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On Uniform Functions on Configuration Spaces of Large-Scale Interacting Systems

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Abstract. Stochastic large-scale interacting systems can be studied via the observables, i.e. functions on the underlying configuration space. In our previous article (Bannai et al., 2024), we introduced the concept of *uniform functions*, which is a suitable class of functions on configuration spaces underlying stochastic systems on infinite graphs. An important consequence is the successful characterization of conserved quantities without introducing the notion of stationary distributions. In this article, we further develop the theory of uniform functions and construct the theory independently of any choice of a base state. Furthermore, we generalize the notion of interactions given in Bannai et al. (2024) to accommodate the case where there are multiple possible state transitions on adjacent vertices. We then prove that if the interaction is *exchangeable*, then any uniform function which gives a global conserved quantity can be expressed as a sum of local conserved quantities of the interaction. Contrary to our previous article, we do not need to assume that the interaction is irreducibly quantified. This shows that our theory of uniform functions on configuration spaces over infinite graphs with transition structure given by an exchangeable interaction is a natural framework to study general stochastic large-scale interacting systems. While some of the ideas in this article are based on Bannai et al. (2024), the current article is logically independent and self-contained.

1. Introduction

Deriving the macroscopic evolution from the dynamics of microscopic systems is a very fundamental and challenging task. As a mathematically rigorous theory, hydrodynamic limits for stochastic large-scale interacting systems (LSIS) have been widely studied. In order to provide a new perspective for the analysis

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of such systems, in our previous article (Bannai et al., 2024), we introduced the concept of configuration space with transition structure and the associated space of *uniform functions*. The uniform functions form a suitable class that includes the conserved quantities on configuration spaces on infinite graphs underlying stochastic LSIS. The theory was extended in Bannai and Sasada (2026+b) under certain assumptions to give a proof of Varadhan's decomposition for hydrodynamic limits. In Bannai et al. (2024) and Bannai and Sasada (2026+b), the theory was constructed using a choice of a fixed base state of the local state space. In this article, we construct the theory independently of any such choice. Furthermore, we expand the notion of interactions to accommodate the case when the transitions between states on adjacent vertices is given by a general symmetric *digraph* (i.e. *directed graph*), which we again call an *interaction*. We then prove that if the interaction is *exchangeable*, then any uniform function invariant under transitions can be expressed as the sum of the local conserved quantities of the interaction. This is a generalization of Bannai et al. (2024, Theorem 3.7). Although we build on ideas initiated in Bannai et al. (2024), our article is logically independent of such results.

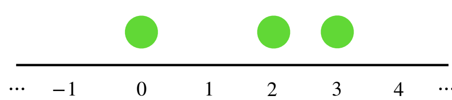

FIGURE 1.1. Example of a configuration in the configuration space $S^X = \{0,1\}^{\mathbb{Z}}$.

Let S be a non-empty set, which we call the *local state space*. The fundamental example is $S = \{0,1\}$, where 0 means that the vertex is vacant and 1 means that the vertex is occupied by a particle. For a symmetric digraph (X,E) with set of vertices X and set of edges $E \subset X \times X$, the configuration space of S on (X,E) is defined as $S^X := \prod_{x\in X} S$. We call any $\eta = (\eta_x)_{x\in X} \in S^X$ a *configuration*. The fundamental example of such graph is given by the *one*-dimensional Euclidean lattice $(\mathbb{Z},\mathbb{E})$, where $\mathbb{E} = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid |i-j| = 1\}$, and the configuration space $S^X = \{0,1\}^{\mathbb{Z}}$ describes all of the possible combination of states on the Euclidean lattice.

The configurations will be quantified via the observables, i.e. functions $f\colon S^X \to \mathbb{R}$ on the configuration space. The premise of our model is that the observables should depend only on the local states in the vicinity of the point of observation. For $x \in X$ and $R > 0$, we say that a function $f\colon S^X \to \mathbb{R}$ is *local at x with radius R*, if $f(\eta)$ depends only on (η_z) satisfying $d_X(x,z) \le R$, where d_X is the graph distance. Given a system of functions $(f_x)_{x\in X}$ with f_x local at x, in order to get a suitable quantity for the entire system, we would like to consider the sum

$$f := \sum_{x\in X} f_x. \tag{1.1}$$

If the original function f_x expresses the number of particles or energy or any other quantity for the local state at x, then f would express the total number of particles or total energy or the total of any other quantity for the entire system on X. In considering the hydrodynamic limits of LSIS, it is useful to consider cases when X is infinite. However, the sum in (1.1) would be an infinite sum and generally not well-defined for such X. Variants of infinite sums of the form given by (1.1) appear in Kipnis and Landim (1999, p.144) and Kipnis et al. (1994, p.1477), but with the caveat that the infinite sum "does not make sense". In Theorem 2.4, we define uniform functions to be a certain sum of local functions on S^X, which gives a rigorous definition for sums such as the one given in (1.1), even for the case when X is infinite.

Our first result, Theorem 2.9, is informally stated as follows: Under the condition that (X,E) is connected and locally finite, if a system of functions $(f_x)_{x\in X}$ is uniformly local (see Theorem 2.8), then the sum given in (1.1) defines a well-defined and well-behaved object, which is a uniform function defined in Theorem 2.4. Moreover, in Theorem 2.10 we also prove that any uniform function can be obtained as the sum of a system of functions on S^X which are uniformly local.

To construct our LSIS, we specify all possible transitions of states on adjacent vertices of the underlying graph. This collection of possible transitions is described as a symmetric digraph $(S\times S,\phi)$, and for such

ϕ, the pair (S, ϕ) is called an *interaction* (see Theorem 3.1). For the case $S = \{0, 1\}$ described in Fig. 1.2, the configuration $(0, 0) \in S \times S$ expresses the state with no particles, $(1, 0) \in S \times S$ expresses the state with a particle in the first vertex and no particles in the second vertex, etc. Then the exclusion

$$\phi_{\mathrm{ex}} := \{((1, 0), (0, 1)), ((0, 1), (1, 0))\} \subset (S \times S) \times (S \times S) \tag{1.2}$$

expresses the rule that a particle can move only if the adjacent vertex is vacant.

FIGURE 1.2. The exclusion ϕ_{ex} for $S = \{0, 1\}$ expresses the rule that a particle can move only if the adjacent vertex is vacant. The diagram on the right expresses the graph $(S\times S, \phi_{\mathrm{ex}})$.

A choice of an interaction gives the transitions of S^X. Namely, for $\eta = (\eta_x)_{x\in X}, \eta' = (\eta'_x)_{x\in X} \in S^X$, the configuration η can transition to η' if and only if there exists an edge $e = (x, y) \in E$ such that $\eta_z = \eta'_z$ for $z \neq x, y$ and $((\eta_x, \eta_y), (\eta'_x, \eta'_y)) \in \phi$. If we denote by $\Phi_E \subset S^X \times S^X$ the set of all such permitted transitions, then (S^X, Φ_E) forms a symmetric digraph. Our construction allows not only nearest-neighbor models but more general models by changing the graph (X, E). For the exclusion ϕ_{ex}, if we take the graph (X, E) to be the Euclidean lattice $(\mathbb{Z}, \mathbb{E})$, then this model underlies the nearest-neighbor exclusion process. The exclusion process is one of the most fundamental models of the interacting particle systems and has been studied extensively (Spohn, 1991; Kipnis and Landim, 1999; Kipnis et al., 1994, 1995; Liggett, 1999; Funaki et al., 1991, 1996; Guo et al., 1988; Varadhan and Yau, 1997).

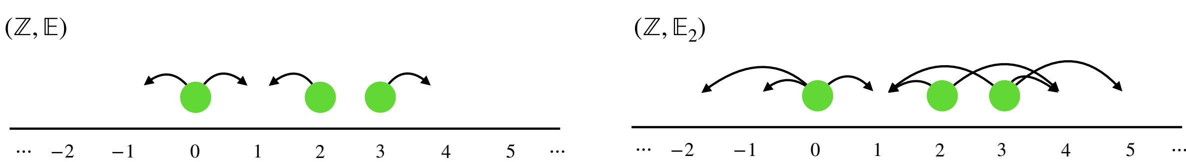

FIGURE 1.3. The transitions on $\{0, 1\}^{\mathbb{Z}}$ induced by the exclusion ϕ_{ex} for the cases where the underlying graphs are $(\mathbb{Z}, \mathbb{E})(= (\mathbb{Z}, \mathbb{E}_1))$ and $(\mathbb{Z}, \mathbb{E}_2)$. The interaction takes place between vertices connected by an edge of the underlying graph.

Moreover, if we take the graph (X, E) to be $(\mathbb{Z}, \mathbb{E}_k)$ with $\mathbb{E}_k := \{(i, j) \in \mathbb{Z}\times\mathbb{Z} \mid 1 \leq |i - j| \leq k\}$ for an integer $k \geq 1$, then this model underlies the exclusion process which allows hopping to vacant vertices of distance up to k. This versatility of the model is one reason that we have separated the interaction from the underlying graph.

Given an interaction (S, ϕ), we say that a function $\xi\colon S \to \mathbb{R}$ is a *conserved quantity* if ξ is a function whose sum is preserved by transitions of an interaction (see Theorem 3.3).

For a conserved quantity ξ, let $\xi_x(\eta) := \xi(\eta_x)$ for any $\eta = (\eta_x)_{x\in X} \in S^X$. Then by definition, ξ_x for any $x \in X$ is local at x with radius 0, hence by our Theorem 2.9,

$$\xi_X := \sum_{x\in X} \xi_x$$

defines a uniform function. For the case $(\{0,1\},\phi_{\mathrm{ex}})$, the function $\xi\colon\{0,1\}\to\mathbb{R}$ given by $\xi(s)=s$ is a conserved quantity of $(\{0,1\},\phi_{\mathrm{ex}})$. If X is finite, then $\xi_X\colon\{0,1\}^X\to\mathbb{R}$ is a function such that $\xi_X(\eta)=\sum_{x\in X}\xi(\eta_x)$ for $\eta\in\{0,1\}^X$ gives the total number of particles of η.

Among properties of the interaction, the concept of exchangeability, introduced in Theorem 3.11, plays a particularly key role in our main results. We observe that the interaction $(\{0,1\},\phi_{\mathrm{ex}})$ is exchangeable. Bannai et al. (2026) gives classifications and examples of various exchangeable interactions, hence there are an abundance of examples of exchangeable interactions.

Since a uniform function may be given by an infinite sum, it is in general not an actual function on the configuration space. However, one can define the difference of values of a uniform function at $\eta,\eta'\in S^X$ with respect to any transition $(\eta,\eta')\in\Phi_E$. Hence a uniform function behaves as a potential on the configuration space.

We say that a uniform function on S^X is invariant via transitions of (S^X,Φ_E), if the difference of values with respect to any transition is *zero*. The main theorem of our article, Theorem 3.12, shows that under the conditions that (S,ϕ) is exchangeable and (X,E) is connected, locally finite and infinite, any uniform function f which is invariant via transitions is expressed as $f=\xi_X$ by some conserved quantity ξ.

A function which is invariant via transitions of S^X can be viewed as a global conserved quantity of S^X. Theorem 3.12 implies that any global conserved quantity expressed by a uniform function is given as the sum of local conserved quantities of the interaction. Hence our theory of uniform functions gives a class of well-behaved global conserved quantities of the configuration space. Contrary to Bannai et al. (2024), we do not need to assume that (S,ϕ) is irreducibly quantified. Heuristically, it was well-known that "the number of" global conserved quantities should be the number of macroscopic variables in the hydrodynamic limit, but before the space of uniform functions was introduced in Bannai et al. (2024), there was no clear understanding of how we can count such a number. With the framework of uniform functions, as discussed in Bannai et al. (2024); Bannai and Sasada (2026+b), we can define the notion of this number as the dimension of the space of these well-behaved global conserved quantities.

Theorem 3.12 can be also interpreted in terms of uniform cohomology. As an analogy of the fact that 0-th cohomology $H^0(X)$ coincides with the space of functions constant on the connected components of the graph (X,E) (see Bannai et al., 2024, Proposition A.8), we can prove that the 0-th uniform cohomology $H^0_{\mathrm{unif}}(S^X)$ (see Section 4 for details) coincides with the space of uniform functions invariant via transitions of (S^X,Φ_E), i.e. constant on the connected components of (S^X,Φ_E), which is precisely stated as Theorem 4.5.

If X is infinite, then (S^X,Φ_E) in general has an infinite number of connected components. Hence the graph cohomology $H^0(S^X)$, which coincides with the space of functions constant on the connected components of (S^X,Φ_E) is in general infinite dimensional. However, if the space of conserved quantities is finite dimensional, as in the case when S is a finite set, then Theorem 4.5 implies that the space $H^0_{\mathrm{unif}}(S^X)$, which can be interpreted as the space of uniform functions constant on the connected components of (S^X,Φ_E), is also finite dimensional. The theory of uniform functions allows for the construction of such a well-behaved cohomology theory. We emphasize that Theorem 3.12 and Theorem 4.5 hold in general only when X is infinite.

The precise contents of this article are as follows. In §2, we define the space of uniform functions on a configuration space S^X of a local state space S over a symmetric digraph (X,E), and prove that the space of uniform functions is independent of any choice of the base state. We then define the notion of a uniform system of local functions and prove in Theorem 2.9 that if (X,E) is connected and locally finite, then the class of uniform functions coincides with the class of functions that can be expressed as the sum of a uniform system of local functions. In §3, we consider an interaction (S,ϕ) and the configuration space with transition structure (S^X,Φ_E) over (X,E). Assuming that the interaction (S,ϕ) is exchangeable and the graph (X,E) is infinite, we prove our main result (Theorem 3.12). In §4, we introduce the differential for uniform functions and prove Theorem 4.5.

2. Uniform Functions on Configuration Spaces

In this section, we give the definition of uniform functions on the space of configurations on a symmetric digraph. We then prove in Theorem 2.5 that the space of uniform functions for distinct choices of the base state $* \in S$ are canonically isomorphic. Recall that S is a non-empty set, which we call the *local state space*. An element of S expresses a state of a system on a single vertex. Furthermore, let (X, E) be a symmetric digraph. In other words, X is a set called the *set of vertices*, $E \subset X \times X$ is a set called the *set of edges*, and we assume that $e = (o_e, t_e) \in E$ if and only if $\bar{e} = (t_e, o_e) \in E$. The set X gives the underlying space for the system. The set E consists of all pairs of vertices in X whose local states directly interact. We say that the graph (X, E) is *locally finite*, if $E_x := \{e \in E \mid o_e = x\}$ is a finite set for any $x \in X$. Here $o_e \in X$ is the origin of the edge e. In what follows, we will assume that the graph (X, E) is connected and locally finite.

Definition 2.1. We define the *configuration space* of states S on (X, E) by $S^X := \prod_{x\in X} S$.

We call any element $\eta = (\eta_x)_{x\in X} \in S^X$ a *configuration*. The configuration space expresses all of the possible configurations of states that our model may take. We say that $f\colon S^X \to \mathbb{R}$ is a *local function*, if there exists a finite $\Lambda \subset X$ such that the value $f(\eta)$ depends only on the coordinates $(\eta_x)_{x\in\Lambda}$. Such function corresponds to a mapping in $C(S^\Lambda)$, where $C(S^\Lambda) := \mathrm{Map}(S^\Lambda, \mathbb{R})$ denotes the space of real-valued functions on S^Λ. If we let $\mathscr{I}$ be the set of finite subsets $\Lambda \subset X$, then the space of local functions is given as $C_{\mathrm{loc}}(S^X) = \bigcup_{\Lambda\in\mathscr{I}} C(S^\Lambda)$.

Fix $* \in S$, which we call the base state, and let S^X_* be the subset of S^X consisting of configurations whose components are $*$ for all but finite $x \in X$. We denote by $\star$ the configuration in S^X whose components are all at base state. Then $\star \in S^X_*$. For $\Lambda \subset X$, we have $C(S^\Lambda) \subset C(S^X_*)$ where $C(S^X_*) := \mathrm{Map}(S^X_*, \mathbb{R})$. For any $f \in C(S^X_*)$ and $\Lambda \subset X$, let $\iota^\Lambda_* f \in C(S^\Lambda)$ be the function

$$\forall \eta \in S^X_*, \quad \iota^\Lambda_* f(\eta) := f(\eta|_\Lambda),$$

where $\eta|_\Lambda \in S^X_*$ is defined as the configuration whose component for $x \in \Lambda$ coincides with that of η, and is at base state $*$ for components outside Λ. This defines a linear operator $\iota^\Lambda_*\colon C(S^X_*) \to C(S^\Lambda) \subset C(S^X_*)$. In particular, if Λ is finite, then $\iota^\Lambda_* f$ is a local function. We define the space of *local functions with exact support* Λ by

$$C_\Lambda(S^X_*) := \left\{ f \in C(S^\Lambda) \mid \forall \text{ finite } \Lambda' \subset X \text{ such that } \Lambda \not\subset \Lambda', \text{ we have } \iota^{\Lambda'}_* f = 0 \right\}.$$

A local function $f \in C(S^\Lambda)$ is in $C_\Lambda(S^X_*)$ if and only if $f(\eta) = 0$ for any configuration $\eta = (\eta_x)_{x\in\Lambda} \in S^\Lambda$ satisfying $\eta_x = *$ for some $x \in \Lambda$. For any $\eta = (\eta_x)_{x\in X} \in S^X$, we define the support of η by $\mathrm{Supp}(\eta) := \{x \in X \mid \eta_x \neq *\}$.

Lemma 2.2. *Suppose (f^*_Λ) is a system of functions such that $f^*_\Lambda \in C_\Lambda(S^X_*)$ for any $\Lambda \in \mathscr{I}$. Then $f = \sum_{\Lambda\in\mathscr{I}} f^*_\Lambda$ gives a well-defined function on S^X_*.*

Proof: For any $\eta \in S^X_*$, we have $f^*_\Lambda(\eta) \neq 0$ if and only if $\Lambda \subset \mathrm{Supp}(\eta)$. Since the support $\mathrm{Supp}(\eta)$ is finite, if we are given functions $f^*_\Lambda \in C_\Lambda(S^X_*)$ for any $\Lambda \in \mathscr{I}$, then we see that

$$f(\eta) = \sum_{\Lambda\in\mathscr{I}} f^*_\Lambda(\eta) = \sum_{\Lambda\subset\mathrm{Supp}(\eta)} f^*_\Lambda(\eta)$$

is a well-defined finite sum. Hence the right-hand side of (2.1) below defines a function on S^X_*. □

The next proposition states that the converse of the previous lemma holds in a certain sense.

Proposition 2.3. *For any function $f \in C(S^X_*)$, there exists a unique system of functions $(f^*_\Lambda)_{\Lambda\in\mathscr{I}}$ satisfying $f^*_\Lambda \in C_\Lambda(S^X_*)$ for any $\Lambda \in \mathscr{I}$ such that*

$$f = \sum_{\Lambda\in\mathscr{I}} f^*_\Lambda \tag{2.1}$$

as a function on S^X_*.

Proof: If such (f^*_Λ) satisfying (2.1) exists, then by definition of $C_\Lambda(S^X_*)$, we see that

$$\iota^\Lambda_* f = \sum_{\Lambda''\subset\Lambda} f^*_{\Lambda''} \tag{2.2}$$

for any $\Lambda \in \mathscr{I}$. We will construct f^*_Λ satisfying (2.2) by induction on the number of elements in $\Lambda \in \mathscr{I}$. If $\Lambda = \emptyset$, then we let $f^*_\emptyset := \iota^\emptyset_* f = f(\star)$. For any $\Lambda \in \mathscr{I}$, assume that there exists $f^*_{\Lambda''}$ for any $\Lambda'' \subsetneq \Lambda$ such that $\iota^{\Lambda'}_* f = \sum_{\Lambda''\subset\Lambda'} f^*_{\Lambda''}$ is satisfied for any proper subset $\Lambda' \subset \Lambda$. We let

$$f^*_\Lambda := \iota^\Lambda_* f - \sum_{\Lambda''\subsetneq\Lambda} f^*_{\Lambda''}.$$

Then for any $\Lambda' \in \mathscr{I}$ such that $\Lambda \not\subset \Lambda'$, we have $\Lambda \cap \Lambda' \subsetneq \Lambda$. Hence

$$\iota^{\Lambda'}_* f^*_\Lambda = \iota^{\Lambda'}_*(\iota^\Lambda_* f) - \sum_{\Lambda''\subsetneq\Lambda} \iota^{\Lambda'}_* f^*_{\Lambda''} = \iota^{\Lambda\cap\Lambda'}_* f - \sum_{\Lambda''\subset\Lambda\cap\Lambda'} \iota^{\Lambda'}_* f^*_{\Lambda''} = 0,$$

where the second equality follows from the definition of $\iota^{\Lambda'}_*$ and the fact that $f^*_{\Lambda''} \in C_{\Lambda''}(S^X)$ and the last equality follows from the induction hypothesis for $\Lambda \cap \Lambda' \subsetneq \Lambda$. This shows that $f^*_\Lambda \in C_\Lambda(S^X)$, and that $\iota^\Lambda_* f = \sum_{\Lambda''\subset\Lambda} f^*_{\Lambda''}$. This gives a construction of a system (f^*_Λ) satisfying (2.2). The uniqueness of f^*_Λ follows from the construction. By Theorem 2.2, the sum $\sum_{\Lambda\in\mathscr{I}} f^*_\Lambda$ defines a well-defined function on S^X_*. The expression in (2.1) follows from the fact that for any $\eta \in S^X_*$, if $\mathrm{Supp}(\eta) \subset \Lambda$, then we have $f(\eta) = \iota^\Lambda_* f(\eta) = \sum_{\Lambda''\subset\Lambda} f^*_{\Lambda''}(\eta) = \sum_{\Lambda''\in\mathscr{I}} f^*_{\Lambda''}(\eta)$ as desired. □

We can use Theorem 2.3 to define uniform functions in $C(S^X_*)$. For any $x, y \in X$, we let $d_X(x, y)$ denote the graph distance of x to y, i.e. the length of the shortest path from x to y, with the convention that $d_X(x, y) = 0$ if $x = y$. For any $\Lambda \subset X$, let $\mathrm{diam}(\Lambda) := \sup_{x,y\in\Lambda} d_X(x, y)$.

Definition 2.4. We say that a function $f \in C(S^X_*)$ is *uniform*, if for the expansion of Theorem 2.3, there exists $R > 0$ such that $f^*_\Lambda = 0$ if $\mathrm{diam}(\Lambda) > R$.

We denote by $C_{\mathrm{unif}}(S^X_*)$ the space of uniform functions. A priori, the uniform functions would seem to depend on the choice of the base state $* \in S$. However, we can prove that there exists a canonical isomorphism between uniform functions for distinct base states $*$ and $*'$ as follows. Let $\mathscr{I}_R$ be the subset of $\mathscr{I}$ consisting of finite $\Lambda \subset X$ such that $\mathrm{diam}(\Lambda) \le R$.

Proposition 2.5. *Let $*, *' \in S$ be any two states of S. Then we have an $\mathbb{R}$-linear isomorphism*

$$C_{\mathrm{unif}}(S^X_*) \cong C_{\mathrm{unif}}(S^X_{*'})$$

which gives the map $f \mapsto f + (f(\star) - f(\star'))$ on the subspace of local functions $C_{\mathrm{loc}}(S^X)$.

Proof: By Theorem 2.3 and the definition of uniform functions, any $f \in C_{\mathrm{unif}}(S^X_*)$ has a decomposition

$$f = \sum_{\Lambda\in\mathscr{I}_R} f^*_\Lambda$$

for some $R > 0$, where $f^*_\Lambda \in C_\Lambda(S^X_*) \subset C(S^\Lambda)$. Since $C(S^\Lambda) \subset C(S^X_{*'})$, we have a decomposition

$$f^*_\Lambda = \sum_{\Lambda'\subset\Lambda} (f^*_\Lambda)^{*'}_{\Lambda'},$$

where $(f^*_\Lambda)^{*'}_{\Lambda'} \in C_{\Lambda'}(S^X_{*'})$. We define $f^{*'}_{\Lambda'}$ for $\Lambda' \in \mathscr{I}_R$ by $f^{*'}_\emptyset := f^*_\emptyset = f(\star)$ for $\Lambda' = \emptyset$ and for $\Lambda' \neq \emptyset$, we let

$$f^{*'}_{\Lambda'} := \sum_{\substack{\Lambda\in\mathscr{I}_R\\ \Lambda'\subset\Lambda}} (f^*_\Lambda)^{*'}_{\Lambda'}. \tag{2.3}$$

The sum is a finite sum since $\Lambda' \neq \emptyset$ is finite and the sum is over $\Lambda \in \mathscr{I}_R$. Hence $f^{*'}_{\Lambda'} \in C_{\Lambda'}(S^X_{*'})$ for any $\Lambda' \in \mathscr{I}$. By Theorem 2.2 and the definition of uniform functions, the infinite sum

$$u'(f) := \sum_{\Lambda' \in \mathscr{I}_R} f^{*'}_{\Lambda'} \tag{2.4}$$

defines a function in $C_{\mathrm{unif}}(S^X_{*'})$. The correspondence $f \mapsto u'(f)$ gives an $\mathbb{R}$-linear map

$$u' : C_{\mathrm{unif}}(S^X_*) \to C_{\mathrm{unif}}(S^X_{*'})$$

which satisfies $u'(f)(\star') = f(\star)$. By construction, we have $u'(f) = f + (f(\star) - f(\star'))$ on the space of local functions.

We next define the inverse in a similar manner. Suppose $*'' \in S$ is another base state. If f' is of the form $f' := u'(f) = \sum_{\Lambda' \in \mathscr{I}_R} f^{*'}_{\Lambda'}$ of (2.4), then for $\Lambda' \in \mathscr{I}_R$ and $\Lambda'' \in \mathscr{I}_R$, we have

$$(f^{*'}_{\Lambda'})^{*''}_{\Lambda''} = \Big(\sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda' \subset \Lambda}} (f^*_\Lambda)^{*'}_{\Lambda'} \Big)^{*''}_{\Lambda''} = \sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda' \subset \Lambda}} ((f^*_\Lambda)^{*'}_{\Lambda'})^{*''}_{\Lambda''}.$$

We define $f^{*''}_{\Lambda''}$ as in (2.3), so that $f^{*''}_\emptyset = f^{*'}_\emptyset = f^*_\emptyset = f(\star)$, and for $\Lambda'' \neq \emptyset$, we let

$$f^{*''}_{\Lambda''} := \sum_{\substack{\Lambda' \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda'}} (f^{*'}_{\Lambda'})^{*''}_{\Lambda''} = \sum_{\substack{\Lambda' \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda'}} \sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda' \subset \Lambda}} ((f^*_\Lambda)^{*'}_{\Lambda'})^{*''}_{\Lambda''} = \sum_{\Lambda \in \mathscr{I}_R} \sum_{\substack{\Lambda' \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda' \subset \Lambda}} ((f^*_\Lambda)^{*'}_{\Lambda'})^{*''}_{\Lambda''} = \sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda}} (f^*_\Lambda)^{*''}_{\Lambda''}.$$

Then by (2.4), we have

$$u''(u'(f)) := \sum_{\Lambda'' \in \mathscr{I}_R} f^{*''}_{\Lambda''} = \sum_{\Lambda'' \in \mathscr{I}_R} \sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda}} (f^*_\Lambda)^{*''}_{\Lambda''},$$

In particular, if $*'' = *$, noting that $(f^*_\Lambda)^*_{\Lambda''} = f^*_\Lambda$ if $\Lambda'' = \Lambda$ and $(f^*_\Lambda)^*_{\Lambda''} = 0$ if $\Lambda'' \neq \Lambda$, we have

$$u''(u'(f)) = \sum_{\Lambda'' \in \mathscr{I}_R} \sum_{\substack{\Lambda \in \mathscr{I}_R \\ \Lambda'' \subset \Lambda}} (f^*_\Lambda)^*_{\Lambda''} = \sum_{\Lambda \in \mathscr{I}_R} f^*_\Lambda = f.$$

This shows that $u'' \circ u' = \mathrm{id}$ as desired. Reversing the roles of $*$ and $*'$, we can also prove that $u' \circ u'' = \mathrm{id}$, hence u' is an isomorphism as desired. □

We will view uniform functions for different base states through the canonical isomorphisms given by Theorem 2.5. The isomorphism is given as a translation by constant functions on the space of local functions. For this reason, it is convenient to consider the space of functions modulo the subspace of constant functions. Let $C^0_{\mathrm{loc}}(S^X)$ and $C^0_{\mathrm{unif}}(S^X_*)$ be the quotients of $C_{\mathrm{loc}}(S^X)$ and $C_{\mathrm{unif}}(S^X_*)$ modulo the subspace of constant functions. Then Theorem 2.5 gives the following.

Corollary 2.6. *Let $*, *' \in S$ be any two states of S. Then the $\mathbb{R}$-linear isomorphism of Theorem 2.5 induces an isomorphism*

$$C^0_{\mathrm{unif}}(S^X_*) \cong C^0_{\mathrm{unif}}(S^X_{*'})$$

which is the identity on the space of local functions $C^0_{\mathrm{loc}}(S^X)$.

Due to Theorem 2.6, we will often simply denote the space $C^0_{\mathrm{unif}}(S^X_*)$ as $C^0_{\mathrm{unif}}(S^X)$ without specifying the base state $* \in S$. For any $f \in C^0_{\mathrm{unif}}(S^X)$, we will call the function in $C^0_{\mathrm{unif}}(S^X_*)$ corresponding to f a *realization of* f for the base state $* \in S$. Similarly, for any $\Lambda \subset X$, we let $C^0(S^\Lambda)$ be the quotient of $C(S^\Lambda)$ modulo the subspace of constant functions. For a base state $* \in S$, any function $f \in C^0(S^\Lambda)$ has a normalized representative $f \in C(S^\Lambda)$ such that $f(\star) = 0$. This normalization depends on the choice of $* \in S$, but the corresponding class in $C^0(S^\Lambda)$ is independent of such choice.

Next, we show that the sum of a system $(f_x)_{x \in X}$ of uniformly local functions on S^X (given in Theorem 2.8) defines a uniform function. As stated in §1, the premise of our model is that the observables of a configuration should *not* depend on the entire configuration space, but should depend only on the local states in the proximity of the point of observation. Hence in order to express such observables, we introduce the notion of a function local at a vertex $x \in X$. Consider the symmetric digraph (X, E). The graph distance d_X of (X, E) expresses the proximity of the vertices in X. For any $R > 0$ and $x \in X$, we let

$$B(x, R) := \{y \in X \mid d_X(x, y) < R\}$$

be the *ball* with center x and radius R. We say that a function $f_x \colon S^X \to \mathbb{R}$ is *local at* x (with radius R), if $f_x \in C(S^{B(x,R)})$ for some $R > 0$. In other words, $f_x(\eta)$ for $\eta = (\eta_z)_{z \in X}$ depends only on the coordinates $(\eta_z)_{z \in B(x,R)}$.

Lemma 2.7. *A function $f \colon S^X \to \mathbb{R}$ is a local function if and only if it is local at any $x \in X$.*

Proof: Suppose f is a local function. Then there exists finite $\Lambda \subset X$ such that $f \in C(S^\Lambda)$. Then, since (X, E) is connected, for any $x \in X$, there exists $R > 0$ such that $\Lambda \subset B(x, R)$. This shows that $f \in C(S^\Lambda) \subset C(S^{B(x,R)})$, proving that f is local at x. Conversely, suppose f is local at $x \in X$. Then there exists $R > 0$ such that $f \in C(S^{B(x,R)})$. Since (X, E) is locally finite, $B(x, R)$ is a finite set. This shows that f is local as desired. □

Although equivalent by Theorem 2.7 to the fact that a function is local, the notion that a function is local at $x \in X$ allows for the definition of the uniformity of the localness as follows.

Definition 2.8. Consider a system of functions $(f_x)_{x \in X}$ in $C^0(S^X)$. We say such system $(f_x)_{x \in X}$ is *uniformly local*, if there exists $R > 0$ independent of $x \in X$ such that f_x is local at x with radius R.

Given a system $(f_x)_{x \in X}$ which is uniformly local, the total observables of the entire system should be given as the infinite sum $\sum_{x \in X} f_x$. Note that $\eta = (\eta_x)_{x \in X} \in S^X_*$ if and only if the support $\mathrm{Supp}(\eta)$ is a finite set. We have the following.

Proposition 2.9. *Let $(f_x)_{x \in X}$ be a uniformly local system of functions in $C^0(S^X)$. Then the infinite sum given in (1.1) defines a uniform function in $C^0_{\mathrm{unif}}(S^X)$.*

Proof: Since $(f_x)_{x \in X}$ is uniformly local, there exists $R > 0$ such that $f_x \in C^0(S^{B(x,R)})$ for any $x \in X$. Take a base state $* \in S$, and a normalized representative of $f_x \in C(S^{B(x,R)})$ satisfying $f_x(\star) = 0$ for any $x \in X$. For $\eta = (\eta_x)_{x \in X} \in S^X_*$, by definition, the support $\mathrm{Supp}(\eta)$ is a finite set. Since we have assumed that (X, E) is locally finite, the set of vertices $x \in X$ such that $B(x, R) \cap \mathrm{Supp}(\eta) \neq \emptyset$ is also a finite set. If $B(x, R) \cap \mathrm{Supp}(\eta) = \emptyset$, then since $f_x \in C\big(S^{B(x,R)}\big)$, we have $f_x(\eta) = f_x(\star) = 0$. This shows that $f(\eta) = \sum_{x \in X} f_x(\eta)$ is a well-defined finite sum, hence defines a function in $C^0(S^X_*)$. Let $R > 0$ such that $f_x \in C(S^{B(x,R)})$. If we take an expansion of f_x, then by Theorem 2.3, we have

$$f_x = \iota_*^{B(x,R)} f_x = \sum_{\Lambda \subset B(x,R)} (f_x)^*_\Lambda.$$

The uniqueness of the expansion of Theorem 2.3 shows that for f^*_Λ such that $f = \sum_{\Lambda \in \mathscr{I}} f^*_\Lambda$, we have

$$f^*_\Lambda = \sum_{x \in X, \Lambda \subset B(x,R)} (f_x)^*_\Lambda$$

where the sum is taken over all $x \in X$ such that $\Lambda \subset B(x, R)$. This shows that $f^*_\Lambda = 0$ if $\mathrm{diam}(\Lambda) > 2R$, hence $f \in C^0_{\mathrm{unif}}(S^X)$ as desired. □

We can also prove the converse of Theorem 2.9.

Proposition 2.10. *Let $f \in C^0_{\mathrm{unif}}(S^X)$. Then there exists a system $(f_x)_{x \in X}$ of uniformly local functions in $C^0(S^X)$ such that $f = \sum_{x \in X} f_x$.*

Proof: We take a base state $* \in S$, and take $f \in C^0_{\mathrm{unif}}(S^X_*)$ normalized so that $f(\star) = 0$. Then by the definition of uniform functions, there exists $R > 0$ such that for the expansion (2.1), $f^*_\Lambda = 0$ if $\mathrm{diam}(\Lambda) > R$. Moreover, $f^*_\emptyset = f(\star) = 0$. For each $x \in X$, let

$$f_x := \sum_{\Lambda \in \mathscr{I},\, x \in \Lambda} \frac{1}{|\Lambda|} f^*_\Lambda,$$

which is a well-defined finite sum such that $f_x \in C(S^{B(x,R)})$ since $\Lambda \subset B(x,R)$ if $\mathrm{diam}(\Lambda) \le R$. We have

$$\sum_{x \in X} f_x = \sum_{x \in X} \sum_{\Lambda \in \mathscr{I},\, x \in \Lambda} \frac{1}{|\Lambda|} f^*_\Lambda = \sum_{\Lambda \in \mathscr{I}} f^*_\Lambda = f$$

as a function on S^X_*, which shows that $f = \sum_{x \in X} f_x$ in $C^0_{\mathrm{unif}}(S^X)$ as desired. □

3. Configuration Space with Transition Structure

In this section, we first define the notion of interactions and conserved quantities. Then we will consider the configuration space on an underlying graph given by the interaction, and prove our main theorem.

Definition 3.1. Let S be a non-empty set. We say that $\phi \subset (S \times S) \times (S \times S)$ is an *interaction*, if $(S \times S, \phi)$ is a symmetric digraph. In other words, we have $(s_1, s_2) \in \phi$ if and only if $(s_2, s_1) \in \phi$. We also refer to the pair (S, ϕ) as an interaction, and the graph $(S \times S, \phi)$ the associated graph.

The set $S \times S$ represents all of the possible configurations on two adjacent vertices of an underlying graph, and the associated graph expresses all of the possible transitions of $S \times S$.

Remark 3.2. In Bannai et al. (2024, Definition 2.4) and Bannai and Sasada (2026+a, §0), we defined an interaction as a map

$$\phi_+ \colon S \times S \to S \times S$$

satisfying $\phi^\iota_+ \circ \phi_+(s_1, s_2) = (s_1, s_2)$ for any $(s_1, s_2) \in S \times S$ such that $\phi_+(s_1, s_2) \neq (s_1, s_2)$. Here, we let $\phi^\iota_+ := \iota \circ \phi_+ \circ \iota$, where $\iota(s_1, s_2) := (s_2, s_1)$ for any $(s_1, s_2) \in S \times S$. In the above articles, we considered the set G_{ϕ_+} given by

$$G_{\phi_+} := \{((s_1, s_2), \phi_+(s_1, s_2)) \mid (s_1, s_2) \in S \times S\} \subset (S \times S) \times (S \times S)$$

expresses all of the possible transitions on two adjacent vertices of an underlying graph. Since the underlying graph (X, E) is symmetric, for any $e = (o_e, t_e) \in E$, $\bar{e} \in E$ holds and so the transitions $(\eta_{o_e}, \eta_{t_e}) \to \phi_+(\eta_{o_e}, \eta_{t_e})$ and $(\eta_{o_e}, \eta_{t_e}) \to \phi^\iota_+(\eta_{o_e}, \eta_{t_e})$ are both possible. The latter one can be understood as we take $s_1 = \eta_{t_e}, s_2 = \eta_{o_e}$ and apply ϕ_+. Hence, $G_{\phi^\iota_+} := \{((s'_1, s'_2), \phi^\iota_+(s'_1, s'_2)) \mid (s'_1, s'_2) \in S \times S\}$ also expresses all of the possible transitions on two adjacent vertices. Then the condition on the map ϕ_+ ensures that $G_{\phi_+} \cup G_{\phi^\iota_+} \subset (S \times S) \times (S \times S)$ is a symmetric digraph, which gives an interaction in our sense of Theorem 3.1. In this way, the definition of interaction in the current paper generalizes the definition given in previous papers (Bannai et al., 2024; Bannai and Sasada, 2026+a).

Next, we define a conserved quantity for an interaction.

Definition 3.3. We say that a function $\xi \colon S \to \mathbb{R}$ is a *conserved quantity* for an interaction (S, ϕ), if the function $\tilde{\xi} \colon S \times S \to \mathbb{R}$ given by

$$\tilde{\xi}(s_1, s_2) := \xi(s_1) + \xi(s_2) \tag{3.1}$$

is constant on the connected components of the associated graph $(S \times S, \phi)$.

We remark that the constant function is trivially a conserved quantity. We denote by $\mathrm{Consv}^\phi(S)$ the space of conserved quantities for an interaction (S, ϕ), modulo the subspace of constant functions.

An example of an interaction is given by the multi-species exclusion interaction, which gives a generalization of the exclusion interaction.

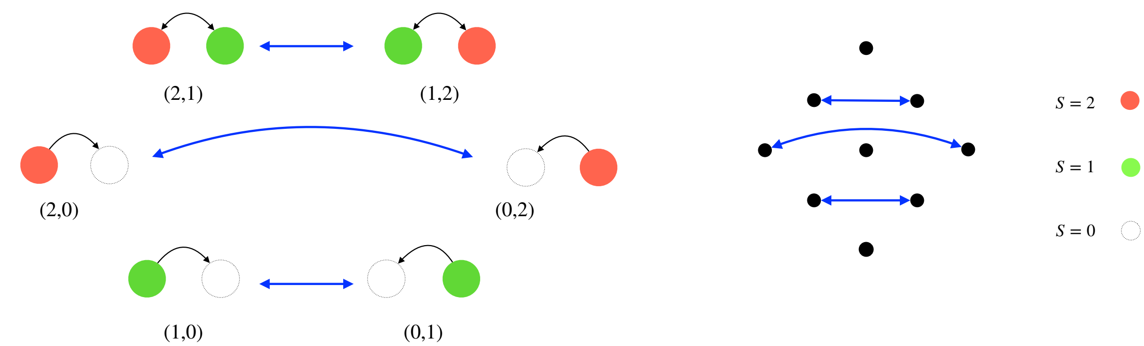

FIGURE 3.4. multi-species exclusion interaction for $\kappa = 2$

Example 3.4. For an integer $\kappa \geq 1$ and $S_\kappa = \{0, 1, \ldots, \kappa\}$, we let

$$\phi^\kappa_{\mathrm{ms}} := \{((j,k),(k,j)) \in S_\kappa \times S_\kappa \mid j, k \in S_\kappa, j \neq k\}.$$

Then the pair $(S_\kappa, \phi^\kappa_{\mathrm{ms}})$ is an interaction, which we call the *multi-species exclusion interaction*. We have $c_{\phi^\kappa_{\mathrm{ms}}} = \dim_{\mathbb{R}} \mathrm{Consv}^{\phi^\kappa_{\mathrm{ms}}}(S_\kappa) = \kappa$, and we have a basis $\xi^1, \ldots, \xi^\kappa$ of $\mathrm{Consv}^{\phi^\kappa_{\mathrm{ms}}}(S_\kappa)$ called the *standard basis* defined as $\xi^i(0) = 0$ and $\xi^i(j) = \delta_{ij}$ for any integer $i, j = 1, \ldots, \kappa$. This interaction underlies the multi-color exclusion process studied in Dermoune and Heinrich (2008) and Halim and Hacène (2009), as well as the multi-species exclusion process studied in Nagahata and Sasada (2011). The case $\kappa = 1$ coincides with the exclusion interaction ϕ_{ex} defined in (1.2), hence $\phi^1_{\mathrm{ms}} = \phi_{\mathrm{ex}}$.

The following two-species exclusion process with annihilation and creation gives an example of an interaction which is *not* an interaction given by a map as in Bannai et al. (2024). See Theorem 3.2 and also compare with Bannai et al. (2024, Example 2.10 (4)).

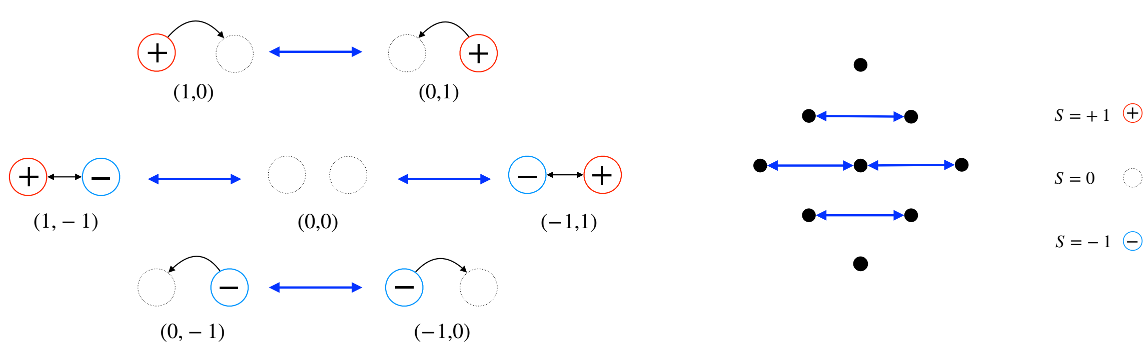

FIGURE 3.5. Interaction of Theorem 3.5 underlying the two-species exclusion process with annihilation and creation studied in Sasada (2010).

Example 3.5. Let $S = \{-1, 0, +1\}$, and we let $\phi \subset (S \times S) \times (S \times S)$ be the interaction given by

$$(-1,0) \leftrightarrow (0,-1), \qquad (1,0) \leftrightarrow (0,1), \qquad (1,-1) \leftrightarrow (-1,1), \qquad (1,-1) \leftrightarrow (0,0) \leftrightarrow (-1,1),$$

where $\leftrightarrow$ denotes the existence of an edge connecting the vertices. Then (S, ϕ) is an interaction underlying the two-species exclusion process with annihilation and creation studied in Sasada (2010). We have $c_\phi = 1$, and $\mathrm{Consv}^\phi(S)$ is spanned by $\xi\colon S \to \mathbb{R}$ given by $\xi(j) = j$ for $j = -1, 0, 1$.

Note that the interactions of Theorem 3.4 and Theorem 3.5 are both exchangeable in the sense of Theorem 3.11. We remark that the two color exclusion process studied in Quastel (1992) is a variant of Theorem 3.4 with $\kappa = 2$ removing the edge $(1,2) \leftrightarrow (2,1)$, hence it is not exchangeable.

Now, let (X, E) be a locally finite symmetric digraph. For any conserved quantity $\xi \in \mathrm{Consv}^{\phi}(S)$, if we let $\xi_x(\eta) := \xi(\eta_x)$ for any $\eta = (\eta_x)_{x\in X} \in S^X$, then ξ_x gives a function which is by definition local at x of radius 0. Hence $(\xi_x)_{x\in X}$ is a system which is uniformly local.

Definition 3.6. For any conserved quantity $\xi \in \mathrm{Consv}^{\phi}(S)$, we let

$$\xi_X := \sum_{x\in X} \xi_x \in C^0_{\mathrm{unif}}(S^X)$$

be the uniform function (due to Theorem 2.9) associated to the uniformly local system $(\xi_x)_{x\in X}$. The correspondence $\xi \mapsto \xi_X$ gives an $\mathbb{R}$-linear homomorphism $\mathrm{Consv}^{\phi}(S) \hookrightarrow C^0_{\mathrm{unif}}(S^X)$.

We next consider the transition structure of an interaction on the configuration space. For any $e \in E$, we let $o_e, t_e \in X$ be the origin and target of e so that $e = (o_e, t_e)$. We define the transition structure Φ_E of an interaction (S, ϕ) on the configuration space S^X as follows.

Definition 3.7. We define the *transition structure* $\Phi_E \subset S^X \times S^X$ on the configuration space S^X by

$$\Phi_E := \left\{(\eta, \eta') \in S^X \times S^X \mid \exists e \in E, \forall x \notin \{o_e, t_e\},\ \eta_x = \eta'_x \text{ and } ((\eta_{o_e}, \eta_{t_e}), (\eta'_{o_e}, \eta'_{t_e})) \in \phi \right\}.$$

Then since $(S \times S, \phi)$ is a symmetric digraph, the configuration space with transition structure (S^X, Φ_E) is also a symmetric digraph.

The topology of the graph (S^X, Φ_E) reflects information concerning the LSIS. If η and η' are in the same connected component of (S^X, Φ_E), then there exists a finite sequence of possible transitions from η to η'. In particular, since the graph (S^X, Φ_E) is symmetric, a microscopic stochastic process associated to the interaction (S, ϕ) would be irreducible on each connected component of (S^X, Φ_E). Therefore, intuitively, the equilibrium states of this stochastic process should be characterized to some extent by the connected components of (S^X, Φ_E). Since the macroscopic variables that appear in the hydrodynamic limit equation are expected to characterize the equilibrium states of the microscopic stochastic process, in order to understand how macroscopic variables are determined by the microscopic dynamics, it is important to study the connected components of the graph (S^X, Φ_E). Knowing the connected components of a graph can be reduced to studying functions that are constant on the connected components. For this reason, we are interested in functions that remain constant on the connected components, or in other words, functions invariant via transitions of (S^X, Φ_E).

The uniform functions of Theorem 2.4, by themselves, are not functions on S^X. Hence we first define what is meant for a uniform function to be invariant via transitions of (S^X, Φ_E). For any $f \in C^0_{\mathrm{unif}}(S^X)$, since f is not generally a function on S^X, the values $f(\eta)$ and $f(\eta')$ can not be defined for $\eta, \eta' \in S^X$. However, given a finite sequence of transitions from η to η', we can define a well-defined difference $f(\eta') - f(\eta)$ as follows. Due to this property, uniform functions can be understood as a kind of potential on the configuration space.

Lemma 3.8. *For any uniform function $f \in C^0_{\mathrm{unif}}(S^X)$ and $\eta, \eta' \in S^X$ such that $\Delta_{\eta,\eta'} := \{x \in X \mid \eta_x \neq \eta'_x\}$ is a finite set, the difference*

$$f(\eta') - f(\eta)$$

gives a well-defined value. In particular, given any η and η' in the same connected component of (S^X, Φ_E), the difference $f(\eta') - f(\eta)$ gives a well-defined value.

Proof: Consider a base state $* \in S$ and an expansion $f = \sum_{\Lambda\in\mathscr{I}_R} f^*_\Lambda$ of Theorem 2.3 for some $R > 0$. Then for any $\Lambda \in \mathscr{I}_R$ such that $\Lambda \cap \Delta_{\eta,\eta'} = \emptyset$, we have $f^*_\Lambda(\eta) = f^*_\Lambda(\eta')$. Hence the sum of the differences

$$f(\eta') - f(\eta) := \sum_{\Lambda\cap\Delta_{\eta,\eta'}\neq\emptyset} (f^*_\Lambda(\eta') - f^*_\Lambda(\eta)) \tag{3.2}$$

gives a well-defined finite sum and is independent of the choice of the base state $* \in S$. By definition, this difference extends the difference for local functions. The statement for a finite sequence of transitions from η to η' follows from the fact that $\Delta_{\eta,\eta'}$ is finite in this case. □

Definition 3.9. We say that a uniform function f is *invariant via transitions of* (S^X, Φ_E), if for any transition $(\eta, \eta') \in \Phi_E$, the difference $f(\eta') - f(\eta) = 0$.

Let f be a uniform function which is invariant via transitions of (S^X, Φ_E). Then by definition, the realization $f \in C^0_{\mathrm{unif}}(S^X_*)$ for any base state $* \in S^X$ is a function which is constant on the connected components of (S^X_*, Φ^*_E) where $\Phi^*_E := \Phi_E \cap (S^X_* \times S^X_*)$. The uniform functions ξ_X obtained from conserved quantities $\xi \in \mathrm{Consv}^\phi(S)$ give examples of uniform functions which are invariant via transitions, due to Theorem 3.10 below.

Lemma 3.10. *For any conserved quantity $\xi \in \mathrm{Consv}^\phi(S)$, the uniform function ξ_X of Theorem 3.6 is invariant via transitions of (S^X, Φ_E).*

Proof: We select a base state $* \in S$. For any $\xi \in \mathrm{Consv}^\phi(S)$, we take the normalization $\xi(*) = 0$. Then $\xi_x \in C_{\{x\}}(S^X_*)$. Hence the sum $\xi_X = \sum_{x \in X} \xi_x$ is in fact the expansion given in (2.1). Suppose $(\eta, \eta') \in \Phi_E$. By the definition of Φ_E, there exists $e = (x, y) \in E$ such that $\eta_z = \eta'_z$ for $z \neq x, y$ and $((\eta_x, \eta_y), (\eta'_x, \eta'_y)) \in \phi$. Hence by the definition of a conserved quantity (3.1), we have $\xi(\eta_x) + \xi(\eta_y) = \xi(\eta'_x) + \xi(\eta'_y)$. This shows that

$$\xi_X(\eta') - \xi_X(\eta) = 0,$$

hence that ξ_X is invariant via transitions of (S^X, Φ_E). □

As mentioned in the Introduction, the following notion of the exchangeability plays an essential role in our main theorem.

Definition 3.11. We say that an interaction (S, ϕ) is *exchangeable* if and only if for any $(s_1, s_2) \in S \times S$, (s_1, s_2) and (s_2, s_1) are in the same connected component of the graph $(S \times S, \phi)$.

We can now state our main theorem.

Theorem 3.12. *Suppose (S, ϕ) is an interaction which is exchangeable, and let (X, E) be a connected and locally finite symmetric digraph such that X is infinite. Let $f \in C^0_{\mathrm{unif}}(S^X)$ be a uniform function which is invariant via transitions of (S^X, Φ_E). Then there exists a conserved quantity $\xi\colon S \to \mathbb{R}$ such that $f = \xi_X$.*

As discussed in the Introduction, the global observable of the entire LSIS expressed by a uniform function which is invariant via transitions can be regarded as a global conserved quantity. The above theorem implies that the global conserved quantity of the entire system is always given as the sum of a local conserved quantity ξ associated to a local interaction.

We will prove Theorem 3.12 at the end of this section. In what follows, we assume that (S, ϕ) is an interaction which is exchangeable and that (X, E) is connected and locally finite. We first prove that the reshuffling of the components of a configuration $\eta = (\eta_x)_{x \in X} \in S^X$ in a configuration space for a connected graph preserves the connected components of (S^X, Φ_E). In order to do so, consider $\eta = (\eta_x)_{x \in X} \in S^X$. For any $x, y \in X$, let $\eta^{x,y}$ be the configuration obtained from η by exchanging the x and y components, namely $\eta^{x,y}_x := \eta_y$ and $\eta^{x,y}_y := \eta_x$. For any $e \in E$, since we have assumed that (S, ϕ) is exchangeable, (η_{o_e}, η_{t_e}) and (η_{t_e}, η_{o_e}) are in the same connected component of $(S \times S, \phi)$. Hence the configurations η^{o_e, t_e} and η are in the same connected component of (S^X, Φ_E). More generally, for $x, y \in X$, since we have assumed that (X, E) is connected, there exists a path $\vec{\gamma} = (e^1, \dots, e^N)$ from x to y. By successively exchanging the components of η with respect to o_{e^i}, t_{e^i} for the edges e^i in $\vec{\gamma}$ from $i = 1$ to N and then in the opposite order from $i = N - 1$ to $i = 1$, we see that $\eta^{x,y}$ and η are in the same connected component of (S^X, Φ_E). This gives the following.

Lemma 3.13. *For a finite $\Lambda \subset X$, let $\sigma\colon \Lambda \to \Lambda$ be a bijection. Then for any configuration $\eta = (\eta_x)_{x \in X} \in S^X$, the configuration η^σ given by $\eta^\sigma_x := \eta_{\sigma(x)}$ for $x \in \Lambda$ and $\eta^\sigma_x = \eta_x$ for $x \notin \Lambda$ is in the same connected component of (S^X, Φ_E) as that of η.*

Proof: This follows from the fact that any bijection $\sigma\colon \Lambda \to \Lambda$ is obtained by successively exchanging two elements of Λ, and that $\eta^{x,y}$ and η are in the same connected component of (S^X, Φ_E) for any $x, y \in X$. □

We next prove the following.

Lemma 3.14. *Consider $f \in C(S^X_*)$ for a base state $* \in S$. For any $x \in X$, let $f^*_{\{x\}}$ be the function with exact support $\{x\}$ in the expansion given in (2.1). If f is invariant via transitions of (S^X, Φ_E), then*

$$\forall x, x' \in X,\ \forall s \in S, \quad f^*_{\{x\}}(s) = f^*_{\{x'\}}(s).$$

Namely, the functions

$$f^*_{\{x\}}\colon S \to \mathbb{R}$$

are equal as functions on S for all $x \in X$.

Proof: Let $s \in S$ and $x, x' \in X$. Consider configurations $\eta = (\eta_y)_{y \in X}, \eta' = (\eta'_z)_{z \in X} \in S^X_*$ such that $\eta_x = \eta'_{x'} = s$ and $\eta_y = \eta'_z = *$ whenever $y \neq x$ and $z \neq x'$. By Theorem 3.13, the configurations η and η' are in the same connected component of $S^X_* \subset S^X$, so there exists a sequence of transitions from η to η'. Since f is invariant via transitions, we have

$$f^*_{\{x\}}(s) = f^*_{\{x\}}(\eta) = f(\eta) - f(\star) = f(\eta') - f(\star) = f^*_{\{x'\}}(\eta') = f^*_{\{x'\}}(s)$$

as desired. For the first and last equalities, we have used the identification of $C(S)$ with $C(S^{\{x\}})$ and $C(S^{\{x'\}}) \subset C(S^X_*)$. □

Lemma 3.15. *Assume that X is infinite, and let $f \in C^0_{\mathrm{unif}}(S^X)$. If f is invariant via transitions of (S^X, Φ_E), then for a base state $* \in S$ and f normalized so that $f(\star) = 0$, we have*

$$f = \sum_{x \in X} f^*_{\{x\}}$$

for the expansion of Theorem 2.3.

Proof: Since f is uniform, there exists $R > 0$ such that $f^*_\Lambda = 0$ if $\operatorname{diam}(\Lambda) > R$. It is sufficient to prove that for any finite $\Lambda \in \mathscr{I}$ such that $|\Lambda| \geq 2$, we have $f^*_\Lambda = 0$. We consider $\Lambda \in \mathscr{I}$, and assume that $f^*_{\Lambda''} = 0$ for any Λ'' such that $2 \leq |\Lambda''| < |\Lambda|$. This condition is trivially true if $|\Lambda| = 2$. Take $\Lambda' \subset X$ such that $|\Lambda'| = |\Lambda|$ and $\operatorname{diam}(\Lambda') > R$. Such Λ' exists since (X, E) is locally finite and infinite. By construction, $f^*_{\Lambda'} = 0$. Consider any $\eta \in S^\Lambda$, and denote again by η the configuration in S^X_* whose components coincide with that of η for $x \in \Lambda$ and are at base state for $x \notin \Lambda$. We fix a bijection $\Lambda \cong \Lambda'$, $x \mapsto x'$ and let η' be the configuration in S^X_* such that $\eta'_{x'} := \eta_x$ for $x' \in \Lambda'$ and is at base state for $x' \notin \Lambda'$. Since η' is obtained from η by rearranging the components, we see from Theorem 3.13 that η and η' are in the same connected component of S^X_*. Hence there exists a finite sequence of transitions from η to η'. Since f is invariant via transitions of (S^X, Φ_E), we have $f(\eta) = f(\eta')$. Then

$$f(\eta) = \iota^\Lambda_* f(\eta) = f^*_\Lambda(\eta) + \sum_{\Lambda'' \subsetneq \Lambda} f^*_{\Lambda''}(\eta) = f^*_\Lambda(\eta) + \sum_{x \in \Lambda} f^*_{\{x\}}(\eta),$$

$$f(\eta') = \iota^{\Lambda'}_* f(\eta') = f^*_{\Lambda'}(\eta') + \sum_{\Lambda'' \subsetneq \Lambda'} f^*_{\Lambda''}(\eta') = \sum_{x \in \Lambda'} f^*_{\{x\}}(\eta')$$

since $f^*_{\Lambda'} = 0$. By Theorem 3.14, we have $\sum_{x \in \Lambda} f^*_{\{x\}}(\eta) = \sum_{x \in \Lambda'} f^*_{\{x\}}(\eta')$. This shows that for any $\eta \in S^\Lambda$, we have $f^*_\Lambda(\eta) = 0$, hence $f^*_\Lambda = 0$. Induction on the number of elements of Λ completes the proof. □

Proof of Theorem 3.12: For $x \in X$, let $\xi := f^*_{\{x\}}\colon S \to \mathbb{R}$ be viewed as a function on S. By Theorem 3.14, the function ξ on S is independent of the choice of x, and we have $\xi_x = f^*_{\{x\}}$ as a function on S^X, and by Theorem 2.3 and Theorem 3.15, we have $f = \sum_{x \in X} \xi_x$. In order to show that ξ is a conserved quantity, consider an edge $e = (x_1, x_2) \in E$. Consider any $(s_1, s_2), (s'_1, s'_2) \in S \times S$ such that $((s_1, s_2), (s'_1, s'_2)) \in \phi$.

Let $\eta, \eta' \in S_*^X$ be the configurations such that $(\eta_{x_1}, \eta_{x_2}) = (s_1, s_2)$, $(\eta'_{x_1}, \eta'_{x_2}) = (s'_1, s'_2)$, and η_x, η'_x are at base state for $x \neq x_1, x_2$. Then by the construction, we have $(\eta, \eta') \in \Phi_E$. This shows that

$$\xi(s_1) + \xi(s_2) = \xi_{x_1}(\eta) + \xi_{x_2}(\eta) = f(\eta) = f(\eta') = \xi_{x_1}(\eta') + \xi_{x_2}(\eta') = \xi(s'_1) + \xi(s'_2).$$

This proves that ξ is a conserved quantity, and $f = \sum_{x \in X} \xi_x = \xi_X$ as desired. □

4. The 0-th Uniform Cohomology of the Configuration Space

In this section, we will interpret our main result Theorem 3.12 in terms of the 0-th cohomology of the configuration space with transition structure (S^X, Φ_E). We first review the cohomology of general graphs (see for example Bannai et al., 2024, Appendix A).

Definition 4.1. For any symmetric digraph (X, E), we let

$$C(X) := \mathrm{Map}(X, \mathbb{R}), \qquad C^1(X) := \mathrm{Map}^{\mathrm{alt}}(E, \mathbb{R}),$$

where $\mathrm{Map}^{\mathrm{alt}}(E, \mathbb{R}) := \{\omega \colon E \to \mathbb{R} \mid \forall (x, y) \in E,\ \omega(y, x) = -\omega(x, y)\}$. Furthermore, we define the differential

$$\partial \colon C(X) \to C^1(X), \qquad f \mapsto \partial f$$

by $\partial f(e) := f(t_e) - f(o_e)$ for any $e \in E$. We define the *cohomology* of (X, E) by

$$H^0(X) := \mathrm{Ker}\, \partial, \qquad H^1(X) := C^1(X)/\partial C(X),$$

and $H^m(X) := \{0\}$ for any $m \in \mathbb{N}$ such that $m \neq 0, 1$.

The 0-th cohomology $H^0(X)$ is known to be the space of functions which are constant on the connected components of (X, E) (see Bannai et al., 2024, Proposition A.8). We will apply the above construction to the configuration space with transition structure. Let (S, ϕ) be an interaction, and let (X, E) be a locally finite connected symmetric digraph. We let (S^X, Φ_E) be the configuration space with transition structure of Theorem 3.7 associated to (S, ϕ) and (X, E). Following Theorem 4.1, the cohomology of the configuration space with transition structure is given by

$$C(S^X) := \mathrm{Map}(S^X, \mathbb{R}), \qquad C^1(S^X) := \mathrm{Map}^{\mathrm{alt}}(\Phi_E, \mathbb{R}),$$

where $\mathrm{Map}^{\mathrm{alt}}(\Phi_E, \mathbb{R}) := \{\omega \colon \Phi_E \to \mathbb{R} \mid \forall (\eta, \eta') \in \Phi_E\ \omega(\eta', \eta) = -\omega(\eta, \eta')\}$. Furthermore, we define the differential

$$\nabla \colon C(S^X) \to C^1(S^X), \qquad f \mapsto \nabla f \tag{4.1}$$

by $\nabla f(\eta, \eta') := f(\eta') - f(\eta)$ for any $(\eta, \eta') \in \Phi_E$. We define the *cohomology* of (S^X, Φ_E) by

$$H^0(S^X) := \mathrm{Ker}\, \nabla, \qquad H^1(S^X) := C^1(S^X)/\nabla C(S^X), \tag{4.2}$$

and $H^m(S^X) := \{0\}$ for any $m \in \mathbb{N}$ such that $m \neq 0, 1$.

When X is infinite, the graph (S^X, Φ_E) generally has an infinite number of connected components. Since the cohomology $H^0(S^X)$ is equivalent to functions on X which are constant on the connected components of (X, E), this implies that $H^0(S^X)$ is infinite dimensional. We will replace the functions $C(S^X) = \mathrm{Map}(S^X, \mathbb{R})$ with the space of uniform functions to construct a suitable definition of uniform cohomology.

First, since the constant functions are mapped to zero, the operator given in (4.1) induces a differential $\nabla \colon C^0(S^X) \to C^1(S^X)$. The restriction to local functions gives

$$\nabla \colon C^0_{\mathrm{loc}}(S^X) \to C^1(S^X). \tag{4.3}$$

We can linearly extend this differential to uniform functions as follows.

Proposition 4.2. *The differential* $\nabla \colon C^0_{\mathrm{loc}}(S^X) \to C^1(S^X)$ *extends to a differential*

$$\nabla \colon C^0_{\mathrm{unif}}(S^X) \to C^1(S^X)$$

on the space of uniform functions.

Proof: For any $(\eta,\eta') \in \Phi_E$, by definition, there exists $(x,y) \in E$ such that $\eta_z = \eta'_z$ for $z \neq x, y$. Hence $\Delta_{\eta,\eta'} = \{x \in X \mid \eta_x \neq \eta'_x\}$ is a finite set. By Theorem 3.8, we have a well-defined difference $\nabla f(\eta,\eta') = f(\eta') - f(\eta)$. Hence a function $\nabla f\colon \Phi_E \to \mathbb{R}$ is well-defined. The map $f \mapsto \nabla f$ gives a differential linearly extending the differential operator given in (4.3). □

Using this differential, we can define the 0-th uniform cohomology as follows.

Definition 4.3. Following (4.2), we define the 0-th uniform cohomology of (S^X, Φ_E) by

$$H^0_{\mathrm{unif}}(S^X) := \mathrm{Ker}\left(\nabla\colon C^0_{\mathrm{unif}}(S^X) \to C^1(S^X)\right).$$

As an analogy of the fact that 0-th cohomology $H^0(X)$ coincides with the space of functions constant on the connected components of (X, E) (see Bannai et al., 2024, Proposition A.8), we can prove that the 0-th uniform cohomology $H^0_{\mathrm{unif}}(S^X)$ coincides with the space of functions constant on the connected components of (S^X, Φ_E), i.e. functions invariant via transitions of (S^X, Φ_E).

Lemma 4.4. *Consider a uniform function $f \in C^0_{\mathrm{unif}}(S^X)$. Then $f \in H^0_{\mathrm{unif}}(S^X)$ if and only if f is invariant via transitions of (S^X, Φ_E).*

Proof: Let $f \in H^0_{\mathrm{unif}}(S^X)$. Then for any $(\eta,\eta') \in \Phi_E$, since $\nabla f = 0$, we have

$$\nabla f(\eta,\eta') = f(\eta') - f(\eta) = 0.$$

This proves that f is invariant via transitions of (S^X, Φ_E). Conversely, suppose f is invariant via transitions of (S^X, Φ_E). Then for any $(\eta,\eta') \in \Phi_E$, we have $\nabla f(\eta,\eta') = f(\eta') - f(\eta) = 0$. This proves that $f \in H^0_{\mathrm{unif}}(S^X)$ as desired. □

Our main theorem gives the following corollary.

Corollary 4.5. *Let (S, ϕ) be an interaction which is exchangeable, and let (X, E) be a connected and locally finite symmetric digraph such that X is infinite. Then we have a canonical isomorphism $\mathrm{Consv}^\phi(S) \cong H^0_{\mathrm{unif}}(S^X)$, given by $\xi \mapsto \xi_X$ for any conserved quantity $\xi \in \mathrm{Consv}^\phi(S)$.*

Proof: Suppose $\xi \in \mathrm{Consv}^\phi(S)$. Then by Theorem 3.10, the uniform function ξ_X is invariant via transitions of (S^X, Φ_E). Hence by Theorem 4.4, we have $\xi_X \in H^0_{\mathrm{unif}}(S^X)$. This gives a homomorphism $\iota\colon \mathrm{Consv}^\phi(S) \to H^0_{\mathrm{unif}}(S^X)$. Fix a base $* \in S$ and take a representative of ξ so that $\xi(*) = 0$, and let $x \in X$. Then for any $s \in S$, let $\eta^s \in S^X$ be the configuration such that $\eta^s_x := s$ and $\eta^s_z := *$ for $z \neq x$. Then $\xi(s) = \xi_X(\eta^s)$. Hence if $\xi_X = 0$ in $H^0_{\mathrm{unif}}(S^X)$, then this implies that $\xi(s) = 0$ for any $s \in S$, hence ι is injective. Next, suppose $f \in H^0_{\mathrm{unif}}(S^X)$. Then by Theorem 4.4, the uniform function f is invariant via transitions of (S^X, Φ_E). Hence by Theorem 3.12, there exists $\xi \in \mathrm{Consv}^\phi(S)$ such that $f = \xi_X$. This proves that ι is surjective, hence that we have an isomorphism $\mathrm{Consv}^\phi(S) \cong H^0_{\mathrm{unif}}(S^X)$ as desired. □

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