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Uniform large deviations for a class of stochastic reaction-diffusion equations on the real line with super-linear drifts driven by multiplicative space-time white noise

Leila Setayeshgar

Assistant Professor of Mathematics, Department of Mathematics and Statistics, College of Charleston, 175 Calhoun St., Charleston, SC 29401
E-mail address: setayeshgarl@cofc.edu

Abstract. We prove a Freidlin-Wentzell uniform large deviation principle for the law of solutions to a class of stochastic reaction-diffusion equations with super-linear drifts without dissipation on the real line $\mathbb{R}$, driven by multiplicative space-time white noise. The uniformity is with respect to initial conditions that are bounded and do not necessarily belong to compact sets. We employ the weak convergence method.

1. Introduction

In this paper we consider a family of stochastic reaction-diffusion equations, perturbed by multiplicative space-time white noise on the real line $\mathbb{R}$, indexed by $0 < \varepsilon \le 1$. The equations read as follows:

$$du^\varepsilon(t,x) = \frac{1}{2}\partial_{xx}u^\varepsilon(t,x)dt + b\big(u^\varepsilon(t,x)\big)dt + \sqrt{\varepsilon}\sigma\big(u^\varepsilon(t,x)\big)W(dt,dx), \tag{1.1}$$

for all $t \in [0,T],\ x \in \mathbb{R}$, with the initial condition

$$u^\varepsilon(0,x) = \xi(x),\ x \in \mathbb{R}.$$

Here, the coefficients b and σ are deterministic and measurable functions from $\mathbb{R}$ to $\mathbb{R}$, whose specific prescribed conditions are given in Section 2. W is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ carried by some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, with the usual definition, and the small parameter $0 < \varepsilon \le 1$, denotes the noise intensity.

Mathematics literature is replete with work on stochastic reaction-diffusion equations on various aspects (see e.g. Sowers, 1992; Peszat, 1994; Salins et al., 2019; Shang and Zhang, 2023; Li et al., 2024; Freidlin, 1988; Dalang et al., 2019; Cerrai and Röckner, 2004; Mytnik and Perkins, 2011; Cerrai, 2003 and references therein). The study of well-posedness for stochastic reaction-diffusion equations on the real line has proven to be challenging, mainly due to the non-compactness of

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the whole space. In Cerrai (2003), the existence and uniqueness of solutions, and existence of an invariant measure for a class of reaction-diffusion systems on bounded domains of $\mathbb{R}^m$, perturbed by multiplicative noise was proven, where the noise was white in time and space if $m = 1$, and colored in space if $m > 1$. In Mytnik et al. (2006), the authors proved existence and pathwise uniqueness for a stochastic heat equation with non-Lipschitz noise coefficients, perturbed by multiplicative colored noise on $\mathbb{R}^m$ for $m \geq 1$. In Mytnik and Perkins (2011), existence and pathwise uniqueness for a stochastic heat equation with Hölder continuous (of index $\gamma > 3/4$) noise coefficients, perturbed by multiplicative noise was proven. In Dalang et al. (2019), a stochastic reaction-diffusion equation on a finite spatial domain was considered. It was proven that, if the drift term is locally Lipschitz and of $\big(|z| \log |z|\big)$ growth, then the stochastic reaction-diffusion equation is globally well-posed. The study of large deviations for stochastic reaction-diffusion equations was pioneered by Freidlin (1988), where additive perturbations were considered. Sowers (1992) was among the first who achieved success in proving a large deviation principle for the law of the solutions to a class of stochastic reaction-diffusion equations more general than that of Freidlin (1988), by using the martingale approach of Walsh (1986). In Peszat (1994), a large deviation principle for a class of reaction-diffusion equations with non-additive random perturbations was proven, where the semigroup approach presented in Da Prato and Zabczyk (1992) was persued. In Cerrai and Röckner (2004), the latter approach was used, and a large deviations for a class of reaction-diffusion equations perturbed by multiplicative noise was obtained, where the diffusion coefficients were globally Lipschitz and unbounded, and the reaction coefficients were locally Lipschitz and of polynomial growth. Their work generalized the previous results.

The existence and uniqueness of strong solutions to Eq. (1.1) was proven by Shang and Zhang (2023). More precisely, it was proven that a strong, unique solution to Eq. (1.1), in the sense of probability theory exists globally, where the drift b, is locally log-Lipschitz and $|b(z)| = O\big(|z| \log |z|\big)$. Essentially, Shang and Zhang (2023) extended the well-posedness result of Dalang et al. (2019), where the space variable belonged to a compact interval to that of the whole real line. The main difficulty that they encountered was that applying the usual truncation procedure were not to be legitimate, which in turn was due to the explosion of the supremum norm of the solution, i.e., $\sup_{x\in\mathbb{R}} |u(t,x)| = \infty$. Due to the presence of logarithmic nonlinearity, a specially designed norm

$$\sup_{t\in[0,T],x\in\mathbb{R}} \left(|u(t,x)| e^{-\lambda|x|e^{\beta t}}\right), \tag{1.2}$$

on $C_{tem}(\mathbb{R})$, where

$$C_{tem} := \big\{ f \in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left(|f(x)| e^{-\lambda|x|}\right) < \infty, \text{ for any } \lambda > 0 \big\},$$

had to be considered. Note that different from the usual norm on $C_{tem}(\mathbb{R})$, the exponent is dependent upon t in a specific way, where the definition of β is given by (2.1) in Section 2. The main contributions of Shang and Zhang (2023) were a new type of Grönwall's inequality, as well as new lower order moment estimates for the stochastic convolution, which led to the proof.

The study of reaction-diffusion equations has been historically highly attractive to scientists and engineers, among others. Besides their sheer mathematical interest, they have numerous applications in modeling natural and physical phenomena. In a stochastic reaction-diffusion equation, the diffusion term describes how the quantity represented by the variable (e.g., u), diffuses in a spatial domain. In this term, the diffusion coefficient represents the rate of diffusion, where the laplacian of u, describes how the quantity represented by u, changes in space. The reaction term, describes the deterministic aspects of evolution of the state's dynamical system. The noise, introduces stochasticity to the model, reflecting the inherent uncertainty in the system dynamics. Stochastic reaction-diffusion equations have wide applications in many areas of science, including but not limited to, physics, chemistry and biology. In biology, they may model metabolic reactions, gene expression, tumor growth and spread, and intracellular signaling. In chemistry, they may model chemical oscillations, and drug delivery. In physics, they may model non-equilibrium transitions, and in mathematical finance, they may model stock price dynamics, and hedging. In short, stochastic reaction-diffusion equations' mathematical richness, as well as their wide applications in modeling natural phenomena, have served as main motivations for studying these equations and therefore, have historically made them appealing to scientists and engineers, among others.

The aim of this work is to prove a Freidlin and Wentzell uniform large deviation principle (FWULDP) for the law of the solutions to Eq. (1.1), where different from the work in Li et al. (2024) in which, a sufficient condition in Matoussi et al. (2021) was employed, a sufficient condition introduced by Salins (2019) is used, where the uniformity is with respect to initial conditions that are only bounded and do not necessarily belong to compact sets. Salins (2019) introduced a definition for an equicontinuous uniform Laplace principle (EULP) and further showed that, the EULP is equivalent to the Freidlin and Wentzell's definition of the uniform large deviation principle. It was then shown that under a sufficient condition, a measurable function of an infinite-dimensional Wiener process satisfies an EULP and therefore an FWULDP with uniformity being over initial conditions that belong to bounded and not necessarily compact sets. We now state the main theorem of our paper.

Theorem 1.1 (Main Theorem). *The processes $u^{\varepsilon}_{\xi} := \{u^{\varepsilon}_{\xi}(t,x) : t \in [0,T],\ x \in \mathbb{R}\}$, where $\varepsilon \in (0,1]$, $\xi \in C_{tem}$, satisfy a ULDP that is uniform over bounded subset of $|\xi|_{(-\lambda)} \leq R$ and the ULDP is valid in the norm*

$$\sup_{t\in[0,T],x\in\mathbb{R}} \left(|u^{\varepsilon}_{\xi}(t,x)| e^{-\lambda|x|e^{\beta t}} \right), \tag{1.3}$$

with rate function I_{ξ} given by (4.2).

Lastly, we conclude the introduction by motivating our work. Our work is partly motivated by the fact that, compactness is present in infinite dimensions, only rarely. Our proof employs the weak convergence method of Dupuis and Ellis (1997), and uses a sufficient condition introduced in Salins (2019) as opposed to that of Budhiraja et al. (2008), which is not legitimate to be applied in this setting, and is further motivated by its potential to study exit time problems.

1.1. *Outline.* The outline of the paper is as follows. In Section 2 we state the preliminaries and assumptions necessary for the formulation of the problem. The existence and uniqueness of a strong solution to Eq. (1.1) is also stated in this section. In Section 3 we state the definition for a uniform large deviation principle, as well as a sufficient condition under which an FWULDP is said to hold. In Section 4 we introduce the controlled process and skeleton equation and assert their existence and uniqueness. In Section 5 we derive a uniform a priori bound on the controlled process, which is used in proving the main theorem. Section 6 is devoted to the statement of the main theorem. Finally, Section 7 presents the proof of the FWULDP and concludes the paper.

1.2. *Remark on Notation.* Unless otherwise noted, we adopt the following notation throughout the paper. C denotes a free, positive constant which may take on different values, and may depend upon different parameters. We use $|r(t,\cdot)|_p := |r(t)|_p$ to denote the $L^p(\mathbb{R})$-norm of a function $r := r(t,x)$ with respect to the variable $x \in \mathbb{R}$. If $r(t,x)$ is only defined for $x \in [0,1]$, then $|r(t)|_p$ denotes the $L^p([0,1])$-norm. If $r := r(t,x)$ is a random field and E is a function space, then saying that r is almost surely in E means that r has a stochastic modification, which is in E almost surely. For any metric space (E, θ), the distance between an element $y \in E$ and a set $A \subset E$ is defined by

$$\mathrm{dist}_E(x, A) := \inf_{y \in B} \theta(x, y). \tag{1.4}$$

Note that at times, we use the abbreviation ULDP in place of FWULDP. Further, any notation not introduced in this section mainly due to impracticality is made so upon usage.

2. Preliminaries

In this section we state the assumptions and preliminaries necessary for the formulation of the problem.

We first introduce the definition of a mild solution to Eq. (1.1), as follows:

Definition 2.1 (Mild solution)**.** A random field, $u^\varepsilon_\xi := \{u^\varepsilon_\xi(t,x) : t \in [0,T], x \in \mathbb{R}\}$, is called a mild solution to Eq. (1.1) with initial condition ξ, if $(t,x) \to u^\varepsilon_\xi(t,x)$ is a jointly measurable and adapted space-time process for any $t \in [0,T]$, and $x \in \mathbb{R}$ and if

$$\begin{aligned} u^\varepsilon_\xi(t,x) = G_t\xi(x) &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) b(u^\varepsilon_\xi(s,y)) dy ds. \\ &+ \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) \sigma(u^\varepsilon_\xi(s,y)) W(dy, ds), \ \mathbb{P} - a.s., \end{aligned}$$

where the function $\{G_t\}_{t\geq 0}$ is the corresponding heat semigroup on $\mathbb{R}$, with explicit form

$$G_t(x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

We remark that the above mild form is equivalent to the weak formulation of stochastic-reaction diffusion equations in the sense of partial differential equations (Walsh, 1986).

Note that the following estimates for the heat kernel hold true (Shang and Zhang, 2023). For any $x \in \mathbb{R}$, and any $t > 0$

(E1) $\displaystyle\int_{\mathbb{R}} G_t(x,y) e^{\gamma|y|} dy \leq 2e^{\frac{\gamma^2 t}{2}} e^{\gamma|x|}, \ \forall\ \gamma \in \mathbb{R},$

(E2) $\displaystyle\int_{\mathbb{R}} G_t^2(x,y) e^{\gamma|y|} dy \leq \frac{1}{\sqrt{\pi t}} e^{\frac{\gamma^2 t}{4}} e^{\gamma|x|}, \ \forall\ \gamma \in \mathbb{R},$

(E3) $\displaystyle\int_{\mathbb{R}} G_t(x,y) e^{\gamma|y|} \gamma|y| dy \leq e^{\frac{\gamma^2 t}{2}} e^{\gamma|x|} \eta|x| + 2e^{\frac{\gamma^2 t}{2}} \left(\gamma^2 t + \gamma\sqrt{\frac{t}{2\pi}}\right) e^{\gamma|x|}, \ \forall\ \gamma \in \mathbb{R},$

Next, we recall the definition of the so-called C_{tem} space

$$C_{tem} := \{f \in C(\mathbb{R}) : \sup_{x \in \mathbb{R}} |f(x)| e^{\lambda|x|} < \infty, \text{ for any } \lambda > 0\},$$

endowed with the metric

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \sup_{x \in \mathbb{R}} |f(x) - g(x)| e^{-\frac{1}{n}|x|}\}, \ \forall\ f, g \in C_{tem}.$$

Note that the space (C_{tem}, d) is a complete, separable, metric space and is thus, Polish. Further for notational convenience, for any $\lambda > 0$ and $g \in C(\mathbb{R})$ we set

$$|g|_{(-\lambda)} := \sup_{x \in \mathbb{R}} \left(|g(x)| e^{-\lambda|x|}\right).$$

In addition, for any $\lambda, \kappa > 0$ we define

$$\beta(\kappa,\lambda) := \max\left\{\frac{\lambda^2}{2}, 4\kappa\right\}, \tag{2.1}$$

$$T^*(\kappa,\lambda) := \frac{1}{2\beta(\kappa,\lambda)}\left[1+\log\left(\frac{4\beta(\kappa,\lambda)}{\lambda^2}\log\frac{\beta(\kappa,\lambda)}{2\kappa}\right)\right]. \tag{2.2}$$

Note that by the definition of β, Eq. (2.2) is well-defined. Moreover, for any $\kappa > 0$ we have

$$\lim_{\lambda\to 0} T^*(\kappa,\lambda) = \infty.$$

Lastly, hereon, and for the sake of simplicity, we write $\beta(c_1,\lambda)$ as β. We are now in a position to state two facts, as follows:

(F1) Eq. (2.2) implies that there exists a positive constant λ_T, such that for all $\lambda \in (0, \lambda_T]$ we have

$$T \le T^*(c_1,\lambda).$$

(F2)
$$\frac{c_1}{\beta}e^{\frac{\lambda^2}{4\beta}}e^{2\beta T-1} \le \frac{1}{2} \iff T \le T^*(c_1,\lambda) = \frac{1}{2\beta}\left[1+\log\left(\frac{4\beta}{\lambda^2}\log\frac{\beta}{2c_1}\right)\right].$$

Further, we impose the following assumptions on the deterministic and measurable functions $b, \sigma : \mathbb{R} \to \mathbb{R}$.

(A1) The function b is continuous. Further, there exist constants $k_1 > 0$, $k_2 > 0$, such that
$$|b(p)| \le k_1|p|\log_+|p| + k_2 \ \forall\ p \in \mathbb{R},$$
where $\log_+|p| := \log(p \vee 1)$ for any $p \ge 0$.

(A2) There exist constants $k_3 > 0$, $k_4 > 0$ and $k_5 > 0$ such that
$$|b(p)-b(q)| \le k_3|p-q|\log_+\frac{1}{|p-q|} + k_4\log_+(|p|\vee|q|)|p-q| + k_5|p-q|,\ \forall\ p,q \in \mathbb{R},$$
where $\log_+|p| := \log(p \vee 1)$ for any $p \ge 0$.

(A3) The function σ is globally Lipschitz and bounded, that is, there exist constants $L_\sigma > 0$ and $K_\sigma > 0$ such that $\forall\ p,q \in \mathbb{R}$

$$|\sigma(p)-\sigma(q)| \le L_\sigma|p-q|,$$
$$|\sigma(p)| \le K_\sigma.$$

Finally, we state the following result (Shang and Zhang, 2023, Theorem 2.5), which asserts the existence and uniqueness of a probabilistically strong solution to Eq. (1.1).

Theorem 2.2 (Existence & uniqueness of solution mapping)**.** *Under the assumptions that σ is bounded and Lipschitz, $\xi \in C_{tem}$, and that (A2) is satisfied, pathwise uniqueness holds for solutions to Eq. (1.1) in $C(\mathbb{R}_+, C_{tem})$. Hence, a unique strong solution to Eq. (1.1) exists in $C(\mathbb{R}_+, C_{tem})$.*

3. Uniform large deviation principle

This section is devoted to providing a sufficient condition under which an EULP is said to hold. Let $(\mathcal{E},\gamma)$ be a Polish space and $\mathcal{E}_0$ a set. For any $\xi \in \mathcal{E}_0$, definition of rate function $I_\xi : \mathcal{E} \to [0,+\infty]$, is given by (4.2). Let $\Lambda_\xi(s)$ denote the level set $\Lambda_\xi(s) := \{\varphi \in \mathcal{E} : I_\xi(\varphi) \le s\}$, and let $\mathscr{J}$ be a collection of subsets of $\mathcal{E}_0$. Moreover, let $\theta(\varepsilon)$ be a function converging to zero as ε approaches zero itself.

Definition 3.1 (FWULDP)**.** A family of $\mathcal{E}$-valued random variables $\{Y^\varepsilon_\xi\}^{\varepsilon\in(0,1]}_{\xi\in\mathcal{E}_0}$, is said to satisfy an FWULDP with speed $\theta(\varepsilon)$ and rate function I_ξ, uniformly over $\mathscr{J}$ if

(1) For any $J \in \mathscr{J}$, $s_0 \geq 0$, and $\delta > 0$

$$\liminf_{\varepsilon\to 0} \inf_{\xi\in J} \inf_{\varphi\in\Lambda_\xi(s_0)} \left(\gamma(\varepsilon)\log\mathbb{P}\Big(\theta(Y^\varepsilon_\xi,\varphi) < \delta\Big) + I_\xi(\varphi)\right) \geq 0. \tag{3.1}$$

(2) For any $J \in \mathscr{J}$, $s_0 \geq 0$, and $\delta > 0$

$$\limsup_{\varepsilon\to 0} \sup_{\xi\in J} \sup_{s\in[0,s_0]} \left(\gamma(\varepsilon)\log\mathbb{P}\Big(\mathrm{dist}_{\mathcal{E}}(Y^\varepsilon_\xi,\Lambda_\xi(s)\Big) \geq \delta) + s\right) \leq 0. \tag{3.2}$$

Note that an EULP is an equivalent formulation to a ULDP (Salins, 2019, Theorem 2.10) . Recall that a family $Q \subset C_b(\mathcal{E})$ of functions is equibounded and equicontinuous if

$$\sup_{z\in Q}\sup_{\varphi\in\mathcal{E}}|z(\varphi)| < \infty \quad \text{and} \quad \lim_{\delta\to 0}\sup_{z\in Q}\sup_{\substack{\varphi,\xi\in\mathcal{E}\\ \theta(\varphi,\xi)<\delta}}|z(\varphi)-z(\xi)| = 0.$$

Definition 3.2 (EULP)**.** A family of $\mathcal{E}$-valued random variables $\{Y^\varepsilon_\xi\}^{\varepsilon\in(0,1]}_{\xi\in\mathcal{E}_0}$, is said to satisfy an EULP with speed $\gamma(\varepsilon)$ and rate function I_ξ, uniformly over $\mathscr{J}$ if for any $J \in \mathscr{J}$ and any equicontinuous and equibounded family $Q \subset C_b(\mathcal{E})$

$$\lim_{\varepsilon\to 0}\sup_{z\in Q}\sup_{\xi\in J}\left|\gamma(\varepsilon)\log\mathbb{E}\exp\left(-\frac{z(Y^\varepsilon_\xi)}{\gamma(\varepsilon)}\right) + \inf_{\varphi\in\mathcal{E}}\{z(\varphi)+I_\xi(\varphi)\}\right| = 0. \tag{3.3}$$

Theorem 3.3 (Salins, 2019, Theorem 2.10)**.** *FWULDP and EULP are equivalent.*

We now present a sufficient condition under which a family of random variables satisfies an EULP and therefore also an FWULDP. To this end, for any $N > 0$ and $T > 0$, let $\mathcal{P}^N_2$ be the collection of $\mathcal{F}_t$-adapted processes such that $\mathbb{P}\big(|\psi|_{L^2([0,T]\times\mathbb{R})} \leq N\big) = 1$.

Assumption 3.4. Assume that there exists a family of measurable maps $\mathscr{H}^\varepsilon : \mathcal{E}_0 \times C\big([0,T]\times\mathbb{R};\mathbb{R}\big) \to \mathcal{E}$ indexed by $\varepsilon \in (0,1]$, $\xi \in \mathcal{E}_0$ and $\psi \in \mathcal{P}^N_2$. Let W be a Brownian sheet and $Y^{\varepsilon,\psi}_\xi := \mathscr{H}^\varepsilon\big(\sqrt{\varepsilon}W + \int_0^\cdot\int_0^\cdot \psi(s)ds\big)$. Let $\mathscr{H}^0$ denote the limiting case of $\mathscr{H}^\varepsilon$ as $\varepsilon \to 0$. Further, Let $\mathscr{J}$ be a collection of subsets of $\mathcal{E}_0$ and assume that for any $\delta > 0$, $J \in \mathscr{J}$ and $N > 0$

$$\lim_{\varepsilon\to 0}\sup_{\xi\in J}\sup_{\psi\in\mathcal{P}^N_2}\mathbb{P}\left(\theta\left(\mathscr{H}^\varepsilon\left(\xi,\sqrt{\varepsilon}W + \int_0^\cdot\int_0^\cdot\psi(s)ds\right),\mathscr{H}^0\left(\xi,\int_0^\cdot\int_0^\cdot\psi(s)ds\right)\right) > \delta\right) = 0. \tag{3.4}$$

Theorem 3.5 (Salins, 2019, Theorem 2.13)**.** *If Assumption 3.4 holds then the family $Y^\varepsilon_\xi := \mathscr{H}^\varepsilon(\xi,\sqrt{\varepsilon}W)$, satisfies an EULP uniformly over $\mathscr{J}$, with speed $\gamma(\varepsilon) = \varepsilon$ and rate function $I_\xi : \mathcal{E} \to \mathbb{R}$*

$$I_\xi(\varphi) = \inf\left\{\frac{1}{2}\int_0^T |\psi(s)|_{L^2([0,T]\times\mathbb{R})}ds : \varphi := \mathscr{H}^0\left(\xi,\int_0^\cdot\int_0^\cdot\psi(s)ds\right)\right\}, \tag{3.5}$$

where the infimum is taken over all $\psi \in L^2([0,T]\times\mathbb{R})$ such that

$$\begin{aligned}\varphi(t,x) = &\int_0^1 G_t(x,y)\xi(y)dy - \int_0^t\int_0^1 \partial_y G_{t-s}(x,y)g(s,y,\varphi(s,y))dyds\\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)f(s,y,\varphi(s,y))dyds\\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(s,y,\varphi(s,y))\psi(s,y)dyds.\end{aligned} \tag{3.6}$$

We use the convention that $\inf(\emptyset) = +\infty$.

4. Controlled process and skeleton equation

In this section we introduce the equation under the change of measure, which we refer to as the controlled process and denote by $v_\xi^{\varepsilon,\psi}$, as well as the deterministic controlled equation in the absence of noise, which we refer to as the so-called skeleton equation, and further denote by $v_\xi^{0,\psi}$. Due to the fact that the unique mild solution to Eq. (1.1) is probabilistically strong, we have that for any $\varepsilon \in (0,1]$ and initial condition $\xi \in C_{tem}$, there exists a measurable map $\mathscr{F}^\varepsilon : C_{tem} \times C([0,T] \times \mathbb{R};\mathbb{R}) \to C\big([0,T];C_{tem})\big)$, such that $u_\xi^\varepsilon := \mathscr{F}^\varepsilon(\xi, \sqrt{\varepsilon}W)$. Next, for any $N > 0$ and $T > 0$, let the space of admissible controls denoted by $\mathcal{P}_2^N$ be the collection of $\mathcal{F}_t$-adapted processes such that $\mathbb{P}(|\psi|_{L^2([0,T]\times\mathbb{R})} \le N) = 1$. Further, let $v_\xi^{\varepsilon,\psi} := \mathscr{F}^\varepsilon\big(\xi, \sqrt{\varepsilon}W + \int_0^\cdot \int_0^\cdot \psi(s)ds\big)$. Note that $v_\xi^{\varepsilon,\psi}$ solves

$$\begin{aligned} v_\xi^{\varepsilon,\psi}(t,x) = G_t\xi(x) &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) b(s,y,v_\xi^{\varepsilon,\psi}(s,y))dyds \\ &+ \sqrt{\varepsilon}\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds) \\ &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))\psi(s,y)dyds. \end{aligned} \tag{4.1}$$

The deterministic controlled equation in the absence of noise, which we refer to as the so-called skeleton equation is $v_\xi^{0,\psi}$. Finally, for $\varphi \in C\big([0,T];C_{tem})\big)$ we define the following action functional or rate function

$$I_\xi(\varphi) := \frac{1}{2}\Big\{\inf_{\psi} \int_0^T \int_{\mathbb{R}} \psi^2(s,y)dyds\Big\}, \tag{4.2}$$

where the infimum is taken over all $\psi \in L^2([0,T]\times\mathbb{R})$ such that

$$\begin{aligned} \varphi(t,x) = G_t\xi(x) &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) b(s,y,\varphi(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,\varphi(s,y))\psi(s,y)dyds, \end{aligned} \tag{4.3}$$

with convention that $\inf(\emptyset) = +\infty$. Finally, we assert that the controlled process and the skeleton equation are well-posed. The existence and uniqueness of the former follows by Theorem 3.2 in Foondun and Setayeshgar (2017), while that of the latter follows by Proposition 3.2. in Li et al. (2024).

5. Uniform estimate on controlled process

In this section we assert and prove a uniform estimate on the controlled process, which will be used in proving the main theorem.

Proposition 5.1 (A priori estimate)**.** *The random field* $\sup_{t\in[0,T],x\in\mathbb{R}} \Big(|v_\xi^{\varepsilon,\psi}(t)|e^{-\lambda|x|e^{\beta t}}\Big)$, *is bounded in probability, i.e., for any given* $T > 0, N > 0, \lambda > 0$ *and* $R > 0$

$$\lim_{\rho\to\infty} \sup_{\varepsilon\in(0,1]} \sup_{|\xi|_{(-\lambda)}\le R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{P}\left(\sup_{t\in[0,T],x\in\mathbb{R}} \Big(|v_\xi^{\varepsilon,\psi}(t)|e^{-\lambda|x|e^{\beta t}}\Big) > \rho \right) = 0. \tag{5.1}$$

Proof: Throughout this proof, C, denotes a free, positive constant, which is independent of ψ, ξ and ε but may depend upon other parameters.

Note that, Fact (F1) implies that there exists a positive constant λ_T such that for all $\lambda \in (0, \lambda_T]$ we have $T \leq T^*(c_1, \lambda)$, where $\beta(c_1, \lambda)$ is written as β for simplicity. This will allow us to use Fact (F2).

In what follows we will prove (5.1) for any $\lambda \in (0, \lambda_T]$, where λ_T is arbitrary. Arbitrariness of λ_T will then yield the same result for all $\lambda > 0$, as desired.

To this end, fix $\lambda \in (0, \lambda_T]$. Further, let $M > 0$ and $\rho \in (0, e^{-1})$. Define the stopping times

$$\begin{aligned}
\tau_{M,\varepsilon} &:= \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\} \\
&\qquad \wedge \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\}, \\
\tau_{\rho,\varepsilon} &:= \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq \rho\right\}, \\
\tau_M^{\rho,\varepsilon} &:= \tau_{M,\varepsilon} \wedge \tau_{\rho,\varepsilon} \wedge T.
\end{aligned} \tag{5.2}$$

with the convention that $\inf \emptyset = +\infty$.

Recall

$$\begin{aligned}
v_\xi^{\varepsilon,\psi}(t,x) &= G_t\xi(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) b(s,y,v_\xi^{\varepsilon,\psi}(s,y))dyds \\
&\quad + \sqrt{\varepsilon}\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds) \\
&\quad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))\psi(s,y)dyds.
\end{aligned}$$

Fix $\lambda \in (0, \lambda_T]$. Further, let

$$V^\varepsilon(z) := \sup_{t\leq z, x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right),\ z \leq T.$$

Therefore

$$\begin{aligned}
V^\varepsilon(z) &\leq \sup_{t\leq z, x\in\mathbb{R}}\left(\left|\int_{\mathbb{R}} G_t\xi(y)dy\right|e^{-\lambda|x|e^{\beta t}}\right) \\
&\quad + \sup_{t\leq z, x\in\mathbb{R}}\left(\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)b(s,y,v_\xi^{\varepsilon,\psi}(s,y))dyds\right|e^{-\lambda|x|e^{\beta t}}\right) \\
&\quad + \sup_{t\leq z, x\in\mathbb{R}}\sqrt{\varepsilon}\left(\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right|e^{-\lambda|x|e^{\beta t}}\right) \\
&\quad + \sup_{t\leq z, x\in\mathbb{R}}\left(\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))\psi(s,y)dyds\right|e^{-\lambda|x|e^{\beta t}}\right) \\
&:= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon.
\end{aligned} \tag{5.3}$$

First consider J_1^ε

$$
\begin{aligned}
J_1^\varepsilon &\le \sup_{t\le z, x\in\mathbb{R}} \left(\left| \int_{\mathbb{R}} G_t(x,y)\xi(y)dy \right| e^{-\lambda|x|} \right) \\
&\le \sup_{y\in\mathbb{R}} \left(|\xi(y)|e^{-\lambda|y|} \right) \times \sup_{t\le z, x\in\mathbb{R}} \left(\int_{\mathbb{R}} G_t(x,y)e^{\lambda|y|}dy e^{-\lambda|x|} \right) \\
&\le 2e^{\frac{\lambda^2 T}{2}} \sup_{y\in\mathbb{R}} \left(|\xi(y)|e^{-\lambda|y|} \right) = 2e^{\frac{\lambda^2 T}{2}} |\xi|_{(-\lambda)},
\end{aligned}
\tag{5.4}
$$

where estimate (E1) for the heat kernel has been used. Next, consider J_4^ε

$$
\begin{aligned}
J_4^\varepsilon &\le \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x,y)dyds \right)^{\frac{1}{2}} \times \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} \sigma^2(s,y,v_\xi^{\varepsilon,\psi}(s,y))\psi^2(s,y) \right)^{\frac{1}{2}} \\
&\le \sqrt{2}\pi^{-\frac{1}{4}} N T^{\frac{1}{4}} K_\sigma,
\end{aligned}
\tag{5.5}
$$

where Hölder's inequality, boundedness of σ, boundedness of controls in $L^2\big([0,T]\times\mathbb{R}\big)$ and estimate (E4) have been used.

Now consider J_2^ε

$$
\begin{aligned}
J_2^\varepsilon &\le \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) \Big(k_1 \big|v_\xi^{\varepsilon,\psi}(s,y)\big| \log_+ \big|v_\xi^{\varepsilon,\psi}(s,y)\big| + k_2 \Big) dsdy e^{-\lambda|x|e^{\beta t}} \right) \\
&\le \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y) e^{\lambda|y|e^{\beta s}} \Big(k_1 \big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} \right. \\
&\quad \left. \times \log_+ \left(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} e^{\lambda|y|e^{\beta s}} \right) + k_2 e^{-\lambda|y|e^{\beta s}} \Big) dsdy e^{-\lambda|x|e^{\beta t}} \right) \\
&\le k_2 T + k_1 \sup_{t\in[0,z], x\in\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\ e^{\lambda|y|e^{\beta s}} \Big(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} \right. \\
&\quad \left. \times \log_+ \left(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} e^{\lambda|y|e^{\beta s}} \right) \Big) dsdy e^{-\lambda|x|e^{\beta t}} \right) \\
&\le k_2 T + k_1 \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \sup_{y\in\mathbb{R}} \left(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} \log_+ \left(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} \right) \right) \right. \\
&\quad \left. \times \int_{\mathbb{R}} G_{t-s}(x,y) e^{\lambda|y|e^{\beta s}} dyds e^{-\lambda|x|e^{\beta t}} \right) \\
&\quad + k_1 \sup_{t\le z, x\in\mathbb{R}} \left(\int_0^t \sup_{y\in\mathbb{R}} \left(\big|v_\xi^{\varepsilon,\psi}(s,y)\big| e^{-\lambda|y|e^{\beta s}} \right) \int_{\mathbb{R}} G_{t-s}(x,y) e^{\lambda|y|e^{\beta s}} \lambda|y|e^{\beta s} dyds e^{-\lambda|x|e^{\beta t}} \right) \\
&\le k_2 T + A + B,
\end{aligned}
\tag{5.6}
$$

where Assumption (A1) and estimate (E1) have been exploited. Note that using estimate (E1) again, for any $0 < s \le t$ we have

$$
\int_{\mathbb{R}} G_{t-s}(x,y) e^{\lambda|y|e^{\beta s}} dy \le e^{\frac{\lambda^2(t-s)e^{2\beta s}}{2}} e^{\lambda|x|e^{\beta s}} \le e^{\frac{\lambda^2}{4\beta} e^{2\beta t - 1}} e^{\lambda|x|e^{\beta t}}.
$$

Therefore

$$
\begin{aligned}
A &\leq 2k_1 e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}} \sup_{t\leq z}\left(\int_0^z \sup_{y\in\mathbb{R}}\left(\left|v_\xi^{\varepsilon,\psi}(s,y)\right|e^{-\lambda|y|e^{\beta s}}\log_+\left(\left|v_\xi^{\varepsilon,\psi}(s,y)\right|e^{-\lambda|y|e^{\beta s}}\right)ds\right)\right. \\
&\leq 2k_1 e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}\int_0^z V^\varepsilon(s)\log_+ V^\varepsilon(s)ds.
\end{aligned}
$$

We now attend to bounding the B term. Note that by estimate (E3) for the heat kernel, for any $0 < s \leq t$ we have

$$
\int_{\mathbb{R}} G_{t-s}(x,y)e^{\lambda|y|e^{\beta s}}\lambda|y|e^{\beta s}dy \leq e^{\frac{\lambda^2(t-s)e^{2\beta s}}{2}}e^{\lambda|x|e^{\beta s}}\lambda|x|e^{\beta s} + C_{\lambda,\beta,t}e^{\lambda|x|e^{\beta s}}. \tag{5.7}
$$

Consequently

$$
\begin{aligned}
B &\leq k_1 \sup_{t\leq z, x\in\mathbb{R}}\left[\int_0^t\left(\sup_{y\in\mathbb{R}}\left|v_\xi^{\varepsilon,\psi}(s,y)\right|e^{-\lambda|y|e^{\beta s}}\right)\right. \\
&\quad \left.\times\left(e^{\frac{\lambda^2(t-s)e^{2\beta s}}{2}}e^{\lambda|x|e^{\beta s}}\lambda|x|e^{\beta s} + C_{\lambda,\beta,t}e^{\lambda|x|e^{\beta s}}\right)ds e^{-\lambda|x|e^{\beta t}}\right] \\
&\leq k_1\sup_{t\leq z}\left[C_{\lambda,\beta,t}\int_0^t\left(\sup_{z\leq s, y\in\mathbb{R}}\left|v_\xi^{\varepsilon,\psi}(z,y)\right|e^{-\lambda|y|e^{\beta z}}\right)ds\right] \\
&\quad + k_1\sup_{t\leq z, x\in\mathbb{R}}\left[\frac{1}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta t-1}}\int_0^t (e^{\lambda|x|e^{\beta s}})'ds\ e^{-\lambda|x|e^{\beta t}}\times\left(\sup_{s\leq t, y\in\mathbb{R}}\left(\left|v_\xi^{\varepsilon,\psi}(s,y)\right|e^{-\lambda|y|e^{\beta s}}\right)\right)\right] \\
&\leq C_{k_1,\lambda,\beta,T}\int_0^z V^\varepsilon(s)ds + \frac{k_1}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}V^\varepsilon(z).
\end{aligned} \tag{5.8}
$$

Therefore

$$
J_2^\varepsilon \leq k_2 T + 2k_1 e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}\int_0^z V^\varepsilon(s)\log_+ V^\varepsilon(s)ds + C_{k_1,\lambda,\beta,T}\int_0^z V^\varepsilon(s)ds + \frac{k_1}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}V^\varepsilon(z).
$$

Note that for any $\lambda \in (0,\lambda_T]$, we have $T \leq T^*(k_1\vee k_2,\lambda)$, and

$$
\frac{k_1\vee k_2}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}} \leq \frac{1}{2} \iff T \leq T^*(k_1\vee k_2,\lambda).
$$

Thus

$$
J_2^\varepsilon \leq k_2 T + 2k_1 e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}\int_0^z V^\varepsilon(s)\log_+ V^\varepsilon(s)ds + C_{k_1,\lambda,\beta,T}\int_0^z V^\varepsilon(s)ds + \frac{1}{2}V^\varepsilon(z). \tag{5.9}
$$

Finally, in view of inequality (5.3), and combining (5.4), (5.9) and (5.5) we have

$$
\begin{aligned}
V^\varepsilon(z) &\leq 2e^{\frac{\lambda^2 T}{2}}|\xi|_{(-\lambda)} + k_2 T + 2k_1 e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}\int_0^z V^\varepsilon(s)\log_+ V^\varepsilon(s)ds + C_{k_1,\lambda,\beta,T}\int_0^z V^\varepsilon(s)ds \\
&\quad + \frac{1}{2}V^\varepsilon(z) + \sup_{t\leq z, x\in\mathbb{R}}\sqrt{\varepsilon}\left(\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right|e^{-\lambda|x|e^{\beta t}}\right) \\
&\quad + \sqrt{2}\pi^{-\frac{1}{4}}NT^{\frac{1}{4}}K_\sigma,\ z\leq T.
\end{aligned}
$$

By the Grönwall-type inequality (Webb, 2003, Theorem 3.1) we have

$$V^\varepsilon(T) \le \Bigg(C_{k_1,k_2,\lambda,\beta,R,K_\sigma,N,T} + \sqrt{\varepsilon} \sup_{t\le T, x\in\mathbb{R}} \left(\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right| e^{-\lambda|x|e^{\beta t}}\right)\Bigg)^{C_{k_1,\lambda,\beta,T}}. \quad (5.10)$$

Note that since $C_{k_1,\lambda,\beta,T} \ge 1$, we may apply the inequality $(p+q)^\alpha \le 2^{\alpha-1}(p^\alpha + q^\alpha)$, with $\alpha \ge 1$ to the RHS of (5.10) to obtain

$$\mathbb{E}\big[V^\varepsilon(T)\big] \le C_{k_1,k_2,\lambda,\beta,R,K_\sigma,N,T} + \varepsilon^{\frac{C_{k_1,\lambda,\beta,T}}{2}} \mathbb{E}\left[\sup_{t\le T,x\in\mathbb{R}} \left(\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right| e^{-\lambda|x|e^{\beta t}}\right)^{C_{k_1,\lambda,\beta,T}}\right].$$

By Proposition 4.2. in Shang and Zhang (2023), for any $\alpha > 0$

$$\mathbb{E}\left[\sup_{t\le T,x\in\mathbb{R}} \left(\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right| e^{-\lambda|x|e^{\beta t}}\right)^{\alpha}\right] \le C_{\lambda,\beta,K_\sigma,T,\alpha}.$$

Note that here $\alpha := C_{k_1,\lambda,\beta,T}$, and since $0<\varepsilon\le 1$ we conclude

$$\mathbb{E}\big[V^\varepsilon(T)\big] \le C_{k_1,k_2,\lambda,\beta,R,K_\sigma,N,T}. \quad (5.11)$$

By Chebyshev's inequality, and that the constant appearing on the RHS of (5.11) is independent of ε, ξ, and ψ, we may conclude that for any given $T>0, N>0, \lambda>0$ and $R>0$

$$\lim_{\rho\to\infty} \sup_{\varepsilon\in(0,1]} \sup_{|\xi|_{(-\lambda)}\le R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{P}\left(\sup_{t\le T,x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t)|e^{-\lambda|x|e^{\beta t}}\right) > \rho\right) = 0. \quad (5.12)$$

Note that (5.12) holds true for any $\lambda\in[0,\lambda_T]$, where λ_T is arbitrary. Arbitrariness of λ_T, will then result in (5.12) being true for any $\lambda>0$. This completes the proof.

□

6. Main Theorem

In this section we restate the main theorem of this paper.

Theorem 6.1 (Uniform Large Deviation Principle)**.** *The processes $u_\xi^\varepsilon := \{u_\xi^\varepsilon(t,x) : t\in[0,T],\ x\in\mathbb{R}\}$, where $\varepsilon\in(0,1]$, and $\xi\in C_{tem}$, satisfy a ULDP that is uniform over bounded subset of $|\xi|_{(-\lambda)}\le R$ and the ULDP is valid in the norm*

$$\sup_{t\in[0,T],x\in\mathbb{R}}\left(|u_\xi^\varepsilon(t,x)|e^{-\lambda|x|e^{\beta t}}\right), \quad (6.1)$$

with rate function I_ξ given by (4.2).

7. Proof of Theorem 6.1

In light of (Salins, 2019, Theorem 2.13) and (Salins, 2019, Theorem 2.10) , it suffices to show that Assumption 3.4 is satisfied. The proposition below, demonstrates the foregoing.

Proposition 7.1 (Uniform convergence in probability)**.** *For any $\delta > 0$, $R > 0$, $N > 0$, $\lambda > 0$, and $T > 0$*

$$\lim_{\varepsilon\to 0} \sup_{|\xi|_{(-\lambda)}\leq R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{P}\left(\sup_{t\leq T, x\in\mathbb{R}} \left|v_\xi^{\varepsilon,\psi} - v_\xi^{0,\psi}\right| e^{-\lambda|x|e^{\beta t}} > \delta\right) = 0, \tag{7.1}$$

Proof: Throughout this proof C denotes a free, positive constant, which is independent of ψ, ξ and ε but may depend upon other parameters.

In what follows, we will prove (7.1) for any $\lambda \in (0, \lambda_T]$, where λ_T is arbitrary. Arbitrariness of λ_T will then yield the same result for all $\lambda > 0$ as desired.

To this end, fix $\lambda \in (0, \lambda_T]$. Further, let $M > 0$ and $\rho \in (0, e^{-1})$. Define the stopping times

$$\begin{aligned}
\tau_{M,\varepsilon} &:= \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\} \\
&\qquad \wedge \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\}, \\
\tau_{\rho,\varepsilon} &:= \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq \rho\right\}, \\
\tau_M^{\rho,\varepsilon} &:= \tau_{M,\varepsilon} \wedge \tau_{\rho,\varepsilon} \wedge T.
\end{aligned} \tag{7.2}$$

with the convention that $\inf\emptyset = +\infty$.

Note that, Proposition 1 holds true for $\sup_{t\leq T, x\in\mathbb{R}}\left(|v_\xi^{0,\psi}(t)|e^{-\lambda|x|e^{\beta t}}\right)$ in an analogous manner. Due to the foregoing and Proposition 1, we have

$$\lim_{M\to\infty} \sup_{\varepsilon\in(0,1]} \sup_{|\xi|_{(-\lambda)}\leq R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{P}(\tau_{M,\varepsilon} < T) = 0. \tag{7.3}$$

Recall that, for any $\psi \in \mathcal{P}_2^N$, $|\xi|_{(-\lambda)} \leq R$, and $\varepsilon \in (0,1]$ we have

$$\begin{aligned}
v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\Big(b(s,y,v_\xi^{\varepsilon,\psi}(s,y)) - b(s,y,v_\xi^{0,\psi}(s,y))\Big)dyds \\
&\quad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\Big(\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y)) - \sigma(s,y,v_\xi^{0,\psi}(s,y))\Big)\psi(s,y)dyds \\
&\quad + \sqrt{\varepsilon}\int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds).
\end{aligned} \tag{7.4}$$

Let

$$R^{\varepsilon,\psi}(z) := \sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}, x\in\mathbb{R}}\left(|v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\right),\ z \leq T.$$

Therefore in view (7.4), we have

$$
\begin{aligned}
R^{\varepsilon,\psi}(z) \le & \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t G_{t-s}(x,y)\big|b(s,y,v_\xi^{\varepsilon,\psi}(s,y)) - b(s,y,v_\xi^{0,\psi}(s,y))\big|dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
& + \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\Big(\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y)) - \sigma(s,y,v_\xi^{0,\psi}(s,y))\Big)\right.\right.\\
& \quad \left.\left.\times\, \psi(s,y)dyds\right| e^{-\lambda|x|e^{\beta t}}\right\}\\
& + \sqrt{\varepsilon}\sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds)\right|e^{-\lambda|x|e^{\beta t}}\right\}\\
& := I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.
\end{aligned}
\tag{7.5}
$$

We now estimate the terms $I_1^\varepsilon, I_2^\varepsilon, I_3^\varepsilon$ separately. First consider I_1^ε. Using Assumption (A2), we obtain

$$
\begin{aligned}
I_1^\varepsilon \le & k_3 \sup_{t\le z\wedge\tau^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|\right.\\
& \left.\times \log_+ \frac{1}{\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|} dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
& + k_4 \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\log_+\left(|v_\xi^{\varepsilon,\psi}(s,y)|\vee|v_\xi^{0,\psi}(s,y)|\right)\right.\\
& \left.\times \big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big| dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
& + k_5 \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
& := I_{11}^\varepsilon + I_{12}^\varepsilon + I_{13}^\varepsilon.
\end{aligned}
\tag{7.6}
$$

First, we estimate the term I_{11}^ε. By the fact that $x\to x\log_+\frac{1}{x}$ is increasing and concave on $(0,e^{-1})$ and (E1), we get

$$
\begin{aligned}
I_{11}^\varepsilon := & k_3 \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)e^{\lambda|y|e^{\beta s}}\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|)e^{-\lambda|y|e^{\beta s}}\right.\\
& \left.\times \log_+ \frac{1}{\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|e^{-\lambda|y|e^{\beta s}}e^{\lambda|y|e^{\beta s}}} dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
\le & k_3 \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)e^{\lambda|y|e^{\beta s}}\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|)e^{-\lambda|y|e^{\beta s}}\right.\\
& \left.\times \log_+ \frac{1}{\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|e^{-\lambda|y|e^{\beta s}}} dyds e^{-\lambda|x|e^{\beta t}}\right\}\\
\le & k_3 \sup_{t\le z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\int_0^t \sup_{y\in\mathbb{R}}\left(\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|e^{-\lambda|y|e^{\beta s}}\right.\right.\\
& \left.\left.\times \log_+ \frac{1}{\big|v_\xi^{\varepsilon,\psi}(s,y) - v_\xi^{0,\psi}(s,y)\big|e^{-\lambda|y|e^{\beta s}}}\right)\int_{\mathbb{R}} G_{t-s}(x,y)e^{\lambda|y|\beta s} dyds e^{-\lambda|x|e^{\beta t}}\right\}
\end{aligned}
$$

$$\leq 2k_3 e^{\frac{\lambda^2}{4\beta}e^{2\beta z}-1}\int_0^z \sup_{r\leq s\wedge\tau_M^{\rho,\varepsilon},y\in\mathbb{R}}\Big(\big|v_\xi^{\varepsilon,\psi}(r,y)-v_\xi^{0,\psi}(r,y)\big|\Big)e^{-\lambda|y|e^{\beta r}}$$

$$\times \log_+ \frac{1}{\sup_{r\leq s\wedge\tau_M^{\rho,\varepsilon},y\in\mathbb{R}}\big(\big|v_\xi^{\varepsilon,\psi}(r,y)-v_\xi^{0,\psi}(r,y)\big|\big)e^{-\lambda|y|e^{\beta r}}}ds\Bigg\}$$

$$\leq 2k_3 e^{\frac{\lambda^2}{4\beta}e^{2\beta z}-1}\int_0^z R^{\varepsilon,\psi}(s)\log_+\frac{1}{R^{\varepsilon,\psi}(s)}ds. \tag{7.7}$$

By a similar argument, we obtain

$$I_{13}^{\varepsilon}\leq 2k_5 e^{\frac{\lambda^2}{4\beta}e^{2\beta T}-1}\int_0^z R^{\varepsilon,\psi}(s)ds. \tag{7.8}$$

Next, consider I_{12}^{ϵ}. Using estimate (E3), (5.7), and that $\log_+(ab)\leq\log_+(a)+\log_+(b)$, for any $a,b>0$ we have

$$I_{12}^{\varepsilon}\leq k_4\cdot\sup_{\substack{t\leq z\wedge\tau_M^{\rho,\varepsilon},\\ x\in\mathbb{R}}}\Bigg\{\int_0^t\int_{\mathbb{R}}G_{t-s}(x,y)e^{\lambda|x|e^{\beta s}}\Big[\log_+\Big(|v_\xi^{\varepsilon,\psi}(s,y)e^{-\lambda|y|e^{\beta s}}|\vee|v_\xi^{0,\psi}(s,y)e^{-\lambda|y|e^{\beta s}}|\Big)+\lambda|y|e^{\beta s}\Big]$$

$$\times\big|v_\xi^{\varepsilon,\psi}(s,y)-v_\xi^{0,\psi}(s,y)\big|e^{-\lambda|y|e^{\beta s}}dyds e^{-\lambda|x|e^{\beta t}}\Bigg\}$$

$$\leq k_4\cdot\sup_{\substack{t\leq z\wedge\tau_M^{\rho,\varepsilon},\\ x\in\mathbb{R}}}\Bigg\{\int_0^t\sup_{y\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(s,y)-v_\xi^{0,\psi}(s,y)|e^{-\lambda|y|e^{\beta s}}\Big)\int_{\mathbb{R}}G_{t-s}(x,y)\lambda|y|e^{\beta s}e^{-\lambda|y|e^{\beta s}}dyds e^{-\lambda|x|e^{\beta t}}\Bigg\}$$

$$+k_4(\log_+ M)\cdot\sup_{\substack{t\leq z\wedge\tau_M^{\rho,\varepsilon},\\ x\in\mathbb{R}}}\Bigg\{\int_0^t\sup_{y\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(s,y)-v_\xi^{0,\psi}(s,y)|e^{-\lambda|y|e^{\beta s}}\Big)\cdot\int_{\mathbb{R}}G_{t-s}(x,y)\lambda|y|e^{\beta s}dyds e^{-\lambda|x|e^{\beta t}}\Bigg\}$$

$$\leq\frac{k_4}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}R^{\varepsilon,\psi}(z)+C_{k_4,\lambda,\beta,T,M}\int_0^z R^{\varepsilon,\psi}(s)ds, \tag{7.9}$$

where the same argument as that used in (5.8) has been employed. In view of (7.6) and combining (7.7), (7.8) and (7.9), we obtain

$$I_1^{\varepsilon}\leq 2k_3 e^{\frac{\lambda^2}{4\beta}e^{2\beta z}-1}\int_0^z R^{\varepsilon,\psi}(s)\log_+\frac{1}{R^{\varepsilon,\psi}(s)}ds+\frac{k_4}{\beta}e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}}R^{\varepsilon,\psi}(z)$$

$$+C_{k_4,\lambda,\beta,T,M}\int_0^z R^{\varepsilon,\psi}(s)ds+2k_5 e^{\frac{\lambda^2}{4\beta}e^{2\beta T}-1}\int_0^z R^{\varepsilon,\psi}(s)ds. \tag{7.10}$$

Next consider I_2^{ε}

$$I_2^{\varepsilon}:=\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\Bigg\{\Bigg|\int_0^t\int_{\mathbb{R}}G_{t-s}(x,y)\Big(\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\Big)\psi(s,y)dyds\Bigg|e^{-\lambda|x|e^{\beta t}}\Bigg\}$$

$$\leq\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\Bigg\{\Bigg[\int_0^t\int_{\mathbb{R}}\Big(G_{t-s}(x,y)e^{\lambda|y|e^{\beta s}}e^{-\lambda|x|e^{\beta t}}\Big)^2dyds\Bigg]^{\frac12}$$

$$\times\Bigg[\int_0^t\int_{\mathbb{R}}\Big(\Big|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\Big|e^{-\lambda|y|e^{\beta s}}|\psi(s,y)|\Big)^2dyds\Bigg]^{\frac12}\Bigg\}$$

$$\leq\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\Bigg\{\Bigg[\int_0^t\int_{\mathbb{R}}\Big(G_{t-s}(x,y)e^{\lambda|y|e^{\beta s}}e^{-\lambda|x|e^{\beta t}}\Big)^2dyds\Bigg]^{\frac12}$$

$$\times\Bigg[\int_0^t\int_{\mathbb{R}}\Big(\Big|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\Big|e^{-\lambda|y|e^{\beta s}}|\psi(s,y)|^{\frac{2}{q}}\Big)^q dyds\Bigg]^{\frac1q}$$

$$\times\left[\int_0^t\int_{\mathbb{R}}\left(|\psi(s,y)|^{1-\frac{2}{q}}\right)^{\frac{2q}{q-2}}dyds\right]^{\frac{q-2}{2q}}\Bigg\}$$

$$\leq \sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{C_{T,\lambda,\beta,N,q}\left[\int_0^t\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|^q e^{-q\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right]^{\frac{1}{q}}\right\},$$

where Hölder's inequality with Hölder conjugates $q/2$ and $q/(q-2)$, with $q>2$, estimate (E2) and boundedness of controls have been used. Next, for any $0<p\leq 2$, raising both sides of the above inequality to the power of p, we obtain that for any $\zeta>0$

$$\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\left|\int_0^t\int_{\mathbb{R}}G_{t-s}(x,y)\Big(\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\Big)\psi(s,y)dyds\right|^p e^{-p\lambda|x|e^{\beta t}}\right\}$$
$$\leq C_{T,\lambda,\beta,N,p,q}\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\left[\int_0^t\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|^q e^{-q\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right]^{\frac{p}{q}}\right\}$$
$$\leq C_{T,\lambda,\beta,N,p,q}\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\sup_{\tau\leq t,y\in\mathbb{R}}\left(\left|\sigma(\tau,y,v_\xi^{\varepsilon,\psi}(\tau,y))-\sigma(\tau,y,v_\xi^{0,\psi}(\tau,y))\right|e^{-\lambda|y|e^{\beta\tau}}\right)^{\frac{(q-p)p}{q}}\right.$$
$$\left.\times\left[\int_0^t\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|^p e^{-p\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right]^{\frac{p}{q}}\right\}$$
$$\leq \sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\zeta\sup_{\tau\leq t,y\in\mathbb{R}}\left(\left|\sigma(\tau,y,v_\xi^{\varepsilon,\psi}(\tau,y))-\sigma(\tau,y,v_\xi^{0,\psi}(\tau,y))\right|e^{-\lambda|y|e^{\beta\tau}}\right)^p\right.$$
$$\left.+C_{T,\lambda,\beta,N,p,q,\zeta}\int_0^T\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|^p e^{-p\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right\},$$

where in the last step Young's inequality, with conjugate exponents $q/(q-p)$ and q/p, has been used. Therefore

$$\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\left\{\left|\int_0^t\int_{\mathbb{R}}G_{t-s}(x,y)\Big(\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\Big)\psi(s,y)dyds\right|^p e^{-p\lambda|x|e^{\beta t}}\right\}$$
$$\leq \sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\zeta\sup_{\tau\leq t,y\in\mathbb{R}}\left(\left|\sigma(\tau,y,v_\xi^{\varepsilon,\psi}(\tau,y))-\sigma(\tau,y,v_\xi^{0,\psi}(\tau,y))\right|e^{-\lambda|y|e^{\beta\tau}}\right)^p\right.$$
$$\left.+C_{T,\lambda,\beta,N,p,q,\zeta}\int_0^T\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|^p e^{-p\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right\}.$$

Taking $p=1$ in the above inequality, by Assumption (A3), and boundedness of controls we get

$$I_2^\varepsilon\leq\zeta\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\sup_{\tau\leq t,y\in\mathbb{R}}\left(\left|\sigma(\tau,y,v_\xi^{\varepsilon,\psi}(\tau,y))-\sigma(\tau,y,v_\xi^{0,\psi}(\tau,y))\right|e^{-\lambda|y|e^{\beta\tau}}\right)\right.$$
$$\left.+C_{T,\lambda,\beta,N,\zeta}\int_0^t\int_{\mathbb{R}}\left|\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))-\sigma(s,y,v_\xi^{0,\psi}(s,y))\right|e^{-\lambda|y|e^{\beta s}}|\psi(s,y)|^2dyds\right\}$$
$$\leq\zeta L_\sigma\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon}}\left\{\sup_{\tau\leq t,y\in\mathbb{R}}\left(\left|v_\xi^{\varepsilon,\psi}(\tau,y)-v_\xi^{0,\psi}(\tau,y)\right|e^{-\lambda|y|e^{\beta\tau}}\right)\right.$$
$$\left.+C_{T,\lambda,\beta,N,L_\sigma,\zeta}\int_0^t\sup_{y\in\mathbb{R}}\left|v_\xi^{\varepsilon,\psi}(s,y)-v_\xi^{0,\psi}(s,y)\right|e^{-\lambda|y|e^{\beta s}}\int_{\mathbb{R}}|\psi(s,y)|^2dyds\right\}$$
$$\leq\zeta L_\sigma R^{\varepsilon,\psi}(z)+C_{T,\lambda,\beta,N,L_\sigma,\zeta}\int_0^z R^{\varepsilon,\psi}(s)\vartheta(s)ds, \tag{7.11}$$

where $\vartheta(s) := \int_{\mathbb{R}} |\psi(s,y)|^2 dy$. Next, let

$$U^\varepsilon(T) := \sup_{t\in[0,T],x\in\mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(s,y,v_\xi^{\varepsilon,\psi}(s,y))W(dy,ds) \right| e^{-\lambda|x|e^{\beta t}} \right\}.$$

Then

$$I_3^\varepsilon \leq \sqrt{\varepsilon}U^\varepsilon(T). \tag{7.12}$$

In view of (7.5) and combining (7.10), (7.11), and (7.12), we obtain for all $z \leq T$

$$\begin{aligned}
R^{\varepsilon,\psi}(z) &\leq 2k_3 e^{\frac{\lambda^2}{4\beta}e^{2\beta z}-1} \int_0^z R^{\varepsilon,\psi}(s)\log_+ \frac{1}{R^{\varepsilon,\psi}(s)} ds + \frac{k_4}{\beta} e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}} R^{\varepsilon,\psi}(z) \\
&\quad + C_{k_4,\lambda,\beta,T,M} \int_0^z R^{\varepsilon,\psi}(s)ds + 2k_5 e^{\frac{\lambda^2}{4\beta}e^{2\beta z}-1} \int_0^z R^{\varepsilon,\psi}(s)ds + \zeta L_\sigma R^{\varepsilon,\psi}(z) \\
&\quad + C_{T,\lambda,\beta,N,L_\sigma,\zeta} \int_0^z R^{\varepsilon,\psi}(s)\vartheta(s)ds + \sqrt{\varepsilon}U^\varepsilon(T) \\
&\leq \frac{k_4}{2(k_1 \vee k_4)} R^{\varepsilon,\psi}(z) + C_{k_1,k_4,k_5,\lambda,M,T} \int_0^z R^{\varepsilon,\psi}(s)ds \\
&\quad + C_{k_1,k_4,k_3,\lambda,T} \int_0^z R^{\varepsilon,\psi}(s)\log_+ \frac{1}{R^{\varepsilon,\psi}(s)} ds + +\zeta L_\sigma R^{\varepsilon,\psi}(z) \\
&\quad + C_{k_1,k_4,T,\lambda,N,L_\sigma,\zeta} \int_0^z R^{\varepsilon,\psi}(s)\vartheta(s)ds,
\end{aligned} \tag{7.13}$$

where in the last step, we have used that

$$\frac{k_1 \vee k_4}{\beta} e^{\frac{\lambda^2}{4\beta}e^{2\beta T-1}} \leq \frac{1}{2} \iff T \leq T^*(k_1 \vee k_4, \lambda). \tag{7.14}$$

Note that this is because there exists $\lambda_T > 0$ such that $T \leq T^*(k_1 \vee k_4, \lambda)$ for all $\lambda \in (0, \lambda_T]$. In other words, Fact (F2) with $c_1 := k_1 \vee k_4$ has been employed, where $\beta(k_1 \vee k_4, \lambda)$ has been written as β for simplicity.

Next, we take $\zeta < \frac{1}{2L_\sigma}$ and apply the Grönwall-type inequality (Li et al., 2024, Lemma 6.2) to obtain

$$R^{\varepsilon,\psi}(T) \leq C_{k_1,k_4,k_3,k_5,\lambda,L_\sigma,N,M,T} \left[\sqrt{\varepsilon}U^\varepsilon(T) + (\sqrt{\varepsilon}U^\varepsilon(T))^{C_{k_1,k_4,k_3,\lambda,L_\sigma,T}} \right],$$

where $0 < C_{k_1,k_4,k_3,\lambda,L_\sigma,T} \leq 1$. Hence

$$\mathbb{E}\big(R^{\varepsilon,\psi}(T)\big) \leq C_{k_1,k_4,k_3,k_5,\lambda,L_\sigma,N,M,T} \left[\sqrt{\varepsilon}\mathbb{E}\big(U^\varepsilon(T)\big) + \mathbb{E}\left(\sqrt{\varepsilon}U^\varepsilon(T) \right)^{C_{k_1,k_4,k_3,\lambda,L_\sigma,T}} \right].$$

Since σ is bounded, (Li et al., 2024, Lemma 4.1) and (Shang and Zhang, 2023, Proposition 4.2) imply that, for any $q > 0$

$$\sup_{\varepsilon\in(0,1]} \sup_{|\xi|_{(-\lambda)}\leq R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{E}[\big(R^{\varepsilon,\psi}(T)\big)^q] \leq C_{\lambda,\beta,T,q}. \tag{7.15}$$

Hence

$$\lim_{\varepsilon\to 0} \sup_{|\xi|_{(-\lambda)}\leq R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{E} \sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}} \left(|v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}} \right) = 0. \tag{7.16}$$

Therefore by Chebyshev's inequality, for any $\delta > 0$

$$\lim_{\varepsilon\to 0} \sup_{|\xi|_{(-\lambda)}\leq R} \sup_{\psi\in\mathcal{P}_2^N} \mathbb{P}\left(\sup_{t\leq z\wedge\tau_M^{\rho,\varepsilon},x\in\mathbb{R}} \left(|v_\xi^{\varepsilon,\psi}(t,x) - v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}} \right) > \delta \right) = 0. \tag{7.17}$$

Note that

$$
\begin{aligned}
&\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\Bigg(\sup_{t\leq\tau_{\rho,\varepsilon}\wedge T,x\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(t,x)-v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\Big)>\delta\Bigg)\\
&\quad\leq\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\Bigg(\sup_{t\leq\tau_M^{\rho,\varepsilon},x\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(t,x)-v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\Big)>\delta\Bigg)\\
&\qquad+\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\Big(\tau_{M,\varepsilon}<T\Big).
\end{aligned}
\tag{7.18}
$$

Therefore, letting $\varepsilon\to 0$ and $M\to\infty$ in (7.18) and exploiting (7.17) and (7.3) we obtain

$$
\lim_{\varepsilon\to 0}\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\Bigg(\sup_{t\leq\tau_{\rho,\varepsilon}\wedge T,x\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(t,x)-v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\Big)>\delta\Bigg)=0. \tag{7.19}
$$

Now by choosing $\delta\leq\rho$ in (7.19), and owing to the definition of $\tau_{\rho,\varepsilon}$ we get

$$
\lim_{\varepsilon\to 0}\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}(\tau_{\rho,\varepsilon}<T)=0. \tag{7.20}
$$

Consequently, by (7.19) and (7.20), for any $\delta>0$, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\left(\sup_{t\leq T,x\in\mathbb{R}}\left|v_\xi^{\varepsilon,\psi}(t,x)-v_\xi^{0,\psi}(t,x)\right|e^{-\lambda|x|e^{\beta t}}>\delta\right)\\
&\quad\leq\lim_{\varepsilon\to 0}\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}\Bigg(\sup_{t\leq\tau_{\rho,\varepsilon}\wedge T,x\in\mathbb{R}}\Big(|v_\xi^{\varepsilon,\psi}(t,x)-v_\xi^{0,\psi}(t,x)|e^{-\lambda|x|e^{\beta t}}\Big)>\delta\Bigg)\\
&\qquad+\lim_{\varepsilon\to 0}\sup_{|\xi|_{(-\lambda)}\leq R}\sup_{\psi\in\mathcal{P}_2^N}\mathbb{P}(\tau_{\rho,\varepsilon}<T)=0.
\end{aligned}
\tag{7.21}
$$

Note that (7.21) holds true for any $\lambda\in[0,\lambda_T]$, where λ_T is arbitrary. Due to the arbitrariness of λ_T, the claim is true for any $\lambda>0$. The proof is now complete. ☐

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