a_{0,j}] > 0$ for all

$j \in J$ and $\mathbf{X} = (X_g)_{g\in G}$ is a family of i.i.d. non-negative random variables such that $\mathbb{E}[X_g] < +\infty$, Theorem 3.3 applies and gives:

$$\exists\gamma \in \overline{\mathbb{R}}_+^*,\ \exists x \geqslant 0,\ X_g \sim \Gamma_\gamma(x).$$

4.3. *Counter-examples.*

Example 4.5 (On two-dimensional random stochastic matrix). Consider the graph $\mathcal{G} = (\{1,2\}, \{(1,1),(1,2),(2,1),(2,2)\})$, with random stochastic matrix $\mathbf{A} = \begin{pmatrix} a_1 & \overline{a_1} \\ a_2 & \overline{a_2} \end{pmatrix}$ (recall that a_1 and a_2 are independent, $\alpha_1 = \mathbb{E}[a_1], \alpha_2 = \mathbb{E}[a_2] \in (0,1)$ and the notation $\overline{x} := 1 - x$ for any $x \in [0,1]$)



Independently of $\mathbf{A}$, let X_1, X_2 be independent non-negative random variables and set $(\widetilde{X}_1, \widetilde{X}_2) := (X_1, X_2)\cdot \mathbf{A}$. Assume $\widetilde{X}_1 \sim X_1$ and $\widetilde{X}_2 \sim X_2$. Then

$$\widetilde{X}_1,\ \widetilde{X}_2 \text{ are independent} \iff X_1 \sim \Gamma(\gamma x_1, \gamma),\ X_2 \sim \Gamma(\gamma x_2, \gamma). \tag{4.1}$$

The direct implication stems from Example 4.1 since first $\mathcal{A} = \mathbb{E}[\mathbf{A}]$ is finite and irreducible, next under the independence of $\widetilde{X}_1, \widetilde{X}_2$, the distribution of $\mathbf{X} = (X_1, X_2)$ is invariant and finally $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2]) = (x_1, x_2)$ is non-elementary $(\alpha_1, \alpha_2 \in (0,1))$. For the converse implication, using independence we have necessarily:

$$x_1 = \mathbb{E}\left[\widetilde{X}_1\right] = \mathbb{E}[a_1]\,\mathbb{E}[X_1] + \mathbb{E}[a_2]\,\mathbb{E}[X_2] = \alpha_1 x_1 + \alpha_2 x_2$$

(and also $x_2 = \overline{\alpha_1} x_1 + \overline{\alpha_2} x_2$), so that Lemma A.14 applies and gives $\widetilde{X}_1 \perp\!\!\!\perp \widetilde{X}_2$.

Moreover, observe that the measure $\mathbf{x} = (x_1, x_2)$ is prime since it is invariant and $\mathrm{Ker}(\mathcal{A} - Id) = \mathrm{Span}(x_1, x_2)$ is one-dimensional by the Perron-Frobenius theorem.

The equivalence (4.1) is an illustration of a partial equivalence between 1a) and 2b) in Theorem 3.5 in this case. But a full equivalence does not hold true since a_1 and a_2 are not necessarily Beta-distributed (as should be the marginals of a Dirichlet distribution). Indeed, set $\alpha_1 = \mathbb{E}[a_1] = \alpha_2 = \mathbb{E}[a_2] = 1/2$, $x_1 = x_2 = 1$ and $\gamma = 1$ so that $X, Y \sim \Gamma(1,1)$. On the one hand, Proposition 3.2 ensures

$$a_1 \sim \mathrm{Beta}(1/2, 1/2),\ a_2 \sim \mathrm{Beta}(1/2,1/2) \implies a_1 X_1 + a_2 X_2 \sim \Gamma(1,1),$$

recovering two-dimensional random stochastic matrices with independent Beta-distributed rows as discussed in Van Assche (1986). On the other hand, inspired by an example due to Bourguin and Tudor (2011), we also have:

$$a_1 \sim \mathrm{Bernoulli}(1/2),\ a_2 = 1/2 \implies a_1 X_1 + a_2 X_2 \sim \Gamma(1,1),$$

since with $\psi_{X_1}(t) = \psi_{X_2}(t) = \frac{1}{1-t}$, we have:

$$\begin{aligned}\psi_{a_1X_1+a_2X_2}(t) &= \mathbb{E}\left[e^{ta_1X_1}\right]\mathbb{E}\left[e^{ta_2X_2}\right] = \left(\frac{1}{2}\,\mathbb{E}\left[e^{tX_1}\right] + \frac{1}{2}\right)\mathbb{E}\left[e^{tX_2/2}\right] \\ &= \left(\frac{1}{2(1-t)} + \frac{1-t}{2(1-t)}\right)\frac{1}{1-t/2} = \frac{1}{1-t}.\end{aligned}$$

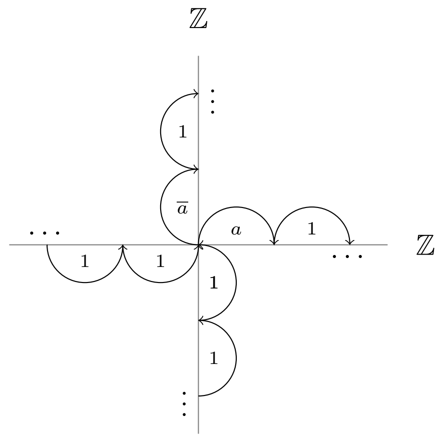
Example 4.6 (Prime necessity). In the following example, we show that the conclusion of Theorem 3.3 does not hold true when we relax the prime condition for $\mathbf{x} = \mathbb{E}[\mathbf{X}]$. Consider the graph

$$\mathcal{G} = \Big(\mathbb{Z}\times\{0\} \cup \{0\}\times\mathbb{Z}, \{((k,0),(k+1,0)), k \in \mathbb{Z}\} \cup \{((0,k),(0,k+1)), k \in \mathbb{Z}\}\Big)$$

with random stochastic matrix $\mathbf{A}$ given by $a_{((k,0),(k+1,0))} = a_{((0,k),(0,k+1))} = 1$ for $k \neq 0$ and for $k = 0$:

$$\left(a_{((0,0),(1,0))}, a_{((0,0),(0,1))}\right) \sim \textsc{Dir}_{\gamma z}(\alpha, \overline{\alpha})$$

with $\alpha \in (0,1)$.



Let Z_1, Z_2 be two independent non-degenerate non-negative random variables such that $Z_1 + Z_2 = Z \sim \Gamma(\gamma z, \gamma)$ but not gamma-distributed (see Bourguin and Tudor, 2011, Section 4). Consider $\mathbf{X} = (X_i)_{i \in \mathcal{I}}$ to be a family of independent random variables with:

$$\begin{aligned}
&\forall k < 0: \quad X_{(k,0)} \sim Z_1, \quad X_{(0,k)} \sim Z_2, \quad X_{(0,0)} \sim Z, \\
&\forall k > 0: \quad X_{(k,0)} \sim Z a_{((0,0),(1,0))}, \quad X_{(0,k)} \sim Z a_{((0,0),(0,1))}.
\end{aligned}$$

Then $\mathbf{X} \cdot \mathbf{A} \sim \mathbf{X}$ since we have:

$$\begin{aligned}
(\mathbf{X} \cdot \mathbf{A})_{(0,0)} &= X_{(-1,0)} a_{(-1,0),(0,0)} + X_{(0,-1)} A_{(0,-1),(0,0)} \sim Z_1 + Z_2 = Z \sim X_{(0,0)}, \\
(\mathbf{X} \cdot \mathbf{A})_{(1,0)} &= X_{(0,0)} a_{(0,0),(1,0)} = Z a_{(0,0),(1,0)} \sim X_{(1,0)}, \\
(\mathbf{X} \cdot \mathbf{A})_{(0,1)} &= X_{(0,0)} a_{(0,0),(0,1)} = Z a_{(0,0),(0,1)} \sim X_{(0,1)},
\end{aligned}$$

and it follows directly that $(\mathbf{X} \cdot \mathbf{A})_i = X_i$ for $i \in \{(\pm k, 0), (0, \pm k) : k \in \mathbb{N} \setminus \{0,1\}\}$. Then, (Inv) derives from independence. In this case, not all the X_i's are gamma-distributed, but the measure $\mathbf{x} = \mathbb{E}[\mathbf{X}]$, given by

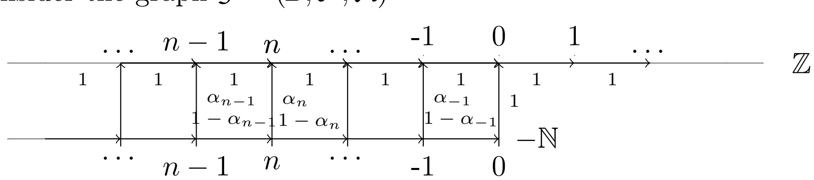
$$\mathbf{x}(-k,0) = \mathbb{E}[Z_1], \ \mathbf{x}(0,-k) = \mathbb{E}[Z_2], \ \mathbf{x}(0,0) = z, \ \mathbf{x}(k,0) = \alpha z, \ \mathbf{x}(0,k) = \overline{\alpha} z, \quad k \geqslant 1, \tag{4.2}$$

is not prime. Indeed, $\mathrm{Ker}(\mathcal{A} - Id)$ is two-dimensional,and is generated by $\mathbf{m}_1$ and $\mathbf{m}_2$ given by:

$$\begin{aligned}
&\mathbf{m}_1(-k,0) = 1, \ \mathbf{m}_1(0,-k-1) = 0, \ \mathbf{m}_1(k+1,0) = \gamma z \alpha, \ \mathbf{m}_1(0,k+1) = \gamma z \overline{\alpha} \quad \forall k \geqslant 0 \\
&\mathbf{m}_2(-k-1,0) = 0, \ \mathbf{m}_2(0,-k) = 1, \ \mathbf{m}_2(k+1,0) = \gamma z \alpha, \ \mathbf{m}_2(0,k+1) = \gamma z \overline{\alpha} \quad \forall k \geqslant 0.
\end{aligned}$$

It is easy to see that $\mathbf{m}_1$, $\mathbf{m}_2$ are prime and not elementary, since $\mathbf{m}_1(1,0)$, $\mathbf{m}_1(0,1)$, $\mathbf{m}_2(1,0)$, $\mathbf{m}_2(0,1)$ are positive. As a consequence all prime invariant measures are non-elementary. The measure $\mathbf{x}$ in (4.2) is thus not prime since it must be written as $\mathbf{x} = \mathbb{E}[Z_1]\,\mathbf{m}_1 + \mathbb{E}[Z_2]\,\mathbf{m}_2$ with $\mathbb{E}[Z_1], \mathbb{E}[Z_2] > 0$.

Example 4.7. This example illustrates the claim in 2) of Remark 3.8 regarding the second part of Theorem 3.7 which does not hold when $\mathbb{E}[\mathbf{X}]$ is a non-elementary prime invariant measure whose support is $\mathcal{I}$. Consider the graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$



given by $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1 := \{i_n : n \in \mathbb{Z}\}$, $\mathcal{I}_2 := \{j_n : n \in -\mathbb{N}\}$ with

$$\mathcal{F} = \{(i_n, i_{n+1}) : n \in \mathbb{Z}\} \cup \{(j_n, i_n) : n \in -\mathbb{N}\} \cup \{(j_n, j_{n+1}) : n \in -\mathbb{N}^*\}$$

and

$$\mathbb{E}\left[a_{(i_n,i_{n+1})}\right] = 1 \ \ \forall n \in \mathbb{Z}, \quad \text{ and } \quad \mathbb{E}\left[a_{(j_n,i_n)}\right] = \alpha_n, \ \mathbb{E}\left[a_{(j_n,j_{n+1})}\right] = 1-\alpha_n \ \ \forall n \in -\mathbb{N},$$

where $\alpha_0 = 1$ and $\alpha_n \in (0,1)$ for $n < 0$ with $\lim_{n\to-\infty} \alpha_n = 0$ and $\sum_{n\in-\mathbb{N}} \alpha_n < +\infty$ (for instance, $\alpha_n = 1/(n^2+1)$). We set

$$\theta := \prod_{n\in-\mathbb{N}^*} (1-\alpha_n) \in (0,1). \tag{4.3}$$

An invariant measure $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$ is a solution to:

$$\begin{array}{ll} x_{i_n} = x_{i_{n-1}} + \alpha_n x_{j_n} \ \ \forall n \leqslant 0, & x_{i_n} = x_{i_{n+1}} \ \ \forall n \geqslant 0 \\ x_{j_{n+1}} = (1-\alpha_n) x_{j_n} \ \ \forall n < 0 & \end{array} \tag{4.4}$$

and is determined by two parameters: $x_{i_{-1}} := a$ and $x_{j_0} := b$. Observing $x_{i_{n-1}} + x_{j_n} = a+b$ for all $n \leqslant 0$, we can solve recursively (4.4) into:

$$x_{j_n} = b \prod_{k=n}^{-1} (1-\alpha_k)^{-1} \ \ \forall n < 0,$$

$$x_{i_n} = (a+b) - b \prod_{k=n+1}^{-1} (1-\alpha_k)^{-1} \ \ \forall n < 1, \quad x_{i_n} = a+b \ \ \forall n \geqslant 0.$$

As a consequence the kernel $\mathrm{Ker}(\mathcal{A} - Id)$ is two-dimensional. Setting $(a,b) = (1,0)$, the corresponding measure reduces to $\mathbf{p} := \mathbf{1}_{\mathcal{I}_1}$, which is clearly positive, invariant, prime and elementary. Setting $(a,b) = (1-\theta, \theta)$ for θ given in (4.3), we observe that the corresponding measure $\mathbf{q}$ is positive and satisfies

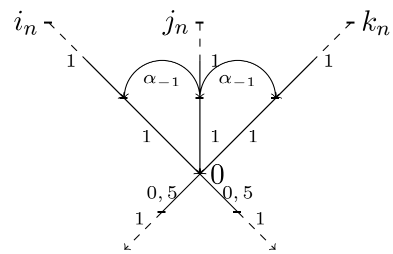
$$\lim_{n\to-\infty} q_{i_n} = 0. \tag{4.5}$$

As a consequence for any $C > 0$, we have neither $\mathbf{q} \leqslant C\mathbf{p}$ (as we can see on $\mathcal{I}_2$ where $p_{j_n} = 0$ and $q_{j_n} > 0$ for any $n < 0$) nor $\mathbf{p} \leqslant C\mathbf{q}$ (as we see from (4.5) for $i_n \in \mathcal{I}_1^-$ when $|n|$ is large enough). The measure $\mathbf{q}$ is invariant, non-elementary, and prime: indeed, consider $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)^+$ such that $\mathbf{x} \leqslant C\mathbf{q}$ for some $C > 0$. First, $\mathbf{x}$ is necessarily of the form $\mathbf{x} = \lambda\mathbf{p} + \mu\mathbf{q}$ with $\lambda, \mu \geqslant 0$ since $0 \leqslant x_{j_n} = \mu \mathbf{q}_{j_n}$ and, by using (4.5), $0 \leqslant x_{i_n} = \lambda + \mu \mathbf{q}_{i_n} \xrightarrow[n\to-\infty]{} \lambda$. Next, for every $i_n \in \mathcal{I}_1^-$, we have $\lambda + \mu q_{i_n} \leqslant C q_{i_n}$ for any $n < 0$, from which (4.5) entails $\lambda = 0$ and, in turns, $\mathbf{x} = \mu\mathbf{q}$.

In this context, consider $\mathbb{E}[\mathbf{X}] = \mathbf{q}$ a prime invariant and $\mathbf{Y} := (Y_i)_{i\in\mathcal{I}}$ where $(Y_i)_{i\in\mathcal{I}_1}$ is i.i.d. with expectation 1 and $Y_j = 0$ for all $j \in \mathcal{I}_2$, so that $\mathbb{E}[\mathbf{Y}] = \mathbf{p}$. The first part of Theorem 3.7 applies (see Remark 3.8), but not the second part, even if $\mathbf{Y}$ satisfies the conditions of this second part ($\mathbf{Y} \cdot \mathbf{A} \sim \mathbf{Y}$, $(Y_i a_{i,j})_{i,j}$ independent, $\mathbb{E}[\mathbf{Y}] = \mathbf{p}$). This observation illustrates that assuming $\mathcal{I} = \mathrm{Supp}(\mathbb{E}[\mathbf{X}])$ and $\mathbb{E}[\mathbf{X}]$ is prime invariant non-elementary is not sufficient for this second part to hold, as mentioned in Point 2) of Remark 3.8.

In Example 4.7, the prime measure $\mathbb{E}[\mathbf{Y}]$ is elementary; however, an easy adaptation of this example can relax this property, for instance, by splitting the graph in two branches at $1 \in \mathcal{I}_1$. Still illustrating Point 2) of Remark 3.8, the next example proposes another graph with $\mathbb{E}[\mathbf{Y}]$ non-elementary and $\mathbb{E}[\mathbf{X}]$ still a prime, invariant, non-elementary measure with support $\mathcal{I}$. In this Example 4.8, there is no elementary prime invariant measure, since 0 always belongs to the support of every such measure; and the set of invariant measures is generated by these three measures.

Example 4.8. Consider the graph $\mathcal{G} = (\mathcal{I}, \mathcal{F})$



specified by the vertices $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ where $\mathcal{I}_1 := \{i_n : n \in \mathbb{Z}\}$, $\mathcal{I}_2 := \{j_n : n \in -\mathbb{N}\}$, $\mathcal{I}_3 := \{k_n : n \in \mathbb{Z}\}$ with $i_0 = j_0 = k_0 = 0$, the edges

$$\mathcal{F} = \{(i_n, i_{n+1}), (k_n, k_{n+1}) : n \in \mathbb{Z}\} \cup \{(j_n, j_{n+1}), (j_n, i_n), (j_n, k_n) : n \in -\mathbb{N}\},$$

and by the following weights:

$$\mathbb{E}\left[a_{(i_n,i_{n+1})}\right] = \mathbb{E}\left[a_{(j_n,j_{n+1})}\right] = 1 \quad \forall n \in \mathbb{Z}^*, \quad \mathbb{E}\left[a_{(0,i_1)}\right] = \mathbb{E}\left[a_{(0,k_1)}\right] = 1/2,$$
$$\mathbb{E}\left[a_{(j_n,i_n)}\right] = \mathbb{E}\left[a_{(j_n,k_n)}\right] = \alpha_n \in (0, 1/2) \quad \forall n \in -\mathbb{N},$$

satisfying

$$\sum_{n \in -\mathbb{N}} \alpha_n < +\infty. \tag{4.6}$$

We then define:

$$\theta = \prod_{n=-2}^{-\infty} (1 - 2\alpha_n). \tag{4.7}$$

An invariant signed measure $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$ satisfies the system:

$$x_{i_n} = x_{i_{n-1}} + \alpha_n x_{j_n} \tag{4.8a}$$
$$x_{j_n} = (1 - 2\alpha_{n-1}) x_{j_{n-1}} \qquad \text{for } n < 0, \tag{4.8b}$$
$$x_{k_n} = x_{k_{n-1}} + \alpha_n x_{j_n} \tag{4.8c}$$

$x_0 = x_{i_{-1}} + (1 - 2\alpha_{-1}) x_{j_{-1}} + x_{k_{-1}}$ for $n = 0$, and $x_{i_n} = x_{k_n} = x_0/2$ for $n > 0$. From (4.8a)–(4.8c), we observe that $x_{i_n} + x_{k_n} + x_{j_{n+1}}$ is constant for $n < -1$, equal to some constant x^*. It follows that:

$$x_{i_n} + x_{k_n} = x^* - x_{j_{-1}} \prod_{j=-2}^{n+1} (1 - 2\alpha_j)^{-1}. \tag{4.9}$$

From (4.8a)–(4.8c), we observe that $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$ is determined recursively by the initial values $x_{i_{-2}} := a$, $x_{j_{-1}} := b$, $x_{k_{-2}} := c$ for $n < 0$. Define $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ as the invariant measures corresponding respectively to the choices $(a, b, c) = (1, 0, 0)$, $(1 - \theta, 2\theta, 1 - \theta)$ and $(0, 0, 1)$.
We derive $\mathbf{m}_1 = \mathbf{1}_{\mathcal{I}_1^-} + \frac{1}{2}\mathbf{1}_{\mathcal{I}_1^+ \cup \mathcal{I}_3^+}$ and $\mathbf{m}_3 = \mathbf{1}_{\mathcal{I}_3^-} + \frac{1}{2}\mathbf{1}_{\mathcal{I}_1^+ \cup \mathcal{I}_3^+}$ where $\mathcal{I}_1^- = \{i_n : n \in -\mathbb{N}\}$, $\mathcal{I}_1^+ = \{i_n : n \geqslant 1\}$, $\mathcal{I}_3^- = \{k_n : n \in -\mathbb{N}\}$, $\mathcal{I}_3^+ = \{k_n : n \geqslant 1\}$. The measure $\mathbf{m}_2$ is more complicated, but its support is $\mathcal{I}$ and, using (4.9), we find that it satisfies:

$$\lim_{n \to -\infty} \mathbf{m}_2(i_n) + \mathbf{m}_2(k_n) = x^* - \frac{b}{\theta} = 0, \tag{4.10}$$

since $x^* = 2$ and $b = 2\theta$ for this measure $\mathbf{m}_2$. Moreover, by monotonicity of the limit, $\mathbf{m}_2(i_n), \mathbf{m}_2(k_n) \geqslant 0$ and $\mathbf{m}_2$ is indeed a (positive) measure because:

$$2\mathbf{m}_2(i_n) = \mathbf{m}_2(i_n) + \mathbf{m}_2(k_n) = 2 - \mathbf{m}_2(j_{n+1})$$

with

$$\mathbf{m}_2(j_{n+1}) = 2 \prod_{k=n}^{-\infty} (1 - 2\alpha_j).$$

The measures $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are prime; indeed, first observe that $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)^+$ can be written $\mathbf{x} = a\mathbf{m}_1 + b\mathbf{m}_2 + c\mathbf{m}_3$ where necessarily $a, b, c \geqslant 0$ since, by (4.10), we have:

$$\begin{aligned}
&0 \leqslant x_{j_n} = b\mathbf{m}_2(j_n) \\
&0 \leqslant x_{j_n} = a\mathbf{m}_1(i_n) + b\mathbf{m}_2(i_n) + c\mathbf{m}_3(i_n) = a + b\mathbf{m}_2(i_n) \xrightarrow[n\to-\infty]{} a, \\
&0 \leqslant x_{k_n} = a\mathbf{m}_1(k_n) + b\mathbf{m}_2(k_n) + c\mathbf{m}_3(k_n) = b\mathbf{m}_2(k_n) + c \xrightarrow[n\to-\infty]{} c.
\end{aligned}$$

- For $\mathbf{m}_1$: Let $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)^+$ with $\mathbf{x} := a\mathbf{m}_1 + b\mathbf{m}_2 + c\mathbf{m}_3 \leqslant C\mathbf{m}_1$ with $a, b, c \geqslant 0$. First, evaluating $\mathbf{x} = a\mathbf{m}_1 + b\mathbf{m}_2 + c\mathbf{m}_3 \leqslant C\mathbf{m}_1$ in $j \in \mathcal{I}_2$ implies that $b = 0$ and next evaluationg $\mathbf{x}$ in $k \in \mathcal{I}_3^-$ implies that $c = 0$; therefore, $\mathbf{x} = a\mathbf{m}_1$ and $\mathbf{m}_1$ is prime.
- For $\mathbf{m}_2$: Let $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)^+$ with $\mathbf{x} := a\mathbf{m}_1 + b\mathbf{m}_2 + c\mathbf{m}_3 \leqslant C\mathbf{m}_2$ with $a, b, c \geqslant 0$. Then for $i_n \in \mathcal{I}_1^-$, we have $a + b\mathbf{m}_2(i_n) \leqslant C\mathbf{m}_2(i_n)$ from which it follows that $a = 0$ since $\lim_{n\to-\infty} \mathbf{m}_2(i_n) = 0$ from (4.10). Similarly, $c = 0$ and then $\mathbf{x} = b\mathbf{m}_2$ which shows that $\mathbf{m}_2$ is prime.
- For $\mathbf{m}_3$, the argument is similar to that for $\mathbf{m}_1$.

Let $\mathbf{X} = (X_i)_{i\in\mathcal{I}}$ be a collection of independent random variables with $\mathbf{x} = \mathbf{m}_2$ satisfying the conditions in 2) in Remark 3.8, see also Theorem 3.7 (in particular, observe that $\mathbf{x}$ is non-elementary, prime, invariant, and has support $\mathcal{I}$).

Consider two independent (non gamma-distributed) random variables A and B such that $A + B \sim \Gamma_\gamma(1)$ (for instance, see Bourguin and Tudor, 2011, Section 4) and take $\mathbf{Y} = (Y_i)_{i\in\mathcal{I}}$ a collection of independent random variables with

$$\begin{aligned}
&Y_{i_n} \sim A \ \ \forall n < 0, \quad Y_0 \sim A + B, \quad Y_{i_n} \sim \frac{A+B}{2} \ \ \forall n > 0, \\
&Y_{j_n} = 0 \ \ \forall n < 0, \quad Y_0 \sim A + B, \\
&Y_{k_n} \sim B \ \ \forall n < 0, \quad Y_0 \sim A + B, \quad Y_{k_n} \sim \frac{A+B}{2} \ \ \forall n > 0.
\end{aligned}$$

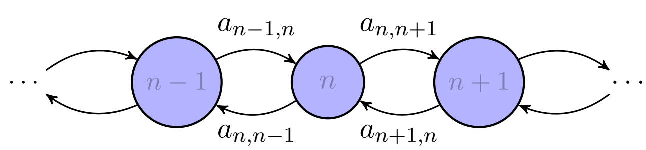
Such a $\mathbf{Y}$ is not of the form appearing in the second part of Theorem 3.7, i.e. for any λ, $\mathbf{Y}$ is not distributed like $\lambda\mathbf{X}$, illustrating 2) in Remark 3.8, and moreover $\mathbb{E}[\mathbf{Y}] = \mathbb{E}[A]\mathbf{m}_1 + \mathbb{E}[B]\mathbf{m}_3$ is not elementary (see vertex 0).

4.4. *Beyond prime measures.* Our main result Theorem 3.3 gives the solutions of (Inv) when the expectation $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_i])_{i\in\mathcal{I}}$ is prime. In this section, we go beyond the prime setting for a particular (infinite) graph and give in Theorem 4.10 the analogue of Theorem 3.3 for this graph (see also Example 4.2 for a finite graph).

Consider a random stochastic graph $\mathcal{G}$ with vertex set $\mathbb{Z}$, edges $\{(n, n+1) : n \in \mathbb{Z}\} \cup \{(n+1, n) : n \in \mathbb{Z}\}$ equipped with a random stochastic matrix $\mathbf{A} = (a_{i,j})_{i,j\in\mathbb{Z}}$ such that:

$$\forall n \in \mathbb{Z}, \quad \alpha_{n,n+1} = \mathbb{E}[a_{n,n+1}] = \delta, \qquad \alpha_{n+1,n} = \mathbb{E}[a_{n+1,n}] = 1 - \delta,$$

where $\delta \in (0, 1/2)$ is fixed.



Proposition 4.9. *Up to multiplication by positive constants, there are only two prime invariant measures:*

$$\mathbf{p} = \mathbf{1}_{\mathbb{Z}}, \quad \mathbf{q} = \left(\frac{\delta^n}{(1-\delta)^n}\right)_{n\in\mathbb{Z}}.$$

Proof: For $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$, solving the recurrence $x_k = x_{k-1}\delta + x_{k+1}(1-\delta)$ for all $k \in \mathbb{Z}$ yields:

$$x_k = x_0 + (x_1 - x_0)\frac{1 - \big(\delta/(1-\delta)\big)^k}{1 - \delta/(1-\delta)}.$$

We obtain the general expression of an invariant measure:

$$\mathbf{x} = \Big(x_0 + \frac{x_1 - x_0}{1 - \delta/(1-\delta)}\Big)\mathbf{p} - \frac{x_1 - x_0}{1 - \delta/(1-\delta)}\mathbf{q}, \quad x_0, x_1 \in \mathbb{R},$$

from which we deduce that $\mathrm{Ker}(\mathcal{A} - I_d)$ is two-dimensional and generated by $\mathbf{p}$ and $\mathbf{q}$, given in the statement. Next, let $\mathbf{x} = \lambda\mathbf{p} + \mu\mathbf{q} \in \mathrm{Ker}(\mathcal{A} - I_d)^+$. Since $\delta \in (0, 1/2)$, we have:

$$q_k = \frac{\delta^k}{(1-\delta)^k} = o(1) = o(p_k) \text{ when } k \to +\infty \quad \text{and} \quad p_k = 1 = o(q_k) \text{ when } k \to -\infty,$$

hence

$$\frac{x_k}{p_k} \xrightarrow[k\to+\infty]{} \lambda \quad \text{and} \quad \frac{x_k}{q_k} \xrightarrow[k\to-\infty]{} \mu, \tag{4.11}$$

and necessarily $\lambda \geqslant 0$, $\mu \geqslant 0$, in the expression of $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - I_d)^+$. In addition, $\mathbf{p}$ and $\mathbf{q}$ are also prime for the following reason:

- If $\mathbf{x} \leqslant \mathbf{p}$, taking the limit as $k \to -\infty$ in
$$\frac{(1-\delta)^k}{\delta^k} = \frac{p_k}{q_k} \geqslant \frac{x_k}{q_k} = \lambda\frac{(1-\delta)^k}{\delta^k} + \mu$$
yields $\mu = 0$, hence $\mathbf{x} = \lambda\mathbf{p}$;
- If $\mathbf{x} \leqslant \mathbf{q}$, taking the limit $k \to +\infty$ in
$$\frac{\delta^k}{(1-\delta)^k} = \frac{q_k}{p_k} \geqslant \frac{x_k}{p_k} = \lambda + \mu\frac{\delta^k}{(1-\delta)^k}$$
yields $\lambda = 0$, hence $\mathbf{x} = \mu\mathbf{q}$.

□

Theorem 4.10. *Independently of the graph $\mathcal{G}$ considered above, consider also a random vector $\mathbf{X} = (X_k)_{k\in\mathbb{Z}}$, whose coordinates X_k are independent, non-negative and non-zero, such that $\mathbf{X}\cdot\mathbf{A} \sim \mathbf{X}$. Then $\mathbf{X}$ is integrable, and $\mathbb{E}[\mathbf{X}] = c\,\mathbf{p} + c'\,\mathbf{q}$ for some $c, c' \geqslant 0$; moreover there exists $\gamma \in \overline{\mathbb{R}}_+^*$ such that $X_k = \Gamma_\gamma(\mathbb{E}[X_k])$.*

Proof: Step 1: Similarly to Step 1 in the proof of Theorem 3.3 (on page 605), for $t < 0$ we obtain:

$$\phi'_{\mathbf{X}}(t) = (\phi'_{X_k}(t))_{k\in\mathbb{Z}} = \left(\frac{\mathbb{E}[X_k e^{tX_k}]}{\mathbb{E}[e^{tX_k}]}\right)_{k\in\mathbb{Z}} \in \mathrm{Ker}(\mathcal{A} - Id)^+.$$

Step 2: Recall $\mathbb{C}^- = \{z \in \mathbb{C} : \Re(z) < 0\}$. Since $\phi'_X(z) \in \mathrm{Ker}(\mathcal{A} - Id)$ for any $z \in \mathbb{C}^-$, Proposition 4.9 implies that there exist $f_{\mathbf{p}}(z)$, $f_{\mathbf{q}}(z)$ such that:

$$\phi'_X(z) = f_{\mathbf{p}}(z)\,\mathbf{p} + f_{\mathbf{q}}(z)\,\mathbf{q}. \tag{4.12}$$

Suitable combinations of the coordinates $k = 0$ and $k = 1$ yield:

$$f_{\mathbf{p}}(z) = \frac{\delta}{2\delta - 1}\phi'_{X_0}(z) + \frac{\delta - 1}{2\delta - 1}\phi'_{X_1}(z), \quad \text{and} \quad f_{\mathbf{q}}(z) = \frac{1-\delta}{1-2\delta}\big(\phi'_{X_0}(z) - \phi'_{X_1}(z)\big).$$

Hence the functions $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$ are analytic on $\mathbb{C}^-$.

Let $K \subset \mathbb{C}^-$ be compact. By using the limits (4.11), we deduce the following uniform convergence results:

$$\frac{{\phi'_{X_k}}^2}{p_k^2} \xrightarrow[k\to+\infty]{} f_{\mathbf{p}}^2, \qquad \frac{\phi''_{X_k}}{p_k} \xrightarrow[k\to+\infty]{} f'_{\mathbf{p}},$$
$$\frac{{\phi'_{X_k}}^2}{q_k^2} \xrightarrow[k\to-\infty]{} f_{\mathbf{q}}^2, \qquad \frac{\phi''_{X_k}}{q_k} \xrightarrow[k\to-\infty]{} f'_{\mathbf{q}}. \tag{4.13}$$

Next, as in (4.11), we have $f_{\mathbf{p}}(t) \geqslant 0$, $f_{\mathbf{q}}(t) \geqslant 0$ for all $t < 0$. Since for all $k \in \mathbb{Z}$ we also have:

$$\phi'_{X_k}(t) \underset{k\to+\infty}{\sim} f_{\mathbf{p}}(t)p_k, \quad \phi'_{X_k}(t) \underset{k\to-\infty}{\sim} f_{\mathbf{q}}(t)q_k,$$

and the function ϕ'_{X_k} is non-decreasing on $(-\infty, 0]$ (Lemma 2.8), we deduce that the functions $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$ are also non-decreasing. If $f_{\mathbf{q}}(-1) = 0$, then we necessarily have $f_{\mathbf{q}}(t) = 0$ for $t \leqslant -1$ and the analytic function $f_{\mathbf{q}}$ must be identically zero. Similarly, if $f_{\mathbf{p}}(-1) = 0$ then $f_{\mathbf{p}} \equiv 0$.
Thus, in any of these cases, we can apply Theorem 3.3 to reach the conclusion. As a consequence, without loss of generality, we assume in the remainder of the proof that $f_{\mathbf{p}}(t) > 0$, $f_{\mathbf{q}}(t) > 0$ for all $t < 0$.

<u>Step 3:</u> As in the Step 3 of the proof of Theorem 3.3, the following equations hold for any $k \in \mathbb{Z}$:

$$\begin{aligned}\frac{\psi''_{X_k}}{\psi_{X_k}} &= \sum_{i\in\mathbb{Z}} \mathbb{E}[a_{i,k}^2]\frac{\psi''_{X_i}}{\psi_{X_i}} + \sum_{i\neq j\in\mathbb{Z}} \alpha_{i,k}\alpha_{j,k}\phi'_{X_i}\phi'_{X_j} \\ &= \mathbb{E}[a_{k-1,k}^2]\frac{\psi''_{X_{k-1}}}{\psi_{X_{k-1}}} + \mathbb{E}[a_{k+1,k}^2]\frac{\psi''_{X_{k+1}}}{\psi_{X_{k+1}}} + 2\alpha_{k-1,k}\alpha_{k+1,k}\phi'_{X_{k-1}}\phi'_{X_{k+1}},\end{aligned}$$

and

$$\begin{aligned}&\frac{\psi''_{X_k}}{\psi_{X_k}} = \phi''_{X_k} + {\phi'_{X_k}}^2 = \phi''_{X_k} + \left(\sum_{i\in\mathcal{I}} \alpha_{i,k}\phi'_{X_i}\right)^2 \\ &= \alpha_{k-1,k}\phi''_{X_{k-1}} + \alpha_{k+1,k}\phi''_{X_{k+1}} + \alpha_{k-1,k}^2{\phi'_{X_{k-1}}}^2 + \alpha_{k+1,k}^2{\phi'_{X_{k+1}}}^2 + 2\alpha_{k-1,k}\alpha_{k+1,k}\phi'_{X_{k-1}}\phi'_{X_{k+1}}.\end{aligned}$$

Setting $\pi_{k,k'} = \mathbb{E}[a_{k,k'}(1-a_{k,k'})] \in [0,1]$, we derive from these equations that for all $k \in \mathbb{Z}$:

$$\pi_{k-1,k}\phi''_{X_{k-1}} + \pi_{k+1,k}\phi''_{X_{k+1}} = \mathrm{Var}(a_{k-1,k}){\phi'_{X_{k-1}}}^2 + \mathrm{Var}(a_{k+1,k}){\phi'_{X_{k+1}}}^2. \tag{E_k}$$

<u>Step 4:</u> Next, observe that

$$\forall k \in \mathbb{Z}, \qquad C := \delta - \delta^2 = \alpha_{k-1,k} - \alpha_{k-1,k}^2 = \pi_{k-1,k} + \mathrm{Var}(a_{k-1,k}) = \pi_{k+1,k} + \mathrm{Var}(a_{k+1,k}) > 0.$$

Since $\pi_{k\pm1,k}, \mathrm{Var}(a_{k\pm1,k})$ are in $[0,1]$, which is compact, by the Bolzano-Weierstrass theorem, there are constants $V, V', \widetilde{V}, \widetilde{V}' \in [0,1]$ and a sequence $(\theta_n)_{n\geqslant1}$ of integers such that $\theta_n \to +\infty$ as $n \to +\infty$ and

$$\mathrm{Var}(a_{\theta_n-1,\theta_n}) \xrightarrow[n\to+\infty]{} V, \quad \pi_{\theta_n-1,\theta_n} \xrightarrow[n\to+\infty]{} \widetilde{V}, \quad \text{with } V + \widetilde{V} = C$$
$$\mathrm{Var}(a_{\theta_n+1,\theta_n}) \xrightarrow[n\to+\infty]{} V', \quad \pi_{\theta_n+1,\theta_n} \xrightarrow[n\to+\infty]{} \widetilde{V}', \quad \text{with } V' + \widetilde{V}' = C.$$

Since $p_{\theta_n} = 1$, taking the limit along $k = \theta_n \xrightarrow[n\to+\infty]{} +\infty$ in equation (E_k) with the uniform convergence (4.13) in Step 2 valid over any compact $K \subset \mathbb{C}^-$, we obtain:

$$(\widetilde{V} + \widetilde{V}')f'_{\mathbf{p}} = (V + V')f_{\mathbf{p}}^2. \tag{4.14}$$

In the remainder of this Step 4, we obtain a similar equation for $f_{\mathbf{q}}$ (see (4.18) below), but more details are required. First, along the same lines as before, there are also constants $W, W', \widetilde{W}', \widetilde{W}' \in [0,1]$ and a sequence $(\vartheta_n)_{n\geqslant 1}$ of integers such that $\vartheta_n \to -\infty$ as $n \to +\infty$ and

$$\mathrm{Var}(a_{\vartheta_n-1,\vartheta_n}) \xrightarrow[n\to+\infty]{} W, \quad \pi_{\vartheta_n-1,\vartheta_n} \xrightarrow[n\to+\infty]{} \widetilde{W}, \quad \text{with } W+\widetilde{W}=C$$
$$\mathrm{Var}(a_{\vartheta_n+1,\vartheta_n}) \xrightarrow[n\to+\infty]{} W', \quad \pi_{\vartheta_n+1,\vartheta_n} \xrightarrow[n\to+\infty]{} \widetilde{W}', \quad \text{with } W'+\widetilde{W}'=C.$$

Dividing equation (E_k) by q_k^2, we have:

$$\pi_{k-1,k}\frac{\phi''_{X_{k-1}}}{q_k^2}+\pi_{k+1,k}\frac{\phi''_{X_{k+1}}}{q_k^2}=\frac{(1-\delta)^2}{\delta^2}\,\mathrm{Var}(a_{k-1,k})\frac{{\phi'_{X_{k-1}}}^2}{q_{k-1}^2}+\frac{\delta^2}{(1-\delta)^2}\,\mathrm{Var}(a_{k+1,k})\frac{{\phi'_{X_{k+1}}}^2}{q_{k+1}^2},$$

and taking limits as $k=\vartheta_n \xrightarrow[n\to+\infty]{} -\infty$ with the uniform convergence (4.13), we obtain:

$$0=W\frac{(1-\delta)^2}{\delta^2}\,f_{\mathbf{q}}^2+W'\frac{\delta^2}{(1-\delta)^2}\,f_{\mathbf{q}}^2,$$

from which we deduce that $W=W'=0$ (since we have assumed $f_{\mathbf{q}}\neq 0$), and in turn, $\widetilde{W}=\widetilde{W}'=C$. Next, dividing equation (E_k) by q_k, we have:

$$\begin{aligned}&\frac{1-\delta}{\delta}\pi_{k-1,k}\frac{\phi''_{X_{k-1}}}{q_{k-1}}+\frac{\delta}{1-\delta}\pi_{k+1,k}\frac{\phi''_{X_{k+1}}}{q_{k+1}}\\&=\frac{1-\delta}{\delta}\,\mathrm{Var}(a_{k-1,k})q_{k-1}\frac{{\phi'_{X_{k-1}}}^2}{q_{k-1}^2}+\frac{\delta}{1-\delta}\,\mathrm{Var}(a_{k+1,k})q_{k+1}\frac{{\phi'_{X_{k+1}}}^2}{q_{k+1}^2}.\end{aligned}\tag{4.15}$$

Taking limits as $k=\vartheta_n \xrightarrow[n\to+\infty]{} -\infty$, the left-hand side of (4.15) goes to

$$C\frac{1-\delta}{\delta}f'_{\mathbf{q}}+C\frac{\delta}{1-\delta}f'_{\mathbf{q}}.\tag{4.16}$$

Since we have seen that $f_{\mathbf{q}}$ is non-decreasing on $(-\infty,0]$ (due to Lemma 2.8, cf. page 617), we also have that $f'_{\mathbf{q}}(t)\geqslant 0$ when $t<0$.

We show now that $\mathrm{Var}(a_{\vartheta_n-1,\vartheta_n})\,q_{\vartheta_n-1}$ and $\mathrm{Var}(a_{\vartheta_n+1,\vartheta_n})\,q_{\vartheta_n+1}$ are bounded. Indeed, let $\varepsilon>0$ be fixed, and $k=\vartheta_n$. Since all the terms of (4.15) are non-negative, for all sufficiently large n, we have:

$$\begin{aligned}\frac{1-\delta}{\delta}\,\mathrm{Var}(a_{k-1,k})q_{k-1}(f_{\mathbf{q}}(-1)^2-\varepsilon)&\leqslant\frac{1-\delta}{\delta}\,\mathrm{Var}(a_{k-1,k})q_{k-1}\frac{\phi'_{X_{k-1}}(-1)^2}{q_{k-1}^2}\\&\leqslant\frac{1-\delta}{\delta}\,\mathrm{Var}(a_{k-1,k})q_{k-1}\frac{\phi'_{X_{k-1}}(-1)^2}{q_{k-1}^2}+\frac{\delta}{1-\delta}\,\mathrm{Var}(a_{k+1,k})q_{k+1}\frac{\phi'_{X_{k+1}}(-1)^2}{q_{k+1}^2}\\&\leqslant C\frac{1-\delta}{\delta}\cdot f'_{\mathbf{q}}(-1)+C\frac{\delta}{1-\delta}f'_{\mathbf{q}}(-1)+\varepsilon,\end{aligned}$$

using the limit (4.16) for the last bound. As a consequence $\mathrm{Var}(a_{\vartheta_n-1,\vartheta_n})\,q_{\vartheta_n-1}$ is bounded. A similar argument applies for $\mathrm{Var}(a_{\vartheta_n+1,\vartheta_n})\,q_{\vartheta_n+1}$. Thus, we can extract a subsequence $(\sigma_{n'})_{n'\geqslant 1}$ from $(\vartheta_n)_{n\geqslant 1}$ such that for some constants Z, Z':

$$\mathrm{Var}(a_{\sigma_{n'}-1,\sigma_{n'}})q_{\sigma_{n'}-1}\xrightarrow[n'\to+\infty]{} Z \quad\text{and}\quad \mathrm{Var}(a_{\sigma_{n'}+1,\sigma_{n'}})q_{\sigma_{n'}+1}\xrightarrow[n'\to+\infty]{} Z'.\tag{4.17}$$

Combining (4.13) and (4.17) to take the limit in the right-hand side of (4.15) along $k=\sigma_{n'}\to-\infty$, and using (4.16), we obtain:

$$\left(C\frac{1-\delta}{\delta}+C\frac{\delta}{1-\delta}\right)\,f'_{\mathbf{q}}=\left(Z\frac{1-\delta}{\delta}+Z'\frac{\delta}{1-\delta}\right)\,f_{\mathbf{q}}^2.\tag{4.18}$$

Note that necessarily $C\frac{1-\delta}{\delta} + C\frac{\delta}{1-\delta} \neq 0$.

Step 5: In this final step, we solve the differential equations (4.14) and (4.18) which are of the form

$$Af' = Bf^2, \quad \text{where } A + B > 0, A \geqslant 0, B \geqslant 0 \text{ and } f \neq 0.$$

There exist positive constants $\gamma, \gamma', c, c' > 0$ such that (see (2.6)):

$$\text{either } f_{\mathbf{p}}(t) = c \text{ or } f_{\mathbf{p}}(t) = \frac{c}{1 - \frac{t}{\gamma}},$$
$$\text{either } f_{\mathbf{q}}(t) = c' \text{ or } f_{\mathbf{q}}(t) = \frac{c'}{1 - \frac{t}{\gamma'}}.$$

If for all $k, k' \in \mathbb{Z}$, $\mathrm{Var}(a_{k,k'}) = 0$, then $V = V' = 0$ and $Z = Z' = 0$ and (4.15), (4.18) entail $f_{\mathbf{p}} = c$, $f_{\mathbf{q}} = c'$; the converse is true. In this case, we have for all $k \in \mathbb{Z}$:

$$\phi'_{X_k}(t) = cp_k + c'q_k, \ \mathbb{E}[e^{tX_k}] = e^{tcp_k + c'q_k} \quad \text{and} \quad X_k = cp_k + c'q_k \quad a.s.$$

and necessarily $A = \mathbb{E}[A]$ a.s. Comparing the singular values of both sides of (E_k), we show that the remaining case must be:

$$f_{\mathbf{p}}(t) = \frac{c}{1 - \frac{t}{\gamma}}, \quad f_{\mathbf{q}}(t) = \frac{c'}{1 - \frac{t}{\gamma}},$$

with the same singular value γ. Indeed, if $f_{\mathbf{q}} = c'$ and $f_{\mathbf{p}}$ has a simple singularity at γ, then using (4.12), ϕ''_{X_k} must have a singularity of order 2 at γ and:

$$\phi'_{X_k}(t)^2 = c'^2 q_k^2 + \underbrace{2cc'q_kp_k}_{>0} \frac{1}{1 - \frac{t}{\gamma}} + \frac{c^2 p_k^2}{(1 - \frac{t}{\gamma})^2},$$

which is incompatible with (E_k) since its left-hand side would have a singularity of order 2 in γ, while it would be of order 1 for its right-hand side. Similarly, if $\gamma \neq \gamma'$, ϕ''_{X_k} and ϕ''_{X_k} have singularities of order 2 in γ and γ' while $\phi'^2_{X_k}$ would have singularities of order 1 in each γ, γ' which leads again to a contradiction with (E_k). Thus, we have necessarily $f_{\mathbf{p}}(t) = \frac{c}{1-\frac{t}{\gamma}}$, $f_{\mathbf{q}}(t) = \frac{c'}{1-\frac{t}{\gamma}}$ and:

$$\forall k \in \mathbb{Z}, \ \phi'_{X_k}(t) = \frac{cp_k + c'q_k}{1 - \frac{t}{\gamma}}, \quad \mathbb{E}[e^{tX_k}] = \left(1 - \frac{t}{\gamma}\right)^{-(cp_k + c'q_k)\gamma} \quad \text{and } X_k \sim \Gamma_\gamma(cp_k + c'q_k).$$

Finally, there exists $\gamma \in \overline{\mathbb{R}}_+^*$ such that $X \sim \Gamma_\gamma(\mathbb{E}[X])$. □

Appendix A.

A.1. *Properties of measures on a graph.* On a graph $\mathcal{G} = (\mathcal{I}, \mathcal{F})$, we consider a partial order relation $\preceq$ on the set of vertices $\mathcal{I}$ defined by $i \preceq j$ when either $i = j$ or there are $n \in \mathbb{N}$ and $(i_0, i_1, \ldots, i_n) \in \mathcal{I}^{n+1}$ such that $i_0 = i$, $i_n = j$ and $(i_{k-1}, i_k) \in \mathcal{F}$ for all $k \in [\![1, n]\!]$. In such a case, we say that i precedes j. As a consequence, we also say that a sequence $(i_k)_{k\in\mathbb{Z}} \in \mathcal{I}^{\mathbb{Z}}$ is ordered (by $\mathcal{F}$) when $\forall k \in \mathbb{Z}$, $(i_k, i_{k+1}) \in \mathcal{F}$. For any $J \subset \mathcal{I}$, we set $\overrightarrow{J} = \{i \in \mathcal{I} : \exists j \in J, j \preceq i\}$ for the set of edges implied by J, and $\overleftarrow{J} = \{i \in \mathcal{I} : \exists j \in J, i \preceq j\}$ for the set of edges implying J. Roughly speaking, $\overleftarrow{J}$ and $\overrightarrow{J}$ are respectively the set of smaller vertices and the set of larger vertices for $\preceq$. In particular, we observe that for any $J \subset \mathcal{I}$, we have $J \subset \overrightarrow{J}$ and $J \subset \overleftarrow{J}$.

The following lemma describes the relationship between the support of an invariant measure and the partial order $\preceq$:

Lemma A.1. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)^+$ be an invariant non-negative measure. The following claims hold true:*

1) If $x_i > 0$ and $i \preceq j$, then $x_j > 0$;
2) If $i \in \mathcal{I}$ with $x_i > 0$, then there is some $j \in \mathcal{I}$ such that $(i,j) \in \mathcal{F}$ and $x_j > 0$;
3) If $x_j > 0$, then there is some $i \in \mathcal{I}$ such that $(i,j) \in \mathcal{F}$ and $x_i > 0$.

Proof: Claim 1) is proved by induction: if for some $n \in \mathbb{N}$, $\exists (i_0, i_1, ..., i_n) \in \mathcal{I}^{n+1}$ with $i_0 = i$, $i_n = j$ and $\forall k \in [\![1,n]\!]$, $(i_{k-1}, i_k) \in \mathcal{F}$ we show inductively $x_{i_k} > 0$ for all $k \in [\![1,n]\!]$ from $x_{i_k} = \sum_{l\in\mathcal{I}} x_l \alpha_{l,i_k} \geqslant x_{i_{k-1}} \alpha_{i_{k-1},i_k} > 0$. Next, if $x_i > 0$ then there exists some $j \in \mathcal{I}$ with $\alpha_{i,j} > 0$, and for this index j, we have $x_j = \sum_{i'\in\mathcal{I}} x_{i'} \alpha_{i',j} \geqslant x_i \alpha_{i,j} > 0$, which proves 2). Finally, if $0 < x_j = \sum_{i'\in\mathcal{I}} x_{i'} \alpha_{i',j}$, then necessarily $x_i \alpha_{i,j} > 0$ for some $i \in \mathcal{I}$, which proves 3). □

The following lemma shows that the restriction of an invariant measure to any connected component of its support is still invariant:

Lemma A.2. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)^+$ be an invariant non-negative measure. Then for any connected component C of $\mathcal{G}$, $\mathbf{x}_C := \mathbf{x}\mathbf{1}_C \in Ker(\mathcal{A} - Id)^+$. In particular, when $\mathbf{x}$ is prime then* $\mathrm{Supp}(\mathbf{x})$ *is connected.*

Proof: Let C be a connected component of $\mathrm{Supp}(\mathbf{x})$ and set $\mathbf{y} = \mathbf{x}_C$. We have:

$$\forall j \in C, \quad \sum_{i\in\mathcal{I}} y_i \alpha_{i,j} = \sum_{i\in C} x_i \alpha_{i,j} = x_j = y_j$$
$$\forall j \notin C, \quad \sum_{i\in\mathcal{I}} y_i \alpha_{i,j} = \sum_{i\notin C} y_i \alpha_{i,j} = \sum_{i\notin C} 0 \alpha_{i,j} = 0 = y_j$$

since $\alpha_{k,l} = 0 = \alpha_{l,k}$ for $k \in C$ and $l \notin C$ by connectivity of C. When $\mathbf{x}$ is prime, since $\mathbf{x}_C \leqslant \mathbf{x}$ for any connected component C of $\mathcal{G}$, we have $\mathbf{x}_C = \lambda_C \mathbf{x}$ for some $\lambda_C \in [0,1]$. Since the supports of the $\mathbf{x}_C$ must all coincide, there is actually a unique connected component. □

When two measures are ordered, their images under $\mathcal{A}$ are also ordered:

Lemma A.3. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and $\mathbf{x}, \mathbf{y}$ be two non-negative measures such that $\mathbf{y} \leqslant \mathbf{x}$. Then: $\mathbf{y} \cdot \mathcal{A} \leqslant \mathbf{x} \cdot \mathcal{A}$.*

Proof: Setting $\widetilde{\mathbf{y}} = \mathbf{y} \cdot \mathcal{A}$ and $\widetilde{\mathbf{x}} = \mathbf{X} \cdot \mathcal{A}$, we have:

$$\forall j \in \mathcal{I}, \quad \widetilde{y}_j = \sum_{i\in\mathcal{I}} y_i \alpha_{i,j} \leqslant \sum_{i\in\mathcal{I}} x_i \alpha_{i,j} = \widetilde{x}_j.$$

□

Lemma A.4. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ a stochastic graph, and let $\mathbf{x} \in Ker(\mathcal{A} - Id)^+$ be an invariant non-negative measure. Let $j^* \in \mathrm{Supp}(\mathbf{x})$ and $J = \overleftarrow{\{j^*\}} \cap \mathrm{Supp}(\mathbf{x})$, and consider $\mathbf{y}(n+1) = \mathbf{y}(n) \cdot \mathcal{A}$ with $\mathbf{y}(0) := \mathbf{x}\mathbf{1}_J = (x_i \mathbf{1}_J(i))_{i\in\mathcal{I}}$. Then: $\forall j \in J$, $\forall n \in \mathbb{N}$, $y_j(n) = x_j$ and $\forall n \in \mathbb{N}$, $\mathbf{0} \leqslant \mathbf{y}(n) \leqslant \mathbf{y}(n+1) \leqslant \mathbf{x}$. Hence, we can define $\mathbf{y}(\infty) := \lim_{n\to+\infty} \mathbf{y}(n)$, and we have:*

$$\mathrm{Supp}(\mathbf{y}(\infty)) = \overrightarrow{J}, \quad \mathbf{0} \leqslant \mathbf{y}(\infty) \leqslant \mathbf{x}, \quad \mathbf{y}(\infty) \in Ker(\mathcal{A} - Id)^+.$$

Proof: Assume that for $n \in \mathbb{N}$, $y_j(n) = x_j$ for all $j \in J$. Observing that for any $j \in J$,

$$j \in J = \overleftarrow{\{j^*\}} \cap \mathrm{Supp}(\mathbf{x}) \implies \overleftarrow{\{j\}} \cap \mathrm{Supp}(\mathbf{x}) \subset \overleftarrow{\{j^*\}} \cap \mathrm{Supp}(\mathbf{x}) = J,$$

we have

$$y_j(n+1) = \sum_{i\in\mathcal{I}} y_i(n) \alpha_{i,j} = \sum_{i\in J} y_i(n) \alpha_{i,j} = \sum_{i\in J} x_i \alpha_{i,j} = x_j$$

since $\alpha_{i,j} = 0$ when $i \notin \overleftarrow{\{j\}}$ and $\mathbf{y}(n+1) = \mathbf{y}(n) \cdot \mathcal{A} \leq \mathbf{x} \cdot \mathcal{A} = \mathbf{x}$ (Lemma A.3) and we have thus $\mathbf{0} \leqslant \mathbf{y}(n) \leqslant \mathbf{y}(n+1) \leqslant \mathbf{x}$. In addition, $\mathbf{y}(\infty) = \lim_{n\to+\infty} \mathbf{y}(n)$ is well defined (pointwisely) and necessarily $\mathbf{0} \leqslant \mathbf{y}(\infty) \leqslant \mathbf{x}$. Furthermore, we have $\mathrm{Supp}(\mathbf{y}(\infty)) = \bigcup_{n\geq 0} \mathrm{Supp}(\mathbf{y}(n))$. We have $\mathbf{y}(\infty) \in \mathrm{Ker}(\mathcal{A} - Id)^+$ since, taking the limit $n \to +\infty$ in

$$y_j(n+1) = \sum_{i\in\mathcal{I}} y_i(n)\alpha_{i,j},$$

we get

$$y_j(\infty) = \sum_{i\in\mathcal{I}} y_i(\infty)\alpha_{i,j}.$$

In particular, $\mathrm{Supp}(\mathbf{y}(\infty)\cdot\mathcal{A}) = \mathrm{Supp}(\mathbf{y}(\infty))$ and $\overrightarrow{\mathrm{Supp}(\mathbf{y}(\infty))} = \mathrm{Supp}(\mathbf{y}(\infty))$. Since

$$\mathrm{Supp}(\mathbf{y}(n)\cdot\mathcal{A}) \subset \overrightarrow{\mathrm{Supp}(\mathbf{y}(n))}$$

and $\overrightarrow{\overrightarrow{J}} = \overrightarrow{J}$, we derive recursively that $\mathrm{Supp}(\mathbf{y}(n)) \subset \overrightarrow{J}$ and conclude that $\mathrm{Supp}(\mathbf{y}(\infty)) = \overrightarrow{J}$ since $J \subset \mathrm{Supp}(\mathbf{y}(\infty))$. □

The next result shows that any vertex in the support of a prime measure admits a common predecessor.

Lemma A.5. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and let $\mathbf{x}$ be a prime invariant non-negative measure. Then:*

1) For any $j \in \mathrm{Supp}(\mathbf{x})$, we have $\overrightarrow{\overleftarrow{\{j\}} \cap \mathrm{Supp}(\mathbf{x})} = \mathrm{Supp}(\mathbf{x})$;
2) For any $j, j' \in \mathrm{Supp}(\mathbf{x})$, there is $i \in \mathrm{Supp}(\mathbf{x})$ such that $i \preceq j$ and $i \preceq j'$.

Proof: **1)** Using the previous Lemma A.4 with $j^* := j \in \mathrm{Supp}(\mathbf{x})$ and $J = \overleftarrow{\{j^*\}} \cap \mathrm{Supp}(\mathbf{x})$, we have $\mathbf{y}(\infty) \leqslant \mathbf{x}$, so that $\mathbf{y}(\infty) = \lambda\,\mathbf{x}$ since $\mathbf{x}$ is prime. Moreover, we necessarily have $\lambda = 1$ since $\mathbf{y}(\infty)$ coincides with $\mathbf{x}$ on J. Then, we have: $\overrightarrow{J} = \mathrm{Supp}(\mathbf{y}(\infty)) = \mathrm{Supp}(\mathbf{x})$.

2) The second part follows if for all $j, j' \in \mathrm{Supp}(\mathbf{x})$, we have $\overleftarrow{\{j\}} \cap \overleftarrow{\{j'\}} \cap \mathrm{Supp}(\mathbf{x}) \neq \emptyset$. If this does not hold for some $j, j' \in \mathrm{Supp}(\mathbf{x})$, then for all $i \in \overleftarrow{\{j'\}} \cap \mathrm{Supp}(\mathbf{x})$ we would have $i \not\preceq j$ which contradicts Claim 1) since in this case $j \notin \overrightarrow{\overleftarrow{\{j'\}} \cap \mathrm{Supp}(\mathbf{x})} = \mathrm{Supp}(\mathbf{x})$. □

Proposition A.6. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and let $\mathbf{x}$ be a prime invariant non-negative measure. Then there exists a sequence $(i_k)_{k\in\mathbb{Z}}$ of $\mathrm{Supp}(\mathbf{x})$ ordered with respect to order $\preceq$ with $(i_k, i_{k+1}) \in \mathcal{F}$ for all $k \in \mathbb{Z}$ such that $\overrightarrow{\{i_k : k \in \mathbb{Z}\}} = \mathrm{Supp}(\mathbf{x})$.*

Proof: First since $\mathbf{x}$ is prime, $\mathrm{Supp}(\mathbf{x})$ is connected and countable by Lemma A.2, we may index $\mathrm{Supp}(\mathbf{x})$ either as $\mathrm{Supp}(\mathbf{x}) = \{j_k : k \in \mathbb{N}\}$ or as $\mathrm{Supp}(\mathbf{x}) = \{j_k : k \in [\![0, M]\!]\}$. Next, we proceed with the recursive construction of the sequence $(i_k)_{k\in\mathbb{Z}} \in \mathrm{Supp}(\mathbf{x})^{\mathbb{Z}}$ from the j_k's by using the axiom of choice:

First, set $i_0 = j_0$, and assume the i_k's are constructed for $k \leqslant N$. Considering j_k for $k \in [\![0, N]\!]$, we can find an ordered sequence $(i_k)_{k\geqslant k_N} \in \mathrm{Supp}(\mathbf{x})^{\mathbb{N}}$ with $k_N \in \mathbb{Z}$ such that:

$$\{j_k : k \in [\![0, N]\!]\} \subset \overrightarrow{\{i_k : k \geqslant k_N\}} \tag{A.1}$$

as follows:

- If $j_{N+1} \in \overrightarrow{\{i_k : k \geqslant k_N\}}$, set $k_{N+1} = k_N$.
- If $j_{N+1} \notin \overrightarrow{\{i_k : k \geqslant k_N\}}$, applying 2) in Lemma A.5 to indices j_{N+1}, i_{k_N}, we find $i \in \mathrm{Supp}(\mathbf{x})$ such that $i \preceq j_{N+1}$ and $i \preceq i_{k_N}$. In particular $\exists i'_0, i'_1, \ldots, i'_n \in \mathrm{Supp}(\mathbf{x})$ with $(i'_l, i'_{l+1}) \in \mathcal{F}$ for all $l = 0, \ldots, n-1$ and $i = i'_0$, $i'_n = i_{k_N}$. Thus, set $k_{N+1} = k_N - n$, and $i_k = i'_{n+k-k_N}$ for $k \in [\![k_{N+1}, k_N]\!]$.

The inclusion (A.1) still holds for $N+1$ and $(i_k)_{k\geqslant k_{N+1}}$ is still ordered. If the sequence $(k_N)_{N\geqslant 1}$ eventually becomes constant, then $\mathrm{Supp}(\mathbf{x}) \subset \overrightarrow{\{i_k : k \geqslant k_N\}}$ and we can still complete the construction of the ordered sequence $(i_k)_{k\in\mathbb{Z}}$ using Claim 3) in Lemma A.1.

Since $\overrightarrow{J\cup J'} = \vec{J}\cup\vec{J'}$, we have:

$$\mathrm{Supp}(\mathbf{x}) = \bigcup_{N\geqslant 1}\{j_k : k\in [\![0,N]\!]\} \subset \bigcup_{N\geqslant 1}\overrightarrow{\{i_k : k\geqslant k_N\}} = \overrightarrow{\bigcup_{N\geqslant 1}\{i_k : k\geqslant k_N\}}$$
$$= \overrightarrow{\{i_k : k\in\mathbb{Z}\}} \subset \overrightarrow{\mathrm{Supp}(\mathbf{x})} = \mathrm{Supp}(\mathbf{x}),$$

where the last equality follows from Claim 1) in Lemma A.1. □

Using Proposition A.6, we now prove Proposition 1.3:

Proof: (Prop. 1.3) Since $\mathbf{x}$ is prime, by Proposition A.6, there is an ordered sequence $(i_k)_{k\in\mathbb{Z}}$ with values in $\mathrm{Supp}(\mathbf{x})$ such that $\vec{I} = \mathrm{Supp}(\mathbf{x})$ where $I := \{i_k : k\in\mathbb{Z}\}$. Moreover, since $\mathbf{x}$ is elementary, it follows that $\overrightarrow{\{i_k\}} = \{i_l : i_k \preceq i_l\} = \{i_l : k\leqslant l\}$ and thus $I = \vec{I} = \mathrm{Supp}(\mathbf{x})$.

Observe that the sequence $(x_{i_k})_{k\in\mathbb{Z}}$ is non-decreasing: indeed, since $(i_k, i_{k+1})\in\mathcal{F}$ and $\mathbf{x}$ is elementary, we have $\alpha_{i_k,i_{k+1}} = 1$ and

$$x_{i_k} = x_{i_k}\alpha_{i_k,i_{k+1}} \leqslant \sum_{i\in\mathcal{I}} x_i\alpha_{i,i_{k+1}} = (\mathbf{x}\cdot\mathcal{A})_{i_{k+1}} = x_{i_{k+1}}. \tag{A.2}$$

There are two cases: when $i_K = i_{K'}$ for some $K<K'$ and when $i_K \neq i_{K'}$ for all $K<K'\in\mathbb{Z}$.

For the first case, assume $i_K = i_{K'}$ for some $K<K'$. Then $(x_{i_k})_{k\in[\![K,K']\!]}$ is constant, equal to some c, and the inequality in (A.2) must be an equality for any $k\in[\![K,K'-1]\!]$. As a consequence, for any $j\in J := \{i_k : k\in[\![K+1,K']\!]\}$, there is a unique $i := i(j)\in\mathcal{I}$ such that $(i,j)\in\mathcal{F}$ (i.e., $i = i_{k-1}$ when $j = i_k$). Thus, we have $\mathbf{1}_J\in\mathrm{Ker}(\mathcal{A}-Id)^+$ since

$$(\mathbf{1}_J\mathcal{A})_{i_k} = \sum_{i\in\mathcal{I}}\mathbf{1}_J(i)\alpha_{i,i_k} = \sum_{i\in J}\alpha_{i,i_k} = \alpha_{i_{k-1},i_k} = 1 = \mathbf{1}_J(i_k) \quad \forall k\in[\![K+1,K']\!]$$
$$(\mathbf{1}_J\mathcal{A})_{i_k} = \sum_{i\in\mathcal{I}}\mathbf{1}_J(i)\alpha_{i,i_k} = \sum_{i\in J}\alpha_{i,i_k} = 0 = \mathbf{1}_J(i_k) \quad \forall k\notin[\![K+1,K']\!],$$

observing in the second line: when $j = i_k$ with $k\notin[\![K+1,K'+1]\!]$, then $i_{k-1}\notin J$ and for $k = K'+1$, use $i_{K'} = i_K\notin J$.

We obtain that $c\mathbf{1}_J\leqslant\mathbf{x}$, and, in turns, the equality $c\mathbf{1}_J = \mathbf{x}$ since $\mathbf{x}$ is prime. In particular, we have $\mathrm{Supp}(\mathbf{x}) = J$ and $\#\mathrm{Supp}(\mathbf{x}) = N < +\infty$ from which we necessarily derive $i_k = i_{k+N}$ $\forall k\in\mathbb{Z}$. Finally, we use the bijection:

$$\begin{array}{ccc} \big(\mathbb{Z}/N\mathbb{Z}, \{(k,k+1) : k\in\mathbb{Z}/N\mathbb{Z}\}\big) & \longrightarrow & \big(\mathrm{Supp}(\mathbf{x}), \mathcal{F}\cap\mathrm{Supp}(\mathbf{x})^2\big) \\ k & \longmapsto & i_k \end{array}$$

to derive the first part of the conclusion.

For the second case, assume that $i_K\neq i_{K'}$ for all $K<K'\in\mathbb{Z}$. Then, necessarily, for all $j = i_k\in\mathrm{Supp}(\mathbf{x}) = I$, there exists a unique $i = i_{k-1}\in\mathrm{Supp}(\mathbf{x})$ such that $(i,j)\in\mathcal{F}$. Then $\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}\in\mathrm{Ker}(\mathcal{A}-Id)^+$ since for all $j = i_k\in\mathrm{Supp}(\mathbf{x})$, we have:

$$(\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}\mathcal{A})_{i_k} = \sum_{i\in\mathcal{I}}\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}(i)\alpha_{i,i_k} = \sum_{i\in\mathrm{Supp}(\mathbf{x})}\alpha_{i,i_k} = \alpha_{i_{k-1},i_k} = 1 = \mathbf{1}_{\mathrm{Supp}(\mathbf{x})}(i_k).$$

Moreover, for all $j\notin\mathrm{Supp}(\mathbf{x})$, we have:

$$(\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}\mathcal{A})_j = \sum_{i\in\mathcal{I}}\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}(i)\alpha_{i,j} = \sum_{i\in\mathrm{Supp}(\mathbf{x})}\alpha_{i,j} = 0 = \mathbf{1}_{\mathrm{Supp}(\mathbf{x})}(j),$$

since any $i \in \mathrm{Supp}(\mathbf{x})$ leads to another vertex in $\mathrm{Supp}(\mathbf{x})$ as follows from

$$x_j = \sum_{i' \in \mathcal{I}} x_{i'} \alpha_{i',j} = \sum_{i' \in \mathrm{Supp}(\mathbf{x})} x_{i'} \alpha_{i',j} \geqslant x_i \alpha_{i,j}.$$

Observe that $(x_{i_k})_{k \in \mathbb{Z}}$ is constant, equal to some $c > 0$. This follows from

$$x_{i_k} = \sum_{i \in \mathcal{I}} x_i \alpha_{i,i_k} = \sum_{i \in \mathrm{Supp}(\mathbf{x})} x_i \alpha_{i,i_k} = x_{i_k} \alpha_{i_{k-1},i_k} = x_{i_k}.$$

Then, we have $c\mathbf{1}_{\mathrm{Supp}(\mathbf{x})} \leqslant \mathbf{x}$ and, in turn, the equality $\mathbf{x} = c\mathbf{1}_{\mathrm{Supp}(\mathbf{x})}$ follows since $\mathbf{x}$ is prime. In particular, since $i_K \neq i_{K'}$ for all $K < K'$, we have $\#\mathrm{Supp}(\mathbf{x}) = +\infty$ and $(\mathrm{Supp}(\mathbf{x}), \preceq)$ is totally ordered; we conclude this case using the bijection:

$$\begin{array}{ccc} \big(\mathbb{Z}, \{(k, k+1) : k \in \mathbb{Z}\}\big) & \longrightarrow & \big(\mathrm{Supp}(\mathbf{x}), \mathcal{F} \cap \mathrm{Supp}(\mathbf{x})^2\big) \\ k & \longmapsto & i_k. \end{array}$$

$\square$

Proposition A.7. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a finite stochastic graph and $\mathbf{m}_1, \mathbf{m}_2$ be two prime invariant measures. Then, either $\mathbf{m}_1 = \lambda\, \mathbf{m}_2$ for some $\lambda > 0$, either* $\mathrm{Supp}(\mathbf{m}_1) \cap \mathrm{Supp}(\mathbf{m}_2) = \emptyset$.

Proof: Since $\mathcal{A}$ is a finite stochastic matrix, there is a finite number of positive recurrence classes $\mathcal{C}_l$, $l \in [\![1, k]\!]$, each $\mathcal{C}_l$ supporting a unique invariant probability π_l. For $i = 1, 2$, $\mathbf{m}_i$ writes $\mathbf{m}_i = \sum_{l=1}^{k} \alpha_{i,l} \pi_l$ for $\alpha_{i,l} \geqslant 0$. Set $\mathbf{m}_3 = \sum_{l=1}^{k} \min(\alpha_{1,l}, \alpha_{2,l}) \pi_l$. If $\mathbf{m}_3 = 0$, then $\min(\alpha_{1,l}, \alpha_{2,l}) = 0$ for all $l \in [\![1, k]\!]$ and necessarily $\mathrm{Supp}(\mathbf{m}_1) \cap \mathrm{Supp}(\mathbf{m}_2) = \emptyset$. Otherwise, since $\mathbf{m}_3 \leqslant \mathbf{m}_1$ and $\mathbf{m}_3 \leqslant \mathbf{m}_2$, we have $\mathbf{m}_3 = \lambda_1\, \mathbf{m}_1 = \lambda_2\, \mathbf{m}_2$ for $\lambda_1, \lambda_2 > 0$ and we conclude with $\lambda = \lambda_2 / \lambda_1$. $\square$

As indicated previously on page 598, the idea behind the notion of prime invariant measure is to recover a canonical basis of $\mathrm{Ker}(\mathcal{A} - Id)^+$, unique up to a multiplicative constant for each basis vector, as in Davis (1954). In order to give a deeper insight into this notion, observe that in the simple case of $[0, +\infty)^2$, the canonical basis $\mathcal{B}_{2,\mathrm{can}}$ is the unique basis (again, up to multiplicative constants) satisfying:

$$(\lambda, \mu) = \lambda\,(1, 0) + \mu\,(0, 1) \in [0, +\infty)^2 \iff \lambda \geqslant 0, \mu \geqslant 0. \tag{A.3}$$

The notion of prime invariant measure is developed along similar lines. We conjecture that for all $K = \mathrm{Ker}(\mathcal{A} - Id)^+$, there exists a unique basis $\mathcal{B} \in K^E$ (again, up to multiplicative constants) where E is either $\emptyset$, $[\![1, N]\!]$, $\mathbb{N}$ or even $\{0, 1\}^{\mathbb{N}}$, containing only invariant prime measures such that $(K, \mathcal{B})$ is isomorphic to $([0, +\infty)^E, \mathcal{B}_{E,\mathrm{can}})$ and thus enjoying the property (A.3), but with some extra summability constraints in case $\#\mathcal{B} = +\infty$.

To gain a better understanding of this conjecture, we can look at $K = \mathrm{Ker}(\mathcal{A} - Id)^+$ as a closed, convex, pointed (i.e. $K \cap -K = \{0\}$) cone. In this context, the half line generated by a prime invariant measure is called an extreme ray.

When $\#\mathcal{I}$ is finite, then K is a polyhedral cone. The Subpart 5 of Davis (1954) develops this point of view for such a basis $\mathcal{B}$ with the notion of $+$-*canonical matrix* which is the matrix whose columns are the vectors of the basis $\mathcal{B}$. In this case, the above conjecture is proved in Example 4.2 but following another approach. However, even when $\#\mathcal{I}$ is countable, $\#\mathcal{B}$ can be zero, finite, countable, or even uncountable[2], thus the term polyhedral may no longer be appropriate. Nevertheless, we derive properties when K is in a finite-dimensional space, from Ito and Lourenço (2017): for instance, Lemma 3 therein shows that $K \neq \{0\}$ contains at least one extreme ray and Lemma 2 states several interesting properties, which are generalizable to all K.

[2]For example, consider the elementary graph that splits in two at each edge when it is run backward for $\preceq$; in this case prime elementary measures are in bijection with $\{0, 1\}^{\mathbb{N}}$.

A.2. *Graphs invariant under a bijection.*

Definition A.8. Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a stochastic graph and let $\tau : \mathcal{I} \to \mathcal{I}$ be a bijective function. The graph $\mathcal{G}$ is said to be τ-invariant with the constant $\alpha > 0$ if

$$\forall i, j \in \mathcal{I}, \quad \alpha_{i,j} = \alpha_{\tau(i),\tau(j)} \text{ and } \alpha_{\tau(i),i} = \alpha.$$

Lemma A.9. *Let $\mathcal{G}$ be a τ-invariant graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)$. Then both $\mathbf{x}_\tau = (x_{\tau(i)})_{i\in\mathcal{I}}$ and $\Delta_\tau \mathbf{x} := \mathbf{x}_\tau - \mathbf{x}$ belong to $Ker(\mathcal{A} - Id)$.*

Proof: Since τ is a bijection and $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$, we have:

$$x_{\tau(j)} = \sum_{i\in\mathcal{I}} x_i \alpha_{i,\tau(j)} = \sum_{i\in\mathcal{I}} x_{\tau(i)} \alpha_{\tau(i),\tau(j)} = \sum_{i\in\mathcal{I}} x_{\tau(i)} \alpha_{i,j} \ \forall j \in \mathcal{I},$$

proving $\mathbf{x}_\tau \in \mathrm{Ker}(\mathcal{A} - Id)$, and, by linearity, $\Delta_\tau \mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$. □

Lemma A.10. *Let $\mathcal{G}$ be a bistochastic τ-invariant graph with the constant α and $\mathbf{x} \in Ker(\mathcal{A} - Id)$ bounded by $M < +\infty$. If $x_j \in [M - \varepsilon, M]$ for some $\varepsilon > 0$, then $x_{\tau(j)} \in [M - \varepsilon/\alpha, M]$.*

Proof: Let $x_j \in [M - \varepsilon, M]$. Since

$$x_j = \sum_{i\in\mathcal{I}} x_i \alpha_{i,j} = \sum_{i\in\mathcal{I}, i\neq\tau(j)} x_i \alpha_{i,j} + x_{\tau(j)} \alpha_{\tau(j),j}$$

and $1 - \sum_{i\in\mathcal{I}, i\neq\tau(j)} \alpha_{i,j} = \alpha_{\tau(j),j} = \alpha$ by the bistochasticity of $\mathcal{G}$, we have:

$$x_{\tau(j)}\alpha = x_j - \sum_{i\in\mathcal{I}, i\neq\tau(j)} x_i \alpha_{i,j} \geqslant M - \varepsilon - M \sum_{i\in\mathcal{I}, i\neq\tau(j)} \alpha_{i,j} = M\alpha - \varepsilon,$$

from which it follows that $M \geqslant x_{\tau(j)} \geqslant M - \varepsilon/\alpha$. □

Lemma A.11. *Let $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathcal{A})$ be a connected, bistochastic, τ-invariant graph and $\mathbf{x} \in Ker(\mathcal{A} - Id)$ be an invariant measure. If $\mathbf{x}$ is bounded, then $\Delta_\tau \mathbf{x} = 0$, i.e. $x_i = x_{\tau(i)}$ $\forall i \in \mathcal{I}$.*

Proof: Let $\mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$ be a bounded invariant measure then $\Delta_\tau \mathbf{x} \in \mathrm{Ker}(\mathcal{A} - Id)$ (Lemma A.9) and is bounded ($|\Delta_\tau x_i| = |x_{\tau(i)} - x_i| \leqslant 2 \sup_{j\in\mathcal{I}} |x_j| < +\infty$).

Assume that $\Delta_\tau \mathbf{x} \neq 0$. Then $M := \sup_{i\in\mathcal{I}}(\Delta_\tau x_i) > 0$ (up to change $\mathbf{x}$ into $-\mathbf{x}$). For any $K \in \mathbb{N}$, with $\varepsilon := M\alpha_\tau^K/2$, $\Delta_\tau x_i \in [M - \varepsilon, M]$ for some $i \in \mathcal{I}$ and using recursively Lemma A.10, we get $\Delta_\tau x_{\tau^k(i)} \in [M - \varepsilon/\alpha_\tau^k, M]$ $\forall k \in \mathbb{N}$. Now, since $\alpha_\tau < 1$, we have:

$$\forall k \in [\![0, K]\!], \quad \Delta_\tau x_{\tau^k(i)} \geqslant M - \frac{\varepsilon}{\alpha_\tau^k} \geqslant M - \frac{M}{2} = \frac{M}{2},$$

which contradicts

$$2 \sup_{i\in\mathcal{I}} |x_i| \geqslant x_{\tau^{K+1}(i)} - x_i = \sum_{k=0}^{K} (x_{\tau^{k+1}(i)} - x_{\tau^k(i)}) = \sum_{k=0}^{K} \Delta_\tau x_{\tau^k(i)} \geqslant (K+1)\frac{M}{2}.$$

We conclude that $\Delta_\tau \mathbf{x} = 0$. □

Proposition A.12. *Let $(G, +)$ be a finitely generated abelian group and $J \subset G$ be a finite set of generators such that $\langle J \rangle = G$. Consider the graph $\mathcal{G} = (\mathcal{I}, \mathcal{F}, \mathbf{A})$ where $\mathcal{I} = G$, $\mathcal{F} = \{(g, g+j) : g \in G, j \in J\}$ and assume that $\mathcal{A} = \mathbb{E}[\mathbf{A}] = (\alpha_{g,g'})_{g,g'\in G}$ satisfies $\alpha_{g,g+j} = \alpha_{0,j} > 0$ $\forall (g, j) \in G \times J$. Then $\mathbb{1}$ is a prime invariant measure of $\mathcal{A}$, i.e. any bounded $\mathbf{x} \in Ker(\mathcal{A} - Id)$ is constant.*

Proof: For any $j \in J$, define the bijection τ^j of G by $\tau^j(g) = g - j$. Then $\mathcal{G}$ is a τ^j-invariant graph with the constant $\alpha_{0,j} > 0$. Lemma A.11 implies $x_g = x_{g+j} = x_{g-j}$ $\forall (g, j) \in G \times J$, so that $\mathbf{x}$ is constant along J and on $\langle J \rangle = G = \mathcal{I}$. □

A.3. *Results ralted to Gamma distributions.* In this section, we gather several results for gamma distributions. First, we recall the original Lukacs' result as given in Lukacs (1955):

Theorem A.13 (Lukacs)**.** *Let X, Y be two non-negative non-degenerate independent random variables. Then $X+Y$ and X/Y are independent if and only if there exist $\alpha,\beta,\gamma>0$ such that $X\sim\Gamma(\alpha,\gamma)$ and $Y\sim\Gamma(\beta,\gamma)$.*

Next, we prove Lemma 2.2:

Proof: (Lemma 2.2) Up to rescaling by the factor γ, we can assume $\gamma=1$. For any bounded measurable functions $F:[0,+\infty)\to\mathbb{R}$ and $G:[0,1]^n\to\mathbb{R}$, we have:

$$\begin{aligned}
&\mathbb{E}\left[F(Y)G\left(\frac{Y_1}{Y},\dots,\frac{Y_n}{Y}\right)\right]\\
&=\int_{[0,+\infty)^n} F(y_1+\dots+y_n)G\left(\frac{y_1}{y_1+\dots+y_n},\dots,\frac{y_n}{y_1+\dots+y_n}\right)\prod_{i=1}^n\left(\frac{y_i^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-y_i}\,dy_i\right)\\
&=\int_{[0,+\infty)\times[0,1]^{n-1}} F(u)G(v_1,\dots,v_n)\left(\prod_{i=1}^{n-1}\frac{(uv_i)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\frac{\left(u\left(1-\sum_{i=1}^{n-1}v_i\right)\right)^{\alpha_n-1}}{\Gamma(\alpha_n)}e^{-u}\,u^{n-1}\,dudv_1\dots dv_{n-1}
\end{aligned}$$

with the change of variables $u=y_1+\dots+y_n$, $v_i=y_i/(y_1+\dots+y_n)$ for $i<n$ and setting $v_n=1-\sum_{i=1}^{n-1}v_i$, with Jacobian:

$$\begin{vmatrix} u & & & -u\\ & \ddots & & \vdots\\ & & u & -u\\ v_1 & \dots & v_{n-1} & 1-\sum_{i=1}^{n-1}v_i\end{vmatrix}=\begin{vmatrix} u & & & 0\\ & \ddots & & \vdots\\ & & u & 0\\ v_1 & \dots & v_{n-1} & 1\end{vmatrix}=u^{n-1}.$$

Thus, by the Fubini's theorem, we have:

$$\begin{aligned}
&\mathbb{E}\left[F(Y)G\left(\frac{Y_1}{Y},\dots,\frac{Y_n}{Y}\right)\right]\\
&=\int_{[0,+\infty)\times[0,1]^{n-1}} F(u)G(v_1,\dots,v_n)\frac{u^{\alpha_1+\dots+\alpha_n-1}e^{-u}}{\Gamma(\alpha_1+\dots+\alpha_n)}\frac{\Gamma(\alpha_1+\dots+\alpha_n)}{\prod_{i=1}^n\Gamma(\alpha_i)}\prod_{i=1}^n v_i^{\alpha_i-1}\,dudv_1\dots dv_{n-1}\\
&=\int_0^{+\infty}F(u)\frac{u^{\alpha_1+\dots+\alpha_n-1}e^{-u}}{\Gamma(\alpha_1+\dots+\alpha_n)}du\int_{[0,1]^{n-1}}G(v_1,\dots,v_n)\frac{\prod_{i=1}^n v_i^{\alpha_i-1}}{B(\alpha_1,\dots,\alpha_n)}\,dv_1\dots dv_{n-1},
\end{aligned}$$

from which (2.1) follows. □

Lemma A.14. *Let $\gamma>0$ and X,Y,A,B be four independent random variables such that:*

$$\begin{aligned}
&A\in[0,1],\quad \mathbb{E}[A]=\alpha,\qquad B\in[0,1],\quad \mathbb{E}[B]=\beta,\\
&X\sim\Gamma(\gamma x,\gamma),\quad Y\sim\Gamma(\gamma y,\gamma),\quad AX+BY\sim\Gamma\left(\gamma(\alpha x+\beta y),\gamma\right).
\end{aligned}$$

Then, setting $\overline{u}=1-u$, for any $u\in[0,1]$, we have:

$$AX+BY\perp\!\!\!\perp\overline{A}X+\overline{B}Y\quad\text{and}\quad \overline{A}X+\overline{B}Y\sim\Gamma\left(\gamma(\overline{\alpha}x+\overline{\beta}y),\gamma\right).$$

Proof: Up to considering $X':=\gamma X$ and $Y':=\gamma Y$ instead of X and Y, we can, and will, assume that $\gamma=1$, and we have from (2.3) $\psi_X(t)=(1-t)^{-x}$, $\psi_Y(t)=(1-t)^{-y}$, $\psi_{AX+BY}(t)=(1-t)^{-\alpha x-\beta y}$. We also have

$$\begin{aligned}
\psi_{AX+BY}(t)&=\mathbb{E}\left[e^{t(AX+BY)}\right]=\mathbb{E}\left[e^{tAX}\right]\mathbb{E}\left[e^{tBY}\right]=\mathbb{E}\left[\mathbb{E}\left[e^{tAX}\,|A\right]\right]\mathbb{E}\left[\mathbb{E}\left[e^{tBY}\,|B\right]\right]\\
&=\mathbb{E}\left[(1-tA)^{-x}\right]\mathbb{E}\left[(1-tB)^{-y}\right],
\end{aligned}$$

from which we deduce for any $t \in (0,1)$:

$$\mathbb{E}\left[(1-tA)^{-x}\right]\ \mathbb{E}\left[(1-tB)^{-y}\right] = (1-t)^{-\alpha x-\beta y}. \tag{A.4}$$

Next, by straightforward computations, we have for $u, v \in [0,1)$ with $u > v$:

$$\begin{aligned}
&\psi_{AX+BY,\overline{A}X+\overline{B}Y}(u,v) = \mathbb{E}\left[e^{u(AX+BY)+v(\overline{A}X+\overline{B}Y)}\right] = \mathbb{E}\left[e^{(uA+v\overline{A})X}e^{(uB+v\overline{B})Y}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[e^{(uA+v\overline{A})X}\ \middle|A\right]\right]\ \mathbb{E}\left[\mathbb{E}\left[e^{(uB+v\overline{B})Y}\ \middle|B\right]\right]\\
&= \mathbb{E}\left[\left(1-(uA+v\overline{A})\right)^{-x}\right]\ \mathbb{E}\left[\left(1-(uB+v\overline{B})\right)^{-y}\right]\\
&= \mathbb{E}\left[(1-v-(u-v)A)^{-x}\right]\ \mathbb{E}\left[(1-v-(u-v)B)^{-y}\right]\\
&= (1-v)^{-x-y}\,\mathbb{E}\left[\left(1-\frac{u-v}{1-v}A\right)^{-x}\right]\ \mathbb{E}\left[\left(1-\frac{u-v}{1-v}B\right)^{-y}\right]\\
&= (1-v)^{-x-y}\left(1-\frac{u-v}{1-v}\right)^{-\alpha x-\beta y} = (1-u)^{-\alpha x-\beta y}\ (1-v)^{-\overline{\alpha}x-\overline{\beta}y}
\end{aligned}$$

by using (A.4) with $t = (u-v)/(1-v) \in (0,1)$. This proves the independence $AX+BY \perp\!\!\!\perp \overline{A}X+\overline{B}Y$ and the distribution of $\overline{A}X+\overline{B}Y$. □

Lemma A.15. *Let $I \subset \mathbb{N}$ with $\#I \geqslant 2$ and $(X_i)_{i\in I}$, Y be independent random variables such that $X_i \sim \Gamma_{\gamma_i}(x_i)$ and $Y \sim \Gamma_\gamma(y)$, with $x_i, y \in (0,+\infty)$ and $\gamma_i, \gamma \in \overline{\mathbb{R}}_+^*$. Then $\sum_{i\in I} X_i \sim Y$ if and only if $\sum_{i\in I} x_i = y$ and $\gamma_i = \gamma$ for all $i \in I$.*

Proof: We give the proof for $I = [\![1,n]\!]$ with $n \geqslant 2$, the case $I = \mathbb{N}$ follows from a straightforward adaptation with adequate limits. The converse implication is straightforward. For the direct implication, first observe that if $\gamma = +\infty$, then Y is almost surely constant and all the $\gamma_i's$ must be $+\infty$ otherwise independence would be violated in $\sum_{i\in I} X_i \sim Y$. In this case, all the X_i's and Y are constant and the equality $\sum_{i=1}^n x_i = y$ then follows. We can thus assume $\gamma < +\infty$ and in this case all the γ_i's must be also positive, otherwise with $\gamma_i = +\infty$, we should have $Y \geqslant X_i = \mathbb{E}[X_i] > 0$ a.s., which is not possible for $Y \sim \Gamma_\gamma(y)$. Then, differentiating the equality $\sum_{i=1}^n \phi_{X_i} = \phi_Y$ for the cumulant generating function in Def. 2.6 of independent X_i's yields:

$$\sum_{i=1}^n \frac{x_i}{1-t/\gamma_i} = \frac{y}{1-t/\gamma} \quad \forall t \in \mathbb{C}\setminus\{\gamma_1,\dots,\gamma_n,\gamma\}.$$

Since all the x_i's are positive, by uniqueness of the decomposition of a rational fraction, all the γ_i's and γ must coincide and the equality $\sum_{i=1}^n x_i = y$ follows readily. □

References

Bobecka, K. and Wesołowski, J. Multivariate Lukacs theorem. *J. Multivariate Anal.*, **91** (2), 143–160 (2004). DOI: 10.1016/j.jmva.2003.09.012.

Bourguin, S. and Tudor, C. A. Cramér theorem for gamma random variables. *Electron. Commun. Probab.*, **16**, 365–378 (2011). DOI: 10.1214/ECP.v16-1639.

Davis, C. Theory of positive linear dependence. *Amer. J. Math.*, **76**, 733–746 (1954). DOI: 10.2307/2372648.

De Masi, A., Galves, A., Löcherbach, E., and Presutti, E. Hydrodynamic limit for interacting neurons. *J. Stat. Phys.*, **158** (4), 866–902 (2015). DOI: 10.1007/s10955-014-1145-1.

DeGroot, M. H. and Rao, M. M. Stochastic give-and-take. *J. Math. Anal. Appl.*, **7**, 489–498 (1963). DOI: 10.1016/0022-247X(63)90071-4.

Düring, B., Georgiou, N., Merino-Aceituno, S., and Scalas, E. Continuum and thermodynamic limits for a simple random-exchange model. *Stochastic Process. Appl.*, **149**, 248–277 (2022). DOI: 10.1016/j.spa.2022.03.015.

Ito, M. and Lourenço, B. F. A bound on the Carathéodory number. *Linear Algebra Appl.*, **532**, 347–363 (2017). DOI: 10.1016/j.laa.2017.06.043.

Koudou, A. E. and Wesołowski, J. Independence preserving property of Kummer laws. *Bernoulli*, **31** (1), 295–311 (2025). DOI: 10.3150/24-bej1728.

Letac, G. and Wesołowski, J. An independence property for the product of GIG and gamma laws. *Ann. Probab.*, **28** (3), 1371–1383 (2000). DOI: 10.1214/aop/1019160339.

Lukacs, E. A characterization of the gamma distribution. *Ann. Math. Statist.*, **26**, 319–324 (1955). DOI: 10.1214/aoms/1177728549.

Matsumoto, H. and Yor, M. An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. I. A time-inversion approach. *Nagoya Math. J.*, **159**, 125–166 (2000). DOI: 10.1017/S0027763000007455.

McKinlay, S. A characterisation of transient random walks on stochastic matrices with Dirichlet distributed limits. *J. Appl. Probab.*, **51** (2), 542–555 (2014). DOI: 10.1239/jap/1402578642.

Muratov, A. and Zuyev, S. Neighbour-dependent point shifts and random exchange models: invariance and attractors. *Bernoulli*, **23** (1), 539–551 (2017). DOI: 10.3150/15-BEJ755.

Ng, K. W., Tian, G.-L., and Tang, M.-L. *Dirichlet and related distributions. Theory, methods and applications.* Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester (2011). ISBN 978-0-470-68819-9. DOI: 10.1002/9781119995784.

Oestereich, A. L., Crokidakis, N., and Cajueiro, D. O. Impact of memory and bias in kinetic exchange opinion models on random networks. *Phys. A*, **607**, Paper No. 128199, 8 (2022). DOI: 10.1016/j.physa.2022.128199.

Van Assche, W. Products of 2×2 stochastic matrices with random entries. *J. Appl. Probab.*, **23** (4), 1019–1024 (1986). DOI: 10.1017/s0021900200118807.

van der Hofstad, R. *Random graphs and complex networks. Vol. 1.* Cambridge Series in Statistical and Probabilistic Mathematics, [43]. Cambridge University Press, Cambridge (2017). ISBN 978-1-107-17287-6. DOI: 10.1017/9781316779422.

West, D. B. *Introduction to graph theory.* Prentice Hall, Inc., Upper Saddle River, NJ (1996). ISBN 0-13-227828-6. MR1367739.