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# Statistics on monotonically ordered non-crossing partitions

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**Abstract.** We study some combinatorial statistics defined on the set $NC^{(\mathrm{mton})}(n)$ of monotonically ordered non-crossing partitions of $\{1,\ldots,n\}$, and on the set $NC_2^{(\mathrm{mton})}(2n)$ of monotonically ordered non-crossing pair-partitions of $\{1,\ldots,2n\}$. Unlike in the analogous results known for unordered non-crossing partitions, the computations of expectations and variances for natural block-counting statistics on $NC^{(\mathrm{mton})}(n)$ and for the expectation of the area statistic on $NC_2^{(\mathrm{mton})}(2n)$ turn out to yield a logarithmic regime. An important role in our study is played by a nice tree structure on the disjoint union of the $NC^{(\mathrm{mton})}(n)$'s, which we use to streamline our arguments. As an illustration of how these ideas can be applied to calculations of cumulants in monotone probability, we discuss some combinatorial aspects of the monotonic Poisson process.

## 1. Introduction

### 1.1. $\boldsymbol{NC(n)}$ *and* $\boldsymbol{NC^{(\mathrm{mton})}(n)}$. *The* $\boldsymbol{NC^{(\mathrm{mton})}}$*-tree structure.*

We will use the customary notation $NC(n)$ for the set of non-crossing partitions of $\{1,\ldots,n\}$. The elements of $NC(n)$ are thus of the form $\pi = \{V_1,\ldots,V_k\}$, where $V_1,\ldots,V_k$ (the *blocks* of the partition $\pi$) are non-empty pairwise disjoint subsets of $\{1,\ldots,n\}$, with $V_1 \cup \cdots \cup V_k = \{1,\ldots,n\}$,

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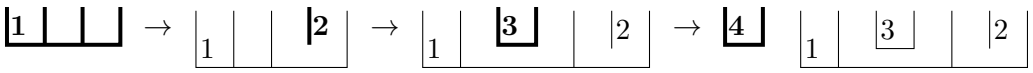



FIGURE 1. $\pi = \{\{1,2\},\{3,4,7,9\},\{5,6\},\{8\}\} \in NC(9)$ *built in 4 steps, by inserting intervals. The building process gives a monotonic ordering* $u$ *of* $\pi$*: every block of* $\pi$ *retains the label assigned to it (in boldface font) at the moment when it was inserted in the picture, thus giving* $u(\{3,4,7,9\}) = 1$, $u(\{8\}) = 2$, $u(\{5,6\}) = 3$, $u(\{1,2\}) = 4$.

and where it is not possible to find $a_1 < a_2 < a_3 < a_4$ in $\{1,\ldots,n\}$ such that $a_1, a_3 \in V_i$ and $a_2, a_4 \in V_j$ for some $i \neq j$. The $NC(n)$'s are one of the many combinatorial structures counted by Catalan numbers. For a quick introduction to some of their basic properties, one can e.g. consult Nica and Speicher (2006, Lecture 9).

This paper is concerned with sets of *monotonically ordered* non-crossing partitions.

**Definition 1.1.** Let $n$ be a positive integer.
$1^o$ Let $\pi = \{V_1,\ldots,V_k\} \in NC(n)$. One has a nesting relation among the blocks of $\pi$, where "$V_i$ nested inside $V_j$" means, by definition, that $\min(V_i) > \min(V_j)$ and $\max(V_i) < \max(V_j)$. We will use the term *monotonic ordering of* $\pi$ to refer to a bijection $u : \pi \to \{1,\ldots,k\}$ having the property that $\big[V_i \text{ nested inside } V_j\big] \Rightarrow \big[u(V_i) > u(V_j)\big]$.
$2^o$ We denote $NC^{(\mathrm{mton})}(n) := \big\{(\pi,u) \mid \pi \in NC(n) \text{ and } u \text{ is a monotonic ordering of } \pi\big\}$.

*Remark* 1.2. For $(\pi,u) \in NC^{(\mathrm{mton})}(n)$, one can think of $u$ as "a possible history of how $\pi$ was created, via successive insertions of intervals". In order to clarify what this means, let us write explicitly $\pi = \{V_1,\ldots,V_k\}$ and let us note that the block $V_{j_o} := u^{-1}(k) \in \pi$ is an interval of $\{1,\ldots,n\}$. (If not, then there would exist $i \neq j_o$ in $\{1,\ldots,k\}$ such that $V_i$ is nested inside $V_{j_o}$, hence such that $u(V_i) > u(V_{j_o}) = k$, in contradiction with $u(V_i) \in \{1,\ldots,k\}$.) One can view the insertion of the interval $V_{j_o}$ as the last step in a $k$-step process which created $\pi$, via successive insertions of intervals which eventually became the blocks $u^{-1}(1),\ldots,u^{-1}(k)$, in this order. For illustration, Figure 1 shows how this process works to create a monotonically ordered partition with 4 blocks in $NC(9)$.

On a heuristic level, one could say that the relation between a $(\pi,u) \in NC^{(\mathrm{mton})}(n)$ and the underlying $\pi \in NC(n)$ is akin to the relation one has, in the representation theory of symmetric groups, between a standard Young tableau and the underlying Young diagram – the tableau records a possible history of how the Young diagram was created, via addition of boxes. From this perspective, the $NC^{(\mathrm{mton})}$-tree that we will next look at is an analogue of the tree of monotonic paths of the well-known Young graph (as presented, for instance, in Section 3.1 of the monograph of Borodin and Olshanski, 2017).

*Remark* 1.3. *(The* $NC^{(\mathrm{mton})}$*-tree.)* In the considerations of the present paper, a crucial role will be played by a tree structure with vertex set $NC^{(\mathrm{mton})} := \sqcup_{n=1}^{\infty} NC^{(\mathrm{mton})}(n)$ (disjoint union), where the unique element of $NC^{(\mathrm{mton})}(1)$ serves as root and where, for every $n \in \mathbb{N}$, the elements of $NC^{(\mathrm{mton})}(n)$ are at distance $n-1$ from the root. The formal description of the edges of this $NC^{(\mathrm{mton})}$-tree is given in Section 2.1 below. Roughly speaking, it is based on the interpretation of every $(\pi,u) \in NC^{(\mathrm{mton})}(n)$ as possible history for how the underlying $\pi = \{V_1,\ldots,V_k\} \in NC(n)$ was created, but where we amend the $k$-step creation process described in Remark 1.2, and make it become an *$n$-step* creation process (use the number $n$ of points that are being partitioned, rather than the number $k$ of blocks of $\pi$). That is: instead of "intervals inserted from above" as in Figure 1, we will now look at insertions of individual points, as shown in Figure 2. At every step, the

new incoming point: either starts a new block of the partition which is being built, or appends itself to the highest-numbered block (necessarily an interval block) that already existed in the picture. Specifically, Figure 2 shows the same partition $\pi \in NC^{(\mathrm{mton})}(9)$ as in Figure 1, where the construction of $\pi$ now takes 9 steps (rather than 4 steps, as before).

*Remark* 1.4. *(Homogeneity property of the $NC^{(\mathrm{mton})}$-tree.)* One has that:

$$\begin{cases} \text{for every } n \in \mathbb{N}, \text{ every } (\pi, u) \in NC^{(\mathrm{mton})}(n) \\ \text{has precisely } n+2 \text{ children in the } NC^{(\mathrm{mton})}\text{-tree.} \end{cases} \tag{1.1}$$

This is a fact that is easy to verify (cf. Remark 2.5 below), but nevertheless has great benefits towards the study of block-counting statistics on $NC^{(\mathrm{mton})}(n)$, because it provides recursions satisfied by such statistics. A number of results obtained in this way are described in the next subsection.

We note that, as a byproduct, the homogeneity from (1.1) can be used as an entry point towards some non-commutative probability considerations. It is relevant to mention here that $NC(n)$ and $NC^{(\mathrm{mton})}(n)$ play crucial roles in the combinatorics of two brands of independence for non-commutative random variables, *free independence* and respectively *monotonic independence*. In particular, each of these two brands of independence uses a notion of *cumulants* for the non-commutative random variables that are considered, where the cumulants of free probability are defined (see e.g. Nica and Speicher, 2006, Lecture 11) with the help of the $NC(n)$'s, while the cumulants of monotonic probability are defined (cf. Hasebe and Saigo, 2011) with the help of the $NC^{(\mathrm{mton})}(n)$'s. The homogeneity (1.1) can be used in order to obtain recursion formulas for monotonic cumulants. The present paper does not aim to detail this direction, but there is one immediate example that felt worth describing, concerning the recursion for the moments of the so-called "monotonic Poisson process" – cf. Section 5 below, where we succinctly review the monotonic Poisson process, and use (1.1) to retrieve some results from Belton (2006) about the cumulants of the said process.

### 1.2. *Block-counting statistics on* $\boldsymbol{NC^{(\mathrm{mton})}(n)}$.

For motivation, we mention that there exists a fair amount of research literature (see e.g. Arizmendi, 2012, Ortmann, 2012, or the more recent presentation made in Kargin, 2019) which is dedicated to

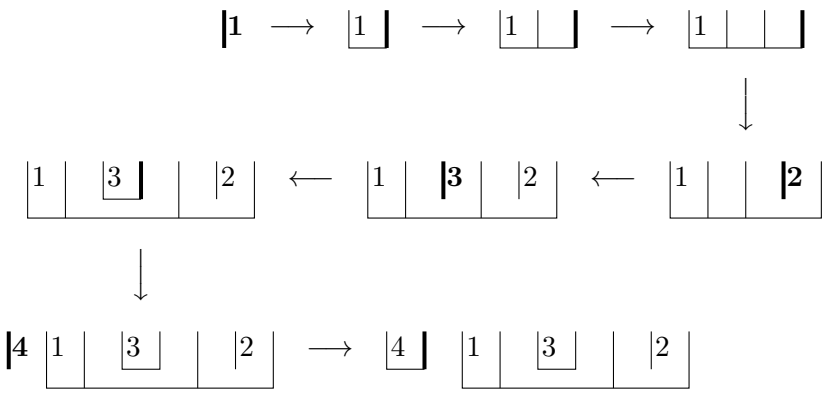



FIGURE 2. $\pi = \{\{1,2\},\{3,4,7,9\},\{5,6\},\{8\}\} \in NC(9)$ *built in 9 steps, by inserting points. The building process gives the path in the $NC^{(\mathrm{mton})}$-tree which starts at the root of the tree and arrives at the monotonically ordered non-crossing partition from Figure 1.*

the study of block-statistics on $NC(n)$, with $NC(n)$ viewed as a uniform probability space. The calculations made in this regime often yield quantities that, asymptotically, grow linearly with $n$. For instance if $X_n : NC(n) \to \mathbb{N}$ is the random variable that counts blocks, $X_n(\pi) := |\pi| = k$ for $\pi = \{V_1, \ldots, V_k\} \in NC(n)$, then the expectation and variance of $X_n$ are found to be

$$E[X_n] = \frac{n+1}{2} \quad \text{and} \quad \mathrm{Var}[X_n] = \frac{n^2-1}{4(2n-1)} \sim \frac{n}{8} \tag{1.2}$$

(see e.g. Kargin, 2019, Theorem 2.1). Another relevant result is that upon fixing an $\ell \in \mathbb{N}$ and considering the random variables $X_n^{(\ell)} : NC(n) \to \mathbb{N} \cup \{0\}$ which are defined by

$$X_n^{(\ell)}(\pi) := \big|\, \{V \in \pi \mid |V| = \ell\} \,\big|, \quad \pi \in NC(n),$$

one gets that $E[X_n^{(\ell)}] \sim n/2^{\ell+1}$ for $n \to \infty$ (see e.g. Ortmann, 2012, Corollary 4.3).

In the present paper we look at the analogous block-counting random variables, in the parallel world of $NC^{(\mathrm{mton})}(n)$.

*Remark and Notation* 1.5. $1^o$ For every $n \in \mathbb{N}$, we view $NC^{(\mathrm{mton})}(n)$ as a uniform probability space, where every $(\pi, u) \in NC^{(\mathrm{mton})}(n)$ has probability $1/|NC^{(\mathrm{mton})}(n)|$. The latter cardinality is $|NC^{(\mathrm{mton})}(n)| = (n+1)!/2$, as had been noted in the literature related to monotonic probability (cf. Arizmendi et al., 2015, Proposition 3.4), and can also be easily derived from the homogeneity property (1.1) of the $NC^{(\mathrm{mton})}$-tree.

$2^o$ For every $n \in \mathbb{N}$, we let $Y_n : NC^{(\mathrm{mton})}(n) \to \mathbb{N}$ be the random variable defined by

$$Y_n(\pi, u) := |\pi| \text{ (number of blocks of } \pi), \quad (\pi, u) \in NC^{(\mathrm{mton})}(n). \tag{1.3}$$

The next theorem gives the expectation and variance of $Y_n$. Unlike the formulas (1.2) concerning the $X_n$'s, the formulas for the $Y_n$'s show a logarithmic regime. In the statements of the latter formulas we will use the standard notation

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n\text{-th harmonic number, for } n \in \mathbb{N}),$$

and $\gamma := \lim_{n\to\infty} H_n - \ln n$ (the Euler-Mascheroni constant).

**Theorem 1.6.** *Let the random variables $Y_n : NC^{(\mathrm{mton})}(n) \to \mathbb{N}$ be as defined in (1.3).*

$1^o$ *One has $E[Y_n] = n - H_n + \frac{3}{2} - \frac{1}{n+1}$, $n \geq 2$. As a consequence, it follows that*

$$E[Y_n] \approx (n - \ln n) + \big(\frac{3}{2} - \gamma\big) \text{ for } n \to \infty, \tag{1.4}$$

*where the meaning of "$\approx$" in (1.4) is that the difference of the quantities on its two sides goes to 0 for $n \to \infty$.*

$2^o$ *One has $\mathrm{Var}[Y_n] = H_n - H_n^{(2)} - \frac{1}{4} + \frac{n}{(n+1)^2}$, $n \geq 2$, where $H_n^{(2)} := 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. As a consequence, it follows that*

$$\mathrm{Var}[Y_n] \approx \ln n - \big(\frac{\pi^2}{6} + \frac{1}{4} - \gamma\big), \text{ for } n \to \infty. \tag{1.5}$$

Suppose next that, for randomly chosen $(\pi, u) \in NC^{(\mathrm{mton})}(n)$, one tallies the blocks of $\pi$ according to their cardinality. The next theorem says that, for large $n$, one should expect $\pi$ to have a lot of blocks of size 1 (about $n - 2\ln n$ of them), a fairly significant number of blocks of size 2 (about $\ln n$ of them), and very few blocks of size $\geq 3$.

**Theorem 1.7.** *For every $n, \ell \in \mathbb{N}$, let $Y_n^{(\ell)} : NC^{(\mathrm{mton})}(n) \to \mathbb{N} \cup \{0\}$ be defined by*

$$Y_n^{(\ell)}(\pi, u) := \big|\, \{V \in \pi \mid |V| = \ell\} \,\big|, \quad (\pi, u) \in NC^{(\mathrm{mton})}(n). \tag{1.6}$$

*There exist constants $c_1, c_2, \ldots, c_\ell, \ldots \in \mathbb{R}$ such that:*

$$\begin{cases} \lim_{n\to\infty} E[Y_n^{(1)}] - (n - 2\ln n) = c_1, \ \lim_{n\to\infty} E[Y_n^{(2)}] - \ln n = c_2, \\ \text{and } \lim_{n\to\infty} E[Y_n^{(\ell)}] = c_\ell, \quad \forall\, \ell \geq 3. \end{cases} \tag{1.7}$$

*Remark* 1.8. $1^o$ Proposition 4.6 below gives the precise values $c_1 = \frac{10}{3} - 2\gamma$ and $c_2 = \gamma - \frac{17}{8}$. The values of $c_\ell$ for $\ell \geq 3$ can also be explicitly computed, but there doesn't appear to be some nice general formula for them (see discussion in Remark 4.7 below).

$2^o$ In the setting of the preceding theorem, one can in fact bundle together all the random variables $Y_n^{(\ell)}$ with $\ell \geq 3$ into $Y_n^{(\geq 3)} : NC^{(\mathrm{mton})}(n) \to \mathbb{N} \cup \{0\}$, where

$$Y_n^{(\geq 3)}(\pi, u) := \big|\, \{V \in \pi \mid |V| \geq 3\} \,\big|, \quad (\pi, u) \in NC^{(\mathrm{mton})}(n). \tag{1.8}$$

It is still the case that the expectations $E[\,Y_n^{(\geq 3)}\,]$ have a finite limit for $n \to \infty$. Indeed, by writing $Y_n^{(\geq 3)} = Y_n - Y_n^{(1)} - Y_n^{(2)}$, with $Y_n$ picked from Theorem 1.6, one immediately finds that $\lim_{n\to\infty} E[\,Y_n^{(\geq 3)}\,] = (\frac{3}{2} - \gamma) - c_1 - c_2 = \frac{7}{24}$.

*Remark* 1.9. We reiterate the fact that the formulas announced in Theorems 1.6 and 1.7 are derived by starting from the homogeneity property (1.1) of the $NC^{(\mathrm{mton})}$-tree. As an intermediate between the observation from (1.1) and explicit formulas for expectations and variances, we will put into evidence two kinds of recursions that may be satisfied by a sequence of random variables $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R})_{n=1}^{\infty}$.

– The recursion of the first kind uses for input a tuple $\underline{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k$, for some $k \in \mathbb{N}$. We will introduce it in Section 2.2, then in Section 4 we will show how this kind of recursion is used to approach the random variables $Y_n$ and $Y_n^{(\ell)}$ from Theorems 1.6 and 1.7 (both these theorems are consequences of the same general result about the recursion of the first kind, given in Theorem 3.2).

– The recursion of the second kind uses for input a triple $(\alpha, \beta; q)$, with $\alpha, \beta, q \in \mathbb{Z}$. We will introduce it in Section 2.3, and we will discuss it further in Section 6. This recursion of the second kind can be applied to the statistics counting outer blocks of $\pi$, in connection to a randomly picked $(\pi, u) \in NC^{(\mathrm{mton})}(n)$. A block $V$ of a partition $\pi \in NC(n)$ is said to be *outer* when it is not possible to find $W \in \pi$ such that $V$ is nested inside $W$. If $Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{N}$ is defined by putting $Z_n(\pi, u) := |\, \{V \in \pi \mid V \text{ is outer}\} \,|$, then (as we will see in Section 6, by invoking the recursion of the second kind) one has the neat formula $E[Z_n] = (2n+1)/3, \ \ \forall\, n \in \mathbb{N}$.

1.3. *The $\boldsymbol{NC_2^{(\mathrm{mton})}}$-tree, and some block-counting statistics on $\boldsymbol{NC_2^{(\mathrm{mton})}(2n)}$.*
In this paper we will also look at some combinatorial statistics on monotonically ordered non-crossing *pair-partitions*. For motivation we mention that the sets of unordered non-crossing pair-partitions, $NC_2(2n) := \big\{\pi = \{V_1, \ldots, V_n\} \in NC(2n) \,\big|\, |V_1| = \cdots = |V_n| = 2\big\}$, play an important role in the combinatorics of free probability, primarily due to their occurrence in the free Central Limit Theorem (see e.g. Nica and Speicher, 2006, Lecture 8). This carries through to monotonic probability, where the sets

$$NC_2^{(\mathrm{mton})}(2n) := \left\{ (\pi, u) \;\middle|\; \begin{array}{r} \pi \in NC_2(2n) \text{ and } u : \pi \to \{1, \ldots, n\}, \\ \text{monotonic ordering} \end{array} \right\} \tag{1.9}$$

appear in the combinatorial description of the monotonic Central Limit Theorem (Muraki, 2001; Hasebe and Saigo, 2011).

Towards studying combinatorial statistics on the $NC_2^{(\mathrm{mton})}(2n)$'s, one can pursue a methodology similar to the one described in Remark 1.9 in connection to the $NC^{(\mathrm{mton})}(n)$'s. This is because the disjoint union $NC_2^{(\mathrm{mton})} := \sqcup_{n=1}^{\infty} NC_2^{(\mathrm{mton})}(2n)$ bears a natural structure of rooted tree analogous to (and in fact easier to spot than) the one of the $NC^{(\mathrm{mton})}$-tree of Remark 1.3. The unique pairing in $NC_2^{(\mathrm{mton})}(2)$ serves as root for the $NC_2^{(\mathrm{mton})}$-tree and, for every $n \in \mathbb{N}$, the elements of $NC_2^{(\mathrm{mton})}(2n)$ are at distance $n-1$ from the root. Moreover, the $NC_2^{(\mathrm{mton})}$-tree has a homogeneity property analogous to the one recorded in (1.1), which takes the following form:

$$\begin{cases} \text{for every } n \in \mathbb{N}, \text{ every } (\pi, u) \in NC_2^{(\mathrm{mton})}(2n) \\ \text{has precisely } 2n+1 \text{ children in the } NC_2^{(\mathrm{mton})}\text{-tree.} \end{cases} \tag{1.10}$$

We note that, as an immediate consequence of (1.10), one has

$$|\, NC_2^{(\mathrm{mton})}(2n)\,| = (2n-1)!!, \quad \forall\, n \in \mathbb{N} \tag{1.11}$$

(showing, in particular, how normalizations have to be done in calculations with uniform random variables on $NC_2^{(\mathrm{mton})}(2n)$).

Two interesting block-counting statistics that can be considered on $NC_2^{(\mathrm{mton})}(2n)$ are

$$\mathrm{Out}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R} \quad \text{and} \quad \mathrm{Int}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}, \tag{1.12}$$

counting the *outer pairs* and respectively the *interval pairs* of $\pi$, for a random $(\pi, u) \in NC_2^{(\mathrm{mton})}(2n)$. Such pairs represent the top and bottom of the natural nesting structure among the pairs of $\pi$: a pair $V \in \pi$ is outer when it is not nested inside any other pair, and is an *interval pair* when it is of the form $V = \{m, m+1\}$ for some $1 \leq m \leq 2n-1$ (which is equivalent to requiring that no $U \in \pi$ is nested inside $V$). In Section 7 of the paper we show how the recursion "of the second kind" from Remark 1.9 can be adapted to the setting of the $NC_2^{(\mathrm{mton})}$-tree, and used to address the random variables $\mathrm{Out}_n, \mathrm{Int}_n$ from (1.12). In particular, one gets nice explicit formulas for their expectations, as follows.

**Theorem 1.10.** *Let* $\mathrm{Int}_n, \mathrm{Out}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}$ *be as defined above.*

$1^o$ *For every* $n \in \mathbb{N}$*, one has that* $E[\,\mathrm{Int}_n\,] = (2n+1)/3$.

$2^o$ *One has*

$$E[\,\mathrm{Out}_n\,] = \frac{2^n \cdot n!}{(2n-1)!!} - 1, \quad n \in \mathbb{N}, \tag{1.13}$$

*with the consequence that* $E[\,\mathrm{Out}_n\,] \sim \sqrt{\pi n}$ *for* $n \to \infty$.

1.4. *Another combinatorial statistic on* $\boldsymbol{NC_2^{(\mathrm{mton})}(2n)}$*: area.*
An interesting statistic, specific to non-crossing pair-partitions, is the one which measures the *area* under a $\pi \in NC_2(2n)$, when $\pi$ is viewed in its incarnation as a Dyck lattice path (see e.g. Nica and Speicher, 2006, Lecture 2) or, in calculus language, when $\pi$ is identified to a piecewise linear function $f_\pi : [0, 2n] \to [0, \infty)$. To be precise, the function $f_\pi$ is determined by the following requirements:

$$\begin{cases} \to\ f_\pi \text{ is continuous on } [0,2n], \text{ and has } f_\pi(0) = 0; \\ \to\ f_\pi \text{ is linear on every interval } [m-1, m],\ 1 \leq m \leq 2n; \\ \to \text{ For every } V = \{i, j\} \in \pi, \text{ with } i < j, \text{ the slope of } f_\pi \mid [i-1, i] \\ \quad \text{is equal to 1 and the slope of } f_\pi \mid [j-1, j] \text{ is equal to } -1. \end{cases} \tag{1.14}$$

The procedure used to define $f_\pi$ in (1.14) has the immediate consequences that $f_\pi(t) \geq 0$ for every $t \in [0, 2n]$ and that $f_\pi(2n) = 0$. It is natural to put

$$\mathrm{area}(\pi) := \int_0^{2n} f_\pi(t)\, dt \in [0, \infty)$$

(where the latter quantity is easily seen, in fact, to always be a positive integer). For illustration, Figure 3 below shows how the graph of $f_\pi$ looks like for an example of pair-partition $\pi \in NC_2(14)$.

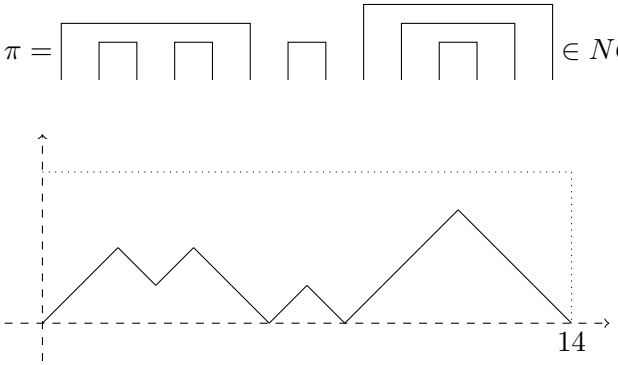



FIGURE 3. $\pi = \{\{1,6\}, \{2,3\}, \{4,5\}, \{7,8\}, \{9,14\}, \{10,13\}, \{11,12\}\}$ *in* $NC_2(14)$, *and graph of the associated function* $f_\pi : [0,14] \to [0,\infty)$.

In the setting of the present paper, the above mentioned notion of area becomes a random variable $A_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}$, defined by $A_n(\pi, u) := \mathrm{area}(\pi)$ for $(\pi, u) \in NC_2^{(\mathrm{mton})}(2n)$.

**Theorem 1.11.** *The expectation of the random variable* $A_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}$ *is*

$$E[A_n] = (2n+1) \sum_{k=1}^{n} \frac{1}{2k+1}, \quad n \in \mathbb{N}. \tag{1.15}$$

*As a consequence of (1.15), it follows that* $E[A_n] \sim n \ln n$ *for* $n \to \infty$.

An interesting aspect of the formula $E[A_n] \sim n \ln n$ from Theorem 1.11 is that it gives a clue on how the graphs of the functions $f_\pi$ ought to be rescaled for asymptotic analysis. Since the rescaling of the horizontal axis must be, clearly, by a factor of $n$ (in order to shrink the domain of $f_\pi$ to the fixed interval $[0, 2]$), it follows that the rescaling of the vertical axis ought to be made by a factor of $\ln n$. This is in contrast to the analogous discussion that one would have for unordered pair-partitions in $NC_2(2n)$, where the calculation of expected areas gives values $\sim \sqrt{\pi} \cdot n^{3/2}$ (see e.g. Merlini et al., 1996, Theorem 2.1), and the suggested vertical rescaling is thus by $\sqrt{n}$ rather than $\ln n$.

1.5. *Organization of the paper.* Besides the Introduction, the paper has 7 sections.

– Section 2 gives the precise description of the $NC^{(\mathrm{mton})}$-tree, and of the two kinds of recursions mentioned in Remark 1.9.

– Sections 3 and 4 deal with the recursion of the first kind for sequences of random variables on the $NC^{(\mathrm{mton})}(n)$'s. The main point here is that, by relying on the $NC^{(\mathrm{mton})}$-tree structure, one can convert the said kind of recursion into a recursion satisfied by a "combinatorial version" of the Laplace transforms of the random variables under consideration. The precise description of how it

is done is given in Theorem 3.2, which is then used in Section 4 in order to prove the Theorems 1.6 and 1.7 stated above.

– In Section 5 we make a concise review of what are monotonic cumulants, and we show how calculations from Section 4.1 can be viewed as giving information on the so-called "monotonic Poisson process".

– In Section 6 we return to the main line of the paper, and we pursue a development analogous to the one from Sections 3 and 4, in connection to the $NC^{(\mathrm{mton})}$-recursion property of the second kind. The resulting recursion for combinatorial Laplace transforms is stated in Theorem 6.1, and its application to the counting of outer blocks is in Corollary 6.3.

– In Section 7 we switch to looking at sequences of random variables on the $NC_2^{(\mathrm{mton})}(2n)$'s. We examine how the formulas obtained in Section 6 about the recursion of the second kind can be adjusted to the $NC_2^{(\mathrm{mton})}$-setting, and we use that in order to prove Theorem 1.10.

– In the final Section 8 we discuss the "area" statistic for monotonically ordered non-crossing pair-partitions, and we give the proof of Theorem 1.11.

## 2. The $NC^{(\mathrm{mton})}$-tree and some recursions related to it

2.1. *Description of the* $\boldsymbol{NC^{(\mathrm{mton})}}$*-tree.*
In this subsection we show how $NC^{(\mathrm{mton})} := \sqcup_{n=1}^{\infty} NC^{(\mathrm{mton})}(n)$ can be organized as set of vertices of a rooted tree, in the way anticipated in Remark 1.3. In order to indicate the edges of this tree we will explain, in Definition 2.2, how one assigns a parent $\mathfrak{p}(\pi, u) \in NC^{(\mathrm{mton})}(n-1)$ to a $(\pi, u) \in NC^{(\mathrm{mton})}(n)$ for $n \geq 2$. First, let us introduce some preliminary notation.

*Notation* 2.1. $1^o$ For every $(\pi, u) \in \sqcup_{n=1}^{\infty} NC^{(\mathrm{mton})}(n)$, we will use the notation "$J(\pi, u)$" for the block of $\pi$ to which $u$ assigns its maximal value, $u\big(J(\pi,u)\big) = |\pi|$. As noticed in Remark 1.2, $J$ is sure to be an interval-block of $\pi$.

$2^o$ A *restrict-and-relabel* notation: for $\pi \in NC(n)$ and $\emptyset \neq W \subseteq \{1, \ldots, n\}$, we let $\pi_W$ denote the partition of $\{1, \ldots, |W|\}$ which is obtained by restricting $\pi$ to $W$ and then re-denoting the elements of $W$ as $1, \ldots, |W|$ in increasing order. It is immediate that the non-crossing property is preserved by this operation, thus $\pi_W \in NC(\,|W|\,)$.

**Definition 2.2.** Let $(\pi, u) \in NC^{(\mathrm{mton})}(n)$, where $n \geq 2$. We will construct a $(\rho, v) \in NC^{(\mathrm{mton})}(n-1)$, which will be called *the parent* of $(\pi, u)$ and will be denoted as $\mathfrak{p}(\pi, u)$. Towards that end, we consider the block $J(\pi, u) = u^{-1}\big(\,|\pi|\,\big) \in \pi$ and we let $m := \max\big(J(\pi, u)\big) \in \{1, \ldots, n\}$. The "$\rho$" part in the desired $(\rho, v)$ is defined as

$$\rho := \pi_{\{1,\ldots,n\}\setminus\{m\}} \in NC(n-1) \tag{2.1}$$

(using the "restrict-and-relabel" notation introduced above). For the "$v$" part in $(\rho, v)$ we start from the monotonic ordering $u : \pi \to \{1, \ldots, |\pi|\}$, and we go as follows.

– Case 1: $|J(\pi, u)| > 1$. In this case we have $|\rho| = |\pi|$, and we can write $\rho = \{\phi(V) \mid V \in \pi\}$, where $\phi$ is the unique increasing bijection from $\{1, \ldots, n\} \setminus \{m\}$ onto $\{1, \ldots, n-1\}$. We define $v : \rho \to \{1, \ldots, |\rho|\}$ by putting $v\big(\phi(V)\big) := u(V)$ for every $V \in \pi$.

– Case 2: $|J(\pi, u)| = 1$, and hence $J(\pi, u) = \{m\}$. Here the set-difference $\{1, \ldots, n\}\setminus\{m\}$ completely eliminates the block $J(\pi, u)$, and thus $|\rho| = |\pi| - 1$. But we still use a procedure similar to the one

described in Case 1: for the same $\phi$ as considered there, we now write $\rho$ as

$$\big\{\phi(V) \mid V \in \pi,\, V \neq \{m\}\big\},$$

and we define the ordering $v : \rho \to \{1, \ldots, |\rho|\}$ by putting $v\big(\phi(V)\big) := u(V)$ for every $V \in \pi$, $V \neq \{m\}$.

**Example 2.3.** $1^o$ Say that $n = 6$, $\pi = \big\{\{1,2,6\}, \{3\}, \{4,5\}\big\}$, and we consider the monotonic ordering $u : \pi \to \{1,2,3\}$ defined by $u\big(\{1,2,6\}\big) = 1,\ u\big(\{3\}\big) = 2,\ u\big(\{4,5\}\big) = 3$, with $J(\pi,u) = \{4,5\}$ and $m := \max\big(J(\pi,u)\big) = 5$. Then $\mathfrak{p}(\pi,u) = (\rho,v) \in NC^{(\mathrm{mton})}(5)$, where

$$\rho = \pi_{\{1,\ldots,6\}\setminus\{5\}} = \big\{\{1,2,5\}, \{3\}, \{4\}\big\} \in NC(5),$$

and the ordering $v$ of the blocks of $\rho$ is: $v\big(\{1,2,5\}\big) = 1,\ v\big(\{3\}\big) = 2,\ v\big(\{4\}\big) = 3$.

$2^o$ Suppose we look at the same $\pi \in NC(6)$ as in $1^o$ above, but considered now with the monotonic ordering $u' : \pi \to \{1,2,3\}$ defined by $u'\big(\{1,2,6\}\big) = 1,\ u'\big(\{4,5\}\big) = 2,\ u'\big(\{3\}\big) = 3$, with $J(\pi,u') = \{3\}$ and $m' := \max\big(J(\pi,u')\big) = 3$. Then $\mathfrak{p}(\pi,u') = (\rho',v') \in NC^{(\mathrm{mton})}(5)$, where

$$\rho' = \pi_{\{1,\ldots,6\}\setminus\{3\}} = \big\{\{1,2,5\}, \{3,4\}\big\} \in NC(5),$$

and the ordering $v'$ of the blocks of $\rho'$ is: $v'\big(\{1,2,5\}\big) = 1,\ v'\big(\{3,4\}\big) = 2$.

*Notation* 2.4. For $n \geq 2$ in $\mathbb{N}$ and $(\rho,v) \in NC^{(\mathrm{mton})}(n-1)$ we will denote

$$C(\rho,v) := \{(\pi,u) \mid \mathfrak{p}(\pi,u) = (\rho,v)\} \subseteq NC^{(\mathrm{mton})}(n); \tag{2.2}$$

that is, $C(\rho,v)$ is the *set of children* of $(\rho,v)$ in the $NC^{(\mathrm{mton})}$-tree.

*Remark* 2.5. *(Homogeneity property of the $NC^{(\mathrm{mton})}$-tree.)* Let $n \geq 2$ be in $\mathbb{N}$ and let $(\rho,v) \in NC^{(\mathrm{mton})}(n-1)$. Then $|C(\rho,v)| = n+1$. More precisely, one can write

$$C(\rho,v) = \big\{(\pi_1,u_1), \ldots, (\pi_n,u_n), (\pi_{n+1},u_{n+1})\big\}, \tag{2.3}$$

where:
– For every $1 \leq m \leq n$, the partition $\pi_m \in NC(n)$ is obtained by inserting a singleton block in the picture of $\rho$, with the added singleton placed at position $m$, and where this singleton block $\{m\}$ gets to have maximal label in the monotonic ordering $u_m$ of the blocks of $\pi_m$ – that is, one has $J(\pi_m,u_m) = \{m\}$.

– The partition $\pi_{n+1}$ is obtained by considering the interval-block $J(\rho,v) = \{p,\ldots,q\}$ with maximal $v$-label, and by "elongating" $\{p,\ldots,q\}$ to become the block with maximal label in the monotonic ordering $u_{n+1}$ of $\pi_{n+1}$ – that is, one has

$$J(\pi_{n+1},u_{n+1}) = \{p,\ldots,q,q+1\} \in \pi_{n+1}. \tag{2.4}$$

## 2.2. *The $NC^{(\mathrm{mton})}$-recursion of the first kind.*

**Definition 2.6.** Let $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R})_{n=1}^{\infty}$ be real random variables, let $k \in \mathbb{N}$, and let $\underline{r} = (r_1,\ldots,r_k) \in \mathbb{R}^k$. We will say that the sequence of $Z_n$'s *satisfies a recursion of the first kind*

with input $\underline{r}$ to mean that the following condition is fulfilled:

$$\left|\begin{array}{l} \text{For every } n \geq 2 \text{ and } (\pi, u) \in NC^{(\mathrm{mton})}(n), \\ \text{with } \mathfrak{p}(\pi, u) = (\rho, v) \in NC^{(\mathrm{mton})}(n-1), \text{ one has:} \\ \\ \begin{array}{ll} Z_n(\pi, u) = Z_{n-1}(\rho, v) + r_1, & \text{if } |J(\pi, u)| = 1, \\ \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ Z_n(\pi, u) = Z_{n-1}(\rho, v) + r_k, & \text{if } |J(\pi, u)| = k, \end{array} \\ \\ \text{and } Z_n(\pi, u) = Z_{n-1}(\rho, v), \qquad \text{if } |J(\pi, u)| > k. \end{array}\right| \tag{2.5}$$

*Remark* 2.7. $1^o$ The way to think about the recursive condition stated in (2.5) is that we generally want to have $Z_n(\pi, u) = Z_{n-1}\big(\mathfrak{p}(\pi, u)\big)$, but we allow exceptions when the special block $J(\pi, u) \in \pi$ has small size – namely, if $|J(\pi, u)| = j \leq k$ then $Z_n(\pi, u) = Z_{n-1}\big(\mathfrak{p}(\pi, u)\big) + r_j$.

$2^o$ In the setting of Definition 2.6, it is immediate that if the input tuple $\underline{r}$ is given and if $Z_1$ is given, then all the $Z_n$'s are uniquely determined. However, the way we would like to use this definition goes, in some sense, in reverse: given a sequence of $Z_n$'s which we deem to be of interest, we want to see if it wouldn't be possible to fit a $k$ and an $\underline{r} = (r_1, \ldots, r_k)$ such that the $Z_n$'s are recursive of the first kind with input $\underline{r}$. This turns out to work for a number of natural examples of block-counting random variables, as shown below.

Note that in the above mentioned process of fitting an input vector $\underline{r} = (r_1, \ldots, r_k)$, there is some freedom in the choice of $k$: we can always replace $k$ with an $\ell > k$, and replace the input tuple $\underline{r} \in \mathbb{R}^k$ with $(r_1, \ldots, r_k, 0, \ldots 0) \in \mathbb{R}^\ell$.

**Example 2.8.** The block-counting random variables $Y_n$ introduced in Notation 1.5 satisfy the condition (2.5), with $k = 1$ and $\underline{r} = (1) \in \mathbb{R}^1$. This amounts to the fact that, for every $n \geq 2$ and $(\pi, u) \in NC^{(\mathrm{mton})}(n)$ with parent $\mathfrak{p}(\pi, u) =: (\rho, v) \in NC^{(\mathrm{mton})}(n-1)$, one has:

$$|\pi| = \begin{cases} |\rho| + 1, & \text{if } |J(\pi, u)| = 1, \\ |\rho|, & \text{if } |J(\pi, u)| > 1. \end{cases} \tag{2.6}$$

For the verification of (2.6), we check the two possible cases: if $|J(\pi, u)| = 1$ then $\pi$ was obtained by inserting a singleton block in $\rho$, and thus $|\pi| = |\rho| + 1$; while if $|J(\pi, u)| \geq 2$ then $\pi$ was obtained by elongating the block $J(\rho, v)$ of $\rho$, and thus $|\pi| = |\rho|$.

**Example 2.9.** Let us fix an $\ell \in \mathbb{N}$ and consider the random variables $\big(Y_n^{(\ell)}\big)_{n=1}^{\infty}$ which were introduced in Equation (1.6) of Theorem 1.7, counting blocks of cardinality $\ell$. We claim that the $Y_n^{(\ell)}$'s satisfy the condition (2.5), with $k = \ell + 1$ and $\underline{r} = (0, \ldots, 0, 1, -1) \in \mathbb{R}^{\ell+1}$. This amounts to the fact that for $n \geq 2$ and $(\pi, u) \in NC^{(\mathrm{mton})}(n)$ with parent $\mathfrak{p}(\pi, u) =: (\rho, v) \in NC^{(\mathrm{mton})}(n-1)$, one has:

$$Y_n^{(\ell)}(\pi, u) = \begin{cases} Y_{n-1}^{(\ell)}(\rho, v) + 1, & \text{if } |J(\pi, u)| = \ell, \\ Y_{n-1}^{(\ell)}(\rho, v) - 1, & \text{if } |J(\pi, u)| = \ell + 1, \\ Y_{n-1}^{(\ell)}(\rho, v), & \text{otherwise.} \end{cases} \tag{2.7}$$

For the verification of (2.7), we check the three possible cases listed there. We will do the verification under the assumption that $\ell \geq 2$ (and leave it as exercise to the reader to supply the slight adjustments needed for $\ell = 1$).

*Case 1:* $|J(\pi, u)| = \ell$. Since $\ell \geq 2$, it must be that $J(\pi, u)$ was obtained by elongating the block $J(\rho, v)$ of $\rho$, which had $|J(\rho, v)| = \ell - 1$. So $\pi$ indeed has one more block of cardinality $\ell$ than $\rho$ did – this is precisely the block $J(\pi, u)$.

*Case 2:* $|J(\pi,u)| = \ell+1$. This is similar to Case 1, but where we now have that $|J(\rho,v)| = \ell \neq |J(\pi,u)|$; hence in this case $\pi$ has one less block of cardinality $\ell$ than $\rho$ did.

*Case 3:* $|J(\pi,u)| \notin \{\ell, \ell+1\}$. In this case, there are no blocks of cardinality $\ell$ that would either appear or disappear in the process of passing from $(\rho,v)$ to $(\pi,u)$, hence $\pi$ has the same number of blocks of cardinality $\ell$ as $\rho$ (as claimed in (2.7)).

**Example 2.10.** Consider the random variables $\left(Y_n^{(\geq 3)}\right)_{n=1}^{\infty}$ which were introduced in Equation (1.8) of Remark 1.8, counting all the blocks which do not have cardinality 1 or 2. The $Y_n^{(\geq 3)}$'s satisfy the condition (2.5), with $k=3$ and $\underline{r} = (0,0,1) \in \mathbb{R}^3$. The verification of this claim is similar to the one shown in the preceding example, and is left to the reader.

## 2.3. *The* $\boldsymbol{NC^{(\mathrm{mton})}}$*-recursion of the second kind.*

**Definition 2.11.** Let $\alpha, \beta, q \in \mathbb{Z}$ be given. We will say that a sequence of random variables $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{Z})_{n=1}^{\infty}$ *satisfies a recursion of the second kind* with input $(\alpha, \beta; q)$ when it has the following property: for every $n \geq 2$ and $(\rho,v) \in NC^{(\mathrm{mton})}(n-1)$, the set $C(\rho,v)$ of children of $(\rho,v)$ has a subset $C_o(\rho,v)$ such that:

$$\begin{cases} (1) & \text{If } (\pi,u) \in C_o(\rho,v), \text{ then } Z_n(\pi,u) = Z_{n-1}(\rho,v) + \alpha; \\ (2) & \text{if } (\pi,u) \in C(\rho,v) \setminus C_o(\rho,v), \text{ then } Z_n(\pi,u) = Z_{n-1}(\rho,v) + \beta; \\ \text{and} & \\ (3) & |C_o(\rho,v)| = Z_{n-1}(\rho,v) + q. \end{cases} \tag{2.8}$$

**Example 2.12.** For every $n \in \mathbb{N}$, let $\mathrm{Out}_n : NC^{(\mathrm{mton})}(n) \to \mathbb{N}$ be defined by

$$\mathrm{Out}_n(\pi,u) := \big|\, \{V \in \pi \mid V \text{ is outer}\} \,\big|, \quad (\pi,u) \in NC^{(\mathrm{mton})}(n), \tag{2.9}$$

where recall that a block $V \in \pi$ is said to be *outer* when there is no $W \in \pi$ with $\min(W) < \min(V)$ and $\max(W) > \max(V)$. We will verify that the sequence of random variables $(\mathrm{Out}_n)_{n=1}^{\infty}$ satisfies a recursion of the second kind, with parameters $\alpha = 1$, $\beta = 0$ and $q = 1$. Towards that end, let us fix an $n \geq 2$ and a $(\rho,v) \in NC^{(\mathrm{mton})}(n-1)$. It is easy to see that the outer blocks of $\rho$ can be ordered from left to right, that is, they can be listed as $W_1, \ldots, W_s$ with $\min(W_1) = 1$, $\min(W_2) = \max(W_1)+1, \ldots, \min(W_s) = \max(W_{s-1})+1$ and $\max(W_s) = n-1$. We consider the set $C(\rho,v)$ of children of $(\rho,v)$ in the $NC^{(\mathrm{mton})}$-tree, and we write it in the way described in Equation (2.3) of Remark 2.5:

$$C(\rho,v) = \{(\pi_1,u_1), \ldots, (\pi_n,u_n), (\pi_{n+1},u_{n+1})\},$$

where for $1 \leq m \leq n$ we have $J(\pi_m,u_m) = \{m\}$, while for $m = n+1$ we get the block $J(\pi_{n+1},u_{n+1})$ of $\pi_{n+1}$ by elongating (that is, by appending one more element to) the interval-block $J(\rho,v)$ of $\rho$. The existing outer blocks of the parent $(\rho,v)$ remain outer in every child $(\pi_j,u_j)$. Hence the only possible change in the number of outer blocks is caused by the inserted singleton. This singleton, placed at position $m$, is an outer block precisely when it is inserted immediately before one of the outer blocks $W_i$, or immediately after the last outer block $W_s$. Equivalently, $m \in \{\min(W_1), \ldots, \min(W_s), n\}$. Therefore

$$\mathrm{Out}_n(\pi_m,u_m) = \begin{cases} \mathrm{Out}_{n-1}(\rho,v)+1, & m \in \{\min(W_1), \ldots, \min(W_s), n\}, \\ \mathrm{Out}_{n-1}(\rho,v), & \text{otherwise}, \end{cases} \tag{2.10}$$

for $1 \leq m \leq n+1$, where the case $m = n+1$ corresponds to the elongation child. This shows that

$$C_o(\rho,v) = \left\{(\pi_m,u_m) \mid m \in \{\min(W_1), \ldots, \min(W_s), n\}\right\}$$

satisfies the conditions in the definition of a recursion of the second kind with parameters $\alpha = 1, \beta = 0, q = 1$. Indeed, the required conditions (1) and (2) from that definition are satisfied due to (2.10), while condition (3) is satisfied because $|C_o(\rho, v)| = s + 1 = \mathrm{Out}_{n-1}(\rho, v) + 1$.

## 3. Laplace transforms for the $NC^{(\mathrm{mton})}$-recursion of the first kind

Let $n \in \mathbb{N}$ and let $Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R}$ be a random variable, considered with respect to uniform distribution on $NC^{(\mathrm{mton})}(n)$. The Laplace transform of $Z_n$ is the complex function

$$\lambda \mapsto E[e^{\lambda Z_n}] = \frac{1}{|NC^{(\mathrm{mton})}(n)|} \sum_{(\pi,u)\in NC^{(\mathrm{mton})}(n)} e^{\lambda Z_n(\pi,u)}, \quad \lambda \in \mathbb{C}.$$

For the purpose of this paper it is convenient that in the latter equation we restrict $\lambda$ to $\mathbb{R}$, put $t = e^{\lambda} \in (0, \infty)$, and set aside the normalization constant $|NC^{(\mathrm{mton})}(n)|$. This leads to the following version of the Laplace transform, which is defined on $(0, \infty)$ and, in our examples of interest where $Z_n$ takes values in $\mathbb{N} \cup \{0\}$, is a polynomial function.

*Definition and Remark* 3.1. Let $n \in \mathbb{N}$ and let $Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R}$ be a random variable, considered with respect to uniform distribution on $NC^{(\mathrm{mton})}(n)$. We will use the term *combinatorial Laplace transform of* $Z_n$ for the function $L_n : (0, \infty) \to \mathbb{R}$ defined by

$$L_n(t) = \sum_{(\pi,u)\in NC^{(\mathrm{mton})}(n)} t^{Z_n(\pi,u)}, \quad t > 0. \tag{3.1}$$

We note that $L_n$ contains significant information about the random variable $Z_n$; in particular, direct calculation shows that the expectation and variance of $Z_n$ are retrieved as

$$E[Z_n] = \frac{L_n'(1)}{L_n(1)} \text{ and } \mathrm{Var}[Z_n] = E[Z_n^2] - \big(E[Z_n]\big)^2 = \frac{L_n''(1) + L_n'(1)}{L_n(1)} - \Big(\frac{L_n'(1)}{L_n(1)}\Big)^2, \tag{3.2}$$

where the $L_n(1)$ in the denominators keeps track of normalizations, $L_n(1) = |NC^{(\mathrm{mton})}(n)| = (n+1)!/2$.

The next theorem says that if a sequence of random variables $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R})_{n=1}^{\infty}$ satisfies a recursion of the first kind (in the sense of Definition 2.6), then the combinatorial Laplace transforms $L(Z_n)$ obey a tractable recursion formula.

**Theorem 3.2.** *Let $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R})_{n=1}^{\infty}$ be a sequence of real random variables which satisfies a recursion of the first kind with input $\underline{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k$, where $k \geq 2$. For every $n \in \mathbb{N}$, let $L_n : (0, \infty) \to \mathbb{R}$ be defined as in Equation (3.1). Then one has:*

$$L_n(t) = (1 + nt^{r_1})L_{n-1}(t) + \sum_{j=2}^{k} (n - j + 1)\,(t^{r_j} - 1)\,t^{r_1+\cdots+r_{j-1}}\,L_{n-j}(t), \tag{3.3}$$

*holding for $n > k$ in $\mathbb{N}$ and $t \in (0, \infty)$.*

Towards the proof of Theorem 3.2, we first prove two lemmas.

**Lemma 3.3.** *Consider the framework and notation of Theorem 3.2. Let $\ell \in \{2, \ldots, k\}$, let $m$ be an integer such that $m \geq \ell$, and let $\mathfrak{p}^{(\ell-1)} : NC^{(\mathrm{mton})}(m) \to NC^{(\mathrm{mton})}(m - \ell + 1)$ be the map obtained by iterating $\ell - 1$ times the parent map $\mathfrak{p}$ of the $NC^{(\mathrm{mton})}$-tree.*

*$1^o$ Let $(\rho, v) \in NC^{(\mathrm{mton})}(m)$ be such that $|J(\rho, v)| = \ell$, and let $(\sigma, w) := \mathfrak{p}^{(\ell-1)}(\rho, v) \in NC^{(\mathrm{mton})}(m-\ell+1)$. One has that*

$$Z_m(\rho, v) = Z_{m-\ell+1}(\sigma, w) + (r_2 + \cdots + r_\ell). \tag{3.4}$$

*$2^o$ The map $\mathfrak{p}^{(\ell-1)}$ establishes a bijection between $\{(\rho, v) \in NC^{(\mathrm{mton})}(m) \mid \ |J(\rho, v)| = \ell\}$ and $\{(\sigma, w) \in NC^{(\mathrm{mton})}(m-\ell+1) \mid \ |J(\sigma, w)| = 1\}$.*

*Proof*: $1^o$ The block $J(\rho, v)$ of $\rho$ is an interval block of cardinality $\ell$, hence it is of the form $\{a, a+1, \ldots, a+\ell-1\}$ for some $a \in \{1, \ldots, m-\ell+1\}$. If we denote

$$(\rho_1, v_1) := \mathfrak{p}(\rho, v) \in NC^{(\mathrm{mton})}(m-1), \quad (\rho_2, v_2) := \mathfrak{p}(\rho_1, v_1) \in NC^{(\mathrm{mton})}(m-2), \ldots$$

$$\ldots, (\rho_{\ell-1}, v_{\ell-1}) := \mathfrak{p}(\rho_{\ell-2}, v_{\ell-2}) \in NC^{(\mathrm{mton})}(m-\ell+1),$$

then (as a consequence of how the parent map is defined) we find that

$$J(\rho_i, v_i) = \{a, a+1, \ldots, a+\ell-i-1\}, \ \forall\, 1 \leq i \leq \ell-1. \tag{3.5}$$

The recursive condition (2.5) satisfied by the $Z_n$'s then implies that:

$$Z_m(\rho, v) = Z_{m-1}(\rho_1, v_1) + r_\ell, \quad Z_{m-1}(\rho_1, v_1) = Z_{m-2}(\rho_2, v_2) + r_{\ell-1}, \ldots$$

$$\ldots, Z_{m-\ell+2}(\rho_{\ell-2}, v_{\ell-2}) = Z_{m-\ell+1}(\rho_{\ell-1}, v_{\ell-1}) + r_2,$$

and concatenating these equalities yields that

$$Z_m(\rho, v) = Z_{m-\ell+1}(\rho_{\ell-1}, v_{\ell-1}) + (r_2 + \cdots + r_\ell).$$

This is precisely the required formula (3.4), since $(\sigma, w)$ from the statement of the lemma coincides with $(\rho_{\ell-1}, v_{\ell-1})$.

$2^o$ By reasoning as in the proof of $1^o$ we see that $\{(\rho, v) \in NC^{(\mathrm{mton})}(m) \mid \ |J(\rho, v)| = \ell\}$ can be written as $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{m-\ell+1}$, disjoint union, where for $a \in \{1, \ldots, m-\ell+1\}$ we put

$$\mathcal{R}_a := \Big\{ (\rho, v) \in NC^{(\mathrm{mton})}(m) \mid \ J(\rho, v) = \{a, a+1, \ldots, a+\ell-1\} \Big\}. \tag{3.6}$$

Let us fix for the moment an $a \in \{1, \ldots, m-\ell+1\}$ and a $(\rho, v) \in \mathcal{R}_a$. The formula (3.5) obtained during the proof of $1^o$ shows (when used with $i = \ell-1$) that the map $\mathfrak{p}^{(\ell-1)}$ sends $(\rho, v)$ to a $(\sigma, w) \in NC^{(\mathrm{mton})}(m-\ell+1)$ such that $J(\sigma, w) = \{a\}$. Upon looking more closely at the iterations of the parent map $\mathfrak{p}$ which take us from $(\rho, v)$ to $(\sigma, w)$, it is easy to see that if we write explicitly

$$\rho = \Big\{ A_1, \ldots A_s, \{a, a+1, \ldots, a+\ell-1\} \Big\}, \tag{3.7}$$

then we will have

$$\sigma = \Big\{ \varphi(A_1), \ldots \varphi(A_s), \{a\} \Big\}, \tag{3.8}$$

where $\varphi$ is the unique order-preserving bijection from $\{1, \ldots, m\} \setminus \{a, \ldots, a+\ell-1\}$ onto $\{1, \ldots, m-\ell+1\} \setminus \{a\}$. Moreover, the ordering $w$ of the blocks of $\sigma$ is related to the ordering $v$ of the blocks of $\rho$ via the formula

$$w\big(\varphi(A_1)\big) = v(A_1), \ldots, w\big(\varphi(A_s)\big) = v(A_s), \tag{3.9}$$

complemented by the fact that $w(\{a\}) = v(\{a, \ldots, a+\ell-1\}) = s+1$.

The considerations of the preceding paragraph show that, for a fixed $a \in \{1, \ldots, m-\ell+1\}$, the map $\mathfrak{p}^{(\ell-1)}$ will send the set $\mathcal{R}_a$ from (3.6) bijectively onto the set

$$\mathcal{S}_a := \Big\{ (\sigma, w) \in NC^{(\mathrm{mton})}(m-\ell+1) \mid \ J(\sigma, w) = \{a\} \Big\}. \tag{3.10}$$

Indeed, for every $(\sigma, w) \in \mathcal{S}_a$, with $\sigma = \{B_1, \ldots, B_s, \{a\}\}$, the unique $(\rho, v) \in \mathcal{R}_a$ such that $\mathfrak{p}^{(\ell-1)}(\rho, v) = (\sigma, w)$ will have

$$\begin{cases} \rho = \Big\{\varphi^{-1}(B_1), \ldots \varphi^{-1}(B_s), \{a, \ldots, a+\ell-1\}\Big\}, \text{ and} \\ \\ v\big(\varphi^{-1}(B_1)\big) = w(B_1), \ldots, v\big(\varphi^{-1}(B_s)\big) = w(B_s), \end{cases} \tag{3.11}$$

where $\varphi$ is the same map as used in Equations (3.8), (3.9).

Finally, by allowing $a$ to run in $\{1, \ldots, m-\ell+1\}$, we find that $\mathfrak{p}^{(\ell-1)}$ sends bijectively $\{(\rho, v) \in NC^{(\mathrm{mton})}(m) \mid |J(\rho, v)| = \ell\}$ onto the disjoint union $\cup_{a=1}^{m-\ell+1} \mathcal{S}_a$, which is equal to $\{(\sigma, w) \in NC^{(\mathrm{mton})}(m-\ell+1) \mid |J(\sigma, w)| = 1\}$. This completes the proof. □

**Lemma 3.4.** *Consider the framework and notation of Theorem 3.2. For every integer $m \geq 2$ and $t \in (0, \infty)$, one has that*

$$\sum_{\substack{(\rho,v) \in NC^{(\mathrm{mton})}(m) \\ \text{with } |J(\rho,v)|=1}} t^{Z_m(\rho,v)} = m\, t^{r_1}\, L_{m-1}(t). \tag{3.12}$$

*Proof*: On the left-hand side of (3.12) we group the terms according to what is the parent of $(\rho, v)$ in the $NC^{(\mathrm{mton})}$-tree, and this converts the quantity indicated there into:

$$\sum_{(\sigma,w) \in NC^{(\mathrm{mton})}(m-1)} \Bigg[ \sum_{\substack{(\rho,v) \in NC^{(\mathrm{mton})}(m) \\ \text{with } \mathfrak{p}(\rho,v)=(\sigma,w) \\ \text{and with } |J(\rho,v)|=1}} t^{Z_m(\rho,v)} \Bigg]. \tag{3.13}$$

Let us now focus on a $(\sigma, w) \in NC^{(\mathrm{mton})}(m-1)$, and examine the inner sum corresponding to this $(\sigma, w)$ in (3.13). In view of the explicit description of the children of $(\sigma, w)$ in the $NC^{(\mathrm{mton})}$-tree observed in Remark 2.5, we can list

$$\big\{(\rho, v) \in NC^{(\mathrm{mton})}(m) \mid \mathfrak{p}(\rho, v) = (\sigma, w) \text{ and } |J(\rho, v)| = 1\big\} = \{(\rho_1, v_1), \ldots, (\rho_m, v_m)\},$$

where, for every $1 \leq i \leq m$, $(\rho_i, v_i)$ is the unique monotonically ordered non-crossing partition in $NC^{(\mathrm{mton})}(m)$ such that $\mathfrak{p}(\rho_i, v_i) = (\sigma, w)$ and $J(\rho_i, v_i) = \{i\}$. Since the recursive condition (2.5) satisfied by the $Z_n$'s assures us that $Z_m(\rho_i, v_i) = Z_{m-1}(\sigma, w) + r_1$, $1 \leq i \leq m$, we infer that

$$\sum_{\substack{(\rho,v) \in NC^{(\mathrm{mton})}(m) \\ \text{with } \mathfrak{p}(\rho,v)=(\sigma,w) \\ \text{and with } |J(\rho,v)|=1}} t^{Z_m(\rho,v)} = \sum_{i=1}^{m} t^{Z_m(\rho_i,v_i)} = m\, t^{Z_{m-1}(\sigma,w)+r_1}. \tag{3.14}$$

Equation (3.14) holds for every $(\sigma, w) \in NC^{(\mathrm{mton})}(m-1)$. This allows us to plug the right-hand side of (3.14) as inner sum in (3.13), in order to conclude that

$$\begin{pmatrix} \text{quantity} \\ \text{from (3.13)} \end{pmatrix} = \sum_{(\sigma,w) \in NC^{(\mathrm{mton})}(m-1)} \Big[ m\, t^{r_1}\, t^{Z_{m-1}(\sigma,w)} \Big] = m\, t^{r_1}\, L_{m-1}(t),$$

as claimed in (3.12). □

3.5. **Proof of Theorem 3.2.** We fix for the proof an $n > k$ in $\mathbb{N}$ and a $t \in (0, \infty)$ for which we will verify that (3.3) holds. In the sum over $(\pi, u) \in NC^{(\mathrm{mton})}(n)$ which defines $L_n(t)$, we organize the terms according to what is the parent of $(\pi, u)$ in the $NC^{(\mathrm{mton})}$-tree:

$$L_n(t) = \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} \Big[ \sum_{\substack{(\pi,u)\in NC^{(\mathrm{mton})}(n) \\ \text{with } \mathfrak{p}(\pi,u)=(\rho,v)}} t^{Z_n(\pi,u)} \Big]. \tag{3.15}$$

Let us fix for the moment an ordered partition $(\rho, v) \in NC^{(\mathrm{mton})}(n-1)$, for which we examine the inner sum corresponding to $(\rho, v)$ in (3.15). This sum is indexed by the set $C(\rho, v) = \{(\pi_1, u_1), \ldots, (\pi_{n+1}, u_{n+1})\}$ of children of $(\rho, v)$ in the $NC^{(\mathrm{mton})}$-tree, with the $(\pi_m, u_m)$'s explicitly described in the paragraph following Equation (2.3) of Remark 2.5. By invoking the recursive property (2.5) in connection to the explicit description of $C(\rho, v)$ reviewed above, we find that for $1 \leq m \leq n$ we have $Z_n(\pi_m, u_m) = Z_{n-1}(\rho, v) + r_1$ (since $(\pi_m, u_m)$ is obtained by inserting a singleton block in $(\rho, v)$). The remaining value $Z_n(\pi_{n+1}, u_{n+1})$ relates to $Z_{n-1}(\rho, v)$ in a way which depends on the cardinality of the block $J(\pi_{n+1}, u_{n+1}) \in \pi_{n+1}$. But the latter cardinality is just $1 + |J(\rho, v)|$ (since $(\pi_{n+1}, u_{n+1})$ is obtained by elongating the block $J(\rho, v) \in \rho$ in order to make it become $J(\pi_{n+1}, u_{n+1})$). The recursive property (2.5) thus gives the following possibilities $(*)$ and $(**)$:

$$\begin{cases} (*) & \text{if } 1 + |J(\rho, v)| > k, \text{ then } Z_n(\pi_{n+1}, u_{n+1}) = Z_{n-1}(\rho, v); \\ (**) & \text{if } 1 + |J(\rho, v)| = j \in \{2, \ldots, k\}, \text{ then } Z_n(\pi_{n+1}, u_{n+1}) = Z_{n-1}(\rho, v) + r_j. \end{cases}$$

Returning to the evaluation of the inner sum which corresponds to $(\rho, v)$ in (3.15), we see that this sum is

$$\begin{aligned} \sum_{m=1}^{n+1} t^{Z_n(\pi_m,u_m)} &= n\, t^{Z_{n-1}(\rho,v)+r_1} + t^{Z_n(\pi_{n+1},u_{n+1})} \\ &= n\, t^{Z_{n-1}(\rho,v)+r_1} + t^{Z_{n-1}(\rho,v)} + T \end{aligned}$$

where the correction term $T$ may appear in connection to the possibility $(**)$, when it is equal to

$$t^{Z_{n-1}(\rho,v)+r_j} - t^{Z_{n-1}(\rho,v)} \qquad (\text{for } j = 1 + |J(\rho, v)|).$$

We now unfix the $(\rho, v)$ that was considered throughout the preceding paragraph, and plug the result of that discussion into the sum over $(\rho, v) \in NC^{(\mathrm{mton})}(n-1)$ appearing in (3.15). Then Equation (3.15) takes the following form:

$$L_n(t) = \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} \Big[ n\, t^{Z_{n-1}(\rho,v)+r_1} + t^{Z_{n-1}(\rho,v)} \Big] \tag{3.16}$$

$$+ \sum_{j=2}^{k} \Big[ \sum_{\substack{(\rho,v)\in NC^{(\mathrm{mton})}(n-1) \\ \text{with } |J(\rho,v)|=j-1}} t^{Z_{n-1}(\rho,v)+r_j} - t^{Z_{n-1}(\rho,v)} \Big].$$

From here we continue as follows.

*Claim 1.* The first sum listed in (3.16) is equal to $(n\, t^{r_1} + 1)\, L_{n-1}(t)$.

*Verification of Claim 1.* The sum in question is $\sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} (n\, t^{r_1} + 1) t^{Z_{n-1}(\rho,v)}$, and, in view of how $L_{n-1}$ is defined, this is indeed $(n\, t^{r_1} + 1)\, L_{n-1}(t)$.

*Claim 2.* The second sum listed in (3.16) is equal to

$$\sum_{j=2}^{k} (n-j+1)\,(t^{r_j}-1)\,t^{r_1+\cdots+r_{j-1}}\,L_{n-j}(t).$$

*Verification of Claim 2.* The sum under consideration can be written as:

$$\sum_{j=2}^{k}(t^{r_j}-1)\Big[\sum_{\substack{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)\\ \text{with } |J(\rho,v)|=j-1}} t^{Z_{n-1}(\rho,v)}\Big],$$

hence it suffices to verify that we have equalities

$$\sum_{\substack{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)\\ \text{with } |J(\rho,v)|=j-1}} t^{Z_{n-1}(\rho,v)} = (n-j+1)\,t^{r_1+\cdots+r_{j-1}}\;L_{n-j}(t), \tag{3.17}$$

holding for every $j\in\{2,\ldots,k\}$. The case $j=2$ of (3.17) says that

$$\sum_{\substack{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)\\ \text{with } |J(\rho,v)|=1}} t^{Z_{n-1}(\rho,v)} = (n-1)\,t^{r_1}\;L_{n-2}(t),$$

which is just Lemma 3.4, used for $m=n-1$. For a $j\in\{3,\ldots,k\}$, the verification of (3.17) goes as follows:

$$\begin{aligned}
\sum_{\substack{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)\\ \text{with } |J(\rho,v)|=j-1}} t^{Z_{n-1}(\rho,v)} &= \sum_{\substack{(\sigma,w)\in NC^{(\mathrm{mton})}(n-j+1)\\ \text{with } |J(\sigma,w)|=1}} t^{Z_{n-j+1}(\sigma,w)+(r_2+\cdots+r_{j-1})}\\
&\quad \text{(by Lemma 3.3, used with } m=n-1 \text{ and } \ell=j-1)\\
&= t^{r_2+\cdots+r_{j-1}}\cdot \sum_{\substack{(\sigma,w)\in NC^{(\mathrm{mton})}(n-j+1)\\ \text{with } |J(\sigma,w)|=1}} t^{Z_{n-j+1}(\sigma,w)}\\
&= t^{r_2+\cdots+r_{j-1}}\cdot (n-j+1)\,t^{r_1}\,L_{n-j}(t) \quad \text{(by Lemma 3.4)}\\
&= (n-j+1)\,t^{r_1+\cdots+r_{j-1}}\;L_{n-j}(t), \quad \text{as required.}
\end{aligned}$$

Finally, substituting the expressions from Claim 1 and Claim 2 on the right-hand side of Equation (3.16) yields the required formula (3.3) from the statement of the theorem. □

## 4. Statistics satisfying an $NC^{(\mathrm{mton})}$-recursion of the first kind

In this section we show how the recursion for combinatorial Laplace transforms found in Theorem 3.2 applies to the statistics considered in Theorems 1.6 and 1.7 of the Introduction, and we provide the proofs of these theorems.

### 4.1. *Block-counting random variables – proof of Theorem 1.6.*

In this subsection we consider the random variables $(Y_n : NC^{(\mathrm{mton})}(n)\to\mathbb{R})_{n=1}^{\infty}$ introduced in Notation 1.5, with $Y_n(\pi,u):=|\pi|$ for $(\pi,u)\in NC^{(\mathrm{mton})}(n)$, and we give the proof of Theorem 1.6.

*Notation and Remark* 4.1. For every $n \in \mathbb{N}$, the combinatorial Laplace transform of $Y_n$ (defined as in Equation (3.1)) will be denoted as $B_n$. That is, we put

$$B_n(t) := \sum_{(\pi,u)\in NC^{(\mathrm{mton})}(n)} t^{|\pi|}, \quad t \in (0,\infty). \tag{4.1}$$

Clearly, every $B_n$ is a polynomial function without constant term. For example, direct inspection shows that $B_1(t) = t$, $B_2(t) = 2t^2 + t$, $B_3(t) = 6t^3 + 5t^2 + t$.

**Lemma 4.2.** *The polynomials $B_n$ introduced above satisfy the recursion*

$$B_n(t) = (1 + nt)\, B_{n-1}(t), \quad n \geq 2, \tag{4.2}$$

*and are consequently given by the formula*

$$B_n(t) = t(1 + 2t)\cdots(1 + nt), \quad \textit{for } n \geq 2 \textit{ in } \mathbb{N} \textit{ and } t \in (0,\infty). \tag{4.3}$$

*Proof*: We will obtain the recursion (4.2) by using Theorem 3.2. This is possible because the polynomials $B_n$ are the combinatorial Laplace transforms of the random variables $Y_n$, which were seen in Example 2.8 to satisfy a recursion of the first kind with input vector $(1) \in \mathbb{R}^1$. Since Theorem 3.2 was stated and proved for an input vector $(r_1, \ldots, r_k)$ with $k \geq 2$, we adjust the input vector for the $Y_n$'s to be $(1, 0) \in \mathbb{R}^2$ instead of $(1) \in \mathbb{R}^1$ (such a trivial adjustment is always allowed, as noted in part $2^o$ of Remark 2.7).

So then: upon invoking Equation (3.3) of Theorem 3.2 in connection to the input vector $\underline{r} = (r_1, r_2) = (1, 0) \in \mathbb{R}^2$, we find that for every $n \geq 3$ we have:

$$B_n(t) = (1 + nt^{r_1})\, B_{n-1}(t) + (n - 2 + 1)\,(t^{r_2} - 1)\, t^{r_1}\, B_{n-2}(t) = (1 + nt)B_{n-1}(t).$$

This is indeed the required formula (4.2), where for $n = 2$ we check by direct inspection that $B_2(t) = (1 + 2t)B_1(t)$.

Finally, Equation (4.3) follows from (4.2) via an immediate induction argument. □

By using Lemma 4.2, it is easy to find the explicit expectation and variance formulas for the $Y_n$'s which were announced in Theorem 1.6 of the Introduction.

4.3. **Proof of Theorem 1.6.** $1^o$ Differentiation of the product formula (4.3) leads to

$$\frac{B_n'(t)}{B_n(t)} = \frac{1}{t} + \frac{2}{1+2t} + \frac{3}{1+3t} + \cdots + \frac{n}{1+nt}, \quad t \in (0,\infty), \tag{4.4}$$

so upon invoking Equation (3.2) of Remark 3.1 we get

$$E[\,Y_n\,] = \frac{B_n'(1)}{B_n(1)} = 1 + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} = n - \sum_{k=3}^{n+1} \frac{1}{k}, \tag{4.5}$$

leading to the formula for $E[Y_n]$ stated in part $1^o$ of Theorem 1.6. The asymptotics stated in Equation (1.4) follows when we take into account that $H_n \approx \ln n + \gamma$.

$2^o$ Let us denote $R_n := B_n'/B_n$. Then direct calculation gives

$$R_n + R_n' = \frac{B_n'}{B_n} + \frac{B_n'' B_n - (B_n')^2}{B_n^2} = \frac{B_n'' + B_n'}{B_n} - \Big[\,\frac{B_n'}{B_n}\,\Big]^2,$$

and comparing this with the second Equation (3.2) in Remark 3.1 we find that

$$\mathrm{Var}(Y_n) = R_n(1) + R_n'(1). \tag{4.6}$$

Now, Equation (4.4) gives an explicit formula for $R_n(t)$, and upon differentiating both sides of that equation we also find the explicit formula for $R_n'(t)$:

$$R_n'(t) = -\frac{1}{t^2} - \frac{4}{(1+2t)^2} - \cdots - \frac{n^2}{(1+nt)^2}, \quad t > 0.$$

Substituting these two explicit formulas into (4.6) gives

$$\begin{aligned}
\mathrm{Var}(Y_n) &= \Big(1 + \frac{2}{3} + \cdots + \frac{n}{n+1}\Big) - \Big(1 + \frac{2^2}{3^2} + \cdots + \frac{n^2}{(n+1)^2}\Big) \\
&= \sum_{k=2}^{n} \Big( \frac{k}{k+1} - \frac{k^2}{(k+1)^2} \Big) = \sum_{k=2}^{n} \frac{k}{(k+1)^2} = \sum_{k=2}^{n} \Big( \frac{1}{k+1} - \frac{1}{(k+1)^2} \Big) \\
&= \Big(H_n + \frac{1}{n+1} - 1 - \frac{1}{2}\Big) - \Big(H_n^{(2)} + \frac{1}{(n+1)^2} - 1 - \frac{1}{4}\Big) \\
&= H_n - H_n^{(2)} + \frac{n}{(n+1)^2} - \frac{1}{4},
\end{aligned}$$

as required. The asymptotics stated in (1.5) follows from the latter formula when we invoke the approximations $H_n \approx \ln n + \gamma$ and $H_n^{(2)} \approx \pi^2/6$. □

4.2. *Block-counting tallied by size – proof of Theorem 1.7.*
In this subsection we consider the random variables $Y_n^{(\ell)}$ defined in Equation (1.6) of Theorem 1.7 – that is, $Y_n^{(\ell)}(\pi, u)$ counts the blocks of cardinality $\ell$ of $\pi$, for $(\pi, u) \in NC^{(\mathrm{mton})}(n)$. The main point of the subsection is to prove Proposition 4.6, which repeats, with some added details, the statement of Theorem 1.7.

*Notation and Remark* 4.4. For every $\ell, n \in \mathbb{N}$, the combinatorial Laplace transform of $Y_n^{(\ell)}$ (defined as in Equation (3.1)) will be denoted as $B_n^{(\ell)}$. Since $Y_n^{(\ell)}$ takes values in $\mathbb{N} \cup \{0\}$, every $B_n^{(\ell)}$ is a polynomial function. For example, direct inspection shows that for $\ell = 1$ we have $B_1^{(1)}(t) = t$, $B_2^{(1)}(t) = 2t^2 + 1$, $B_3^{(1)}(t) = 6t^3 + 5t + 1$, while for $\ell = 2$ we have $B_1^{(2)}(t) = 1$, $B_2^{(2)}(t) = t + 2$, $B_3^{(2)}(t) = 5t + 7$.

**Lemma 4.5.** *Let the polynomials $B_n^{(\ell)}$ be as defined above.*

$1^o$ *For $\ell = 1$, the polynomials $\big(B_n^{(1)}\big)_{n=1}^{\infty}$ satisfy the recursion*

$$B_n^{(1)}(t) = (nt+1)\, B_{n-1}^{(1)}(t) + (n-1)\,(1-t)\, B_{n-2}^{(1)}(t), \quad n \geq 3. \tag{4.7}$$

$2^o$ *For every $\ell \geq 2$, the polynomials $\big(B_n^{(\ell)}\big)_{n=1}^{\infty}$ satisfy the recursion*

$$B_n^{(\ell)}(t) = (n+1)\, B_{n-1}^{(\ell)}(t) + (t-1)\Big( (n-\ell+1)\, B_{n-\ell}^{(\ell)}(t) - (n-\ell)\, B_{n-\ell-1}^{(\ell)}(t) \Big), \tag{4.8}$$

*holding for $n \geq \ell + 2$.*

*Proof*: Both Equations (4.7) and (4.8) are obtained as special cases of the general recursion formula (3.3) from Theorem 3.2, where we take into account that (as seen in Example 2.9) the random variables $\big(Y_n^{(1)}\big)_{n=1}^{\infty}$ satisfy a recursion of the first kind with input vector $(1, -1)$, while for every $\ell \geq 2$ the random variables $\big(Y_n^{(\ell)}\big)_{n=1}^{\infty}$ satisfy a recursion of the first kind with input vector $(0, \ldots, 0, 1, -1) \in \mathbb{R}^{\ell+1}$. □

**Proposition 4.6.** *Consider the random variables* $Y_n^{(\ell)}$ *introduced in Theorem 1.7.*

$1^o$ *For* $\ell = 1$, *one has* $E[\,Y_n^{(1)}\,] = n - 2H_n + \frac{10}{3} - \frac{3}{n+1}$, $\quad n \geq 3$, *with the consequence that*

$$E[\,Y_n^{(1)}\,] \approx n - 2\,\ln n + \big(\frac{10}{3} - 2\gamma\big) \quad \textit{for } n \to \infty. \tag{4.9}$$

$2^o$ *For* $\ell = 2$, *one has* $E[\,Y_n^{(2)}\,] = H_n - \frac{51}{24} + \frac{6n-1}{2n(n+1)}$, $\quad n \geq 4$, *with the consequence that*

$$E[\,Y_n^{(2)}\,] \approx \ln n - \big(\frac{51}{24} - \gamma\big) \quad \textit{for } n \to \infty. \tag{4.10}$$

$3^o$ *For any fixed* $\ell \geq 3$ *and* $n \to \infty$, *the sequence* $\big(\,E[Y_n^{(\ell)}]\,\big)_{n=1}^{\infty}$ *has a limit* $c_\ell \in [0, \infty)$.

$4^o$ *Same as in Remark 1.8 of the Introduction, let* $Y_n^{(\geq 3)} : NC^{(\mathrm{mton})}(n) \to \mathbb{R}$ *be the random variable which counts all blocks of cardinality* $\geq 3$,

$$Y_n^{(\geq 3)}(\pi, u) := \big|\,\{V \in \pi \mid |V| \geq 3\}\,\big|\,, \quad \forall\,(\pi, u) \in NC^{(\mathrm{mton})}(n).$$

*Then* $E[Y_n^{(\geq 3)}] = \frac{7}{24} - \frac{2n-1}{2n(n+1)}$, $n \geq 4$, *with the consequence that* $\lim_{n\to\infty} E[\,Y_n^{(\geq 3)}\,] = \frac{7}{24}$.

*Proof*: $1^o$ When we differentiate both sides of Equation (4.7) and evaluate the result at $t = 1$, we find that

$$\begin{aligned}[B_n^{(1)}]'(1) &= nB_{n-1}(1) + (n+1)[B_{n-1}^{(1)}]'(1) - (n-1)\,B_{n-2}^{(1)}(1),\\ &= n \cdot \frac{n!}{2} + (n+1)[B_{n-1}^{(1)}]'(1) - (n-1)\,\frac{(n-1)!}{2}, \quad n \geq 3,\end{aligned} \tag{4.11}$$

where the second equality takes into account that $B_{n-1}(1) = |NC^{(\mathrm{mton})}(n-1)| = n!/2$, and likewise for $B_{n-2}(1) = (n-1)!/2$. But, as noted in Equation (3.2) from Remark 3.1, the derivative at 1 of a combinatorial Laplace transform is related to the expectation of the corresponding random variable, in a way which allows us to write

$$[B_n^{(1)}]'(1) = \frac{(n+1)!}{2}\,E[\,Y_n^{(1)}\,], \quad [B_{n-1}^{(1)}]'(1) = \frac{n!}{2}\,E[\,Y_{n-1}^{(1)}\,]. \tag{4.12}$$

Substituting (4.12) into (4.11) leads to the recursion

$$E[\,Y_n^{(1)}\,] = E[\,Y_{n-1}^{(1)}\,] + 1 + \frac{1}{n} - \frac{3}{n+1}, \quad n \geq 3, \tag{4.13}$$

which is initialized with the value $E[\,Y_2^{(1)}\,] = \frac{4}{3}$. From (4.13), an easy induction on $n$ leads to the explicit formula for $E[\,Y_n^{(1)}\,]$ stated in the proposition. This, in turn, immediately entails the asymptotics (4.9).

$2^o\&3^o$ For every $\ell \geq 2$, one can perform the same steps as indicated in the proof of $1^o$ above, but starting now from Equation (4.8) of Lemma 4.5. Upon performing the necessary differentiation, followed by setting $t = 1$ and by replacing the derivatives $[B_n^{(\ell)}]'(1)$, $[B_{n-1}^{(\ell)}]'(1)$ in terms of the expectations $E[Y_n^{(\ell)}]$, $E[Y_{n-1}^{(\ell)}]$, we find that

$$[B_n^{(\ell)}]'(1) = (n+1)\,[B_{n-1}^{(\ell)}]'(1) + (n-\ell+1) \cdot \frac{(n-\ell+1)!}{2} - (n-\ell) \cdot \frac{(n-\ell)!}{2},$$

and hence that

$$E[Y_n^{(\ell)}] = E[Y_{n-1}^{(\ell)}] + \frac{(n-\ell)!}{(n+1)!}\big[(n-\ell+1)^2 - (n-\ell)\big], \quad n \geq \ell + 2. \tag{4.14}$$

For the rest of the discussion, the case $\ell = 2$ goes differently from $\ell \geq 3$.

For $\ell \geq 3$, one only needs to note that (4.14) implies the inequality

$$\left| E[Y_n^{(\ell)}] - E[Y_{n-1}^{(\ell)}] \right| \leq \frac{(n-\ell)!}{(n+1)!}\big[(n-\ell+1)(n-\ell+2)\big] \leq \frac{1}{n(n+1)}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$, this implies that the sequence of $E[Y_n^{(\ell)}]$'s converges for $n \to \infty$.

For $\ell = 2$, the recursion (4.14) reduces to

$$E[\,Y_n^{(2)}\,] = E[\,Y_{n-1}^{(2)}\,] + \frac{n^2-3n+3}{(n-1)n(n+1)} \qquad = E[\,Y_{n-1}^{(2)}\,] + \Big[\,\frac{1}{2}\cdot\frac{1}{n-1} - 3\cdot\frac{1}{n} + \frac{7}{2}\cdot\frac{1}{n+1}\,\Big],$$

holding for $n \geq 4$. Telescoping the latter formula leads to

$$E[\,Y_n^{(2)}\,] = E[\,Y_3^{(2)}\,] - \frac{11}{24} + \Big[\,\frac{1}{5} + \cdots + \frac{1}{n-1}\,\Big] + \Big[\,\frac{1}{2}\cdot\frac{1}{n} + \frac{7}{2}\cdot\frac{1}{n+1}\,\Big], \tag{4.15}$$

Finally, upon performing the direct computation for $E[\,Y_3^{(2)}\,] = \frac{5}{12}$ and upon completing the harmonic number $H_n$ in (4.15), one finds the concrete expression for $E[Y_n^{(2)}]$ stated in $2^o$, which has the asymptotics (4.10) as an immediate consequence.

$4^o$ This follows by taking expectations in the relation $Y_n^{(\geq 3)} = Y_n - Y_n^{(1)} - Y_n^{(2)}$ (with $Y_n$ as in Theorem 1.6), and then by invoking the formulas for expectations obtained in Theorem 1.6 and in $1^o, 2^o$ of the present proposition. □

*Remark* 4.7. $1^o$ The limits $c_\ell$ in part $3^o$ of Proposition 4.6 can be explicitly computed by doing partial fraction decomposition in the formula for $E[Y_n^{(\ell)}] - E[Y_{n-1}^{(\ell)}]$ which comes out of Equation (4.14). For instance: for $\ell = 3$, this partial fraction decomposition is

$$E[Y_n^{(3)}] - E[Y_{n-1}^{(3)}] = \frac{1}{6}\cdot\frac{1}{n-2} - \frac{3}{2}\cdot\frac{1}{n-1} + \frac{7}{2}\cdot\frac{1}{n} - \frac{13}{6}\cdot\frac{1}{n+1}, \quad n \geq 5.$$

We leave it as an exercise to the patient reader to check that, upon telescoping the latter expression (in order to reduce $E[Y_n^{(3)}]$ to $E[Y_{n-1}^{(3)}]$, then to $E[Y_{n-2}^{(3)}], \ldots$, all the way to $E[Y_4^{(3)}] = 1/10$), one finds that $\lim_{n\to\infty} E[Y_n^{(3)}] = 23/90$.

$2^o$ The result in part $4^o$ of Proposition 4.6 can be obtained by the same method as shown in the proof of parts $1^o - 3^o$. Indeed, since the sequence $\big(\,Y_n^{(\geq 3)}\,\big)_{n=1}^{\infty}$ satisfies a recursion of the first kind with input vector $\underline{r} = (0,0,1) \in \mathbb{R}^3$, Theorem 3.2 gives us that the corresponding combinatorial Laplace transforms $B_n^{(\geq 3)}$ satisfy the recursion

$$B_n^{(\geq 3)}(t) = (n+1)\,B_{n-1}^{(\geq 3)}(t) + (t-1)\,(n-2)\,B_{n-3}^{(\geq 3)}(t), \quad n \geq 4.$$

This further implies a recursion for the expectations $E[Y_n^{(\geq 3)}]$, and entails the formulas stated in part $4^o$ of Proposition 4.6.

## 5. Moments of the monotonic Poisson process

In this section we make a brief detour from the main line of the paper, to point out that the considerations on the block-counting statistic from Section 4.1 can be cast in a non-commutative probability setting, where they become considerations on *monotonic cumulants*. In order to review the latter notion, we will use (as commonly done by people working in this area) a simplified algebraic version for what is a probability distribution on the real line. We want to look at such distributions where every polynomial is integrable, which makes it convenient to go with the following definition.

**Definition 5.1.** By *algebraic distribution* we will mean a linear functional $\mu : \mathbb{C}[X] \to \mathbb{C}$ with the property that $\mu(1) = 1$. For such a functional, it is customary that the number $\mu(X)$ is called *expectation* of $\mu$; and in general, for every $n \in \mathbb{N}$, the number $\mu(X^n) \in \mathbb{C}$ will be referred to as the *moment of order* $n$ of $\mu$.

*Remark and Definition* 5.2. Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be an algebraic distribution. There exists a sequence of complex numbers $\big(\, c_n(\mu) \,\big)_{n=1}^{\infty}$, uniquely determined, such that

$$\mu(X^n) = \sum_{(\pi,u)\in NC^{(\mathrm{mton})}(n)} \Big[ \frac{1}{|\pi|!} \cdot \prod_{V\in\pi} c_{|V|}(\mu) \Big], \ \forall\, n \in \mathbb{N}. \tag{5.1}$$

The numbers $c_n(\mu)$ are called *monotonic cumulants* of $\mu$, and Equation (5.1) goes under the name of "moment-cumulant formula" (for monotonic cumulants). Upon grouping terms on the right-hand side, we can also write it in the form:

$$\mu(X^n) = \sum_{\pi\in NC(n)} \Big[ \frac{\mathrm{monord}(\pi)}{|\pi|!} \cdot \prod_{V\in\pi} c_{|V|}(\mu) \Big], \ \ n \in \mathbb{N}, \tag{5.2}$$

where $\mathrm{monord}(\pi)$ stands for the number of monotonic orderings of $\pi \in NC(n)$. The ratio $\mathrm{monord}(\pi)/|\pi|!$ can be thought of as the probability that a randomly picked ordering of the blocks of $\pi$ is monotonic. This ratio is equal to 1 for an interval partition (when every $V \in \pi$ is a subinterval of $\{1,\ldots,n\}$), but is $< 1$ when nestings among the blocks of $\pi$ impose restrictions on what orderings of $\pi$ are monotonic.

The existence of a sequence $\big(\, c_n(\mu) \,\big)_{n=1}^{\infty}$, uniquely determined, such that Equation (5.1) holds, is easily seen by isolating in the sum on the right-hand side the unique term which has $\pi = \big\{\, \{1,\ldots,n\} \,\big\} =: 1_n$ (the partition with only one block). That special term is equal to $c_n(\mu)$, which allows us to rewrite (5.2) in the form

$$c_n(\mu) = \mu(X^n) - \sum_{\substack{\pi\in NC(n),\\ \pi\neq 1_n}} \Big[ \frac{\mathrm{monord}(\pi)}{|\pi|!} \cdot \prod_{V\in\pi} c_{|V|}(\mu) \Big], \tag{5.3}$$

and then use induction in order to write an explicit formula for $c_n(\mu)$ in terms of the moments $\mu(X^k),\ 1 \leq k \leq n$. For instance, for the first few values of $n$ we get:

$$\mu(X) = c_1(\mu),\ \ \mu(X^2) = c_2(\mu) + \big(\, c_1(\mu) \,\big)^2,\ \ \mu(X^3) = c_3(\mu) + \frac{5}{2} c_1(\mu)\, c_2(\mu) + \big(\, c_1(\mu) \,\big)^3,$$

which gives

$$c_1(\mu) = \mu(X),\ \ c_2(\mu) = \mu(X^2) - \big(\, \mu(X) \,\big)^2,\ \ c_3(\mu) = \mu(X^3) - \frac{5}{2}\mu(X)\, \mu(X^2) + \frac{3}{2}\big(\, \mu(X) \,\big)^3.$$

Another easily derived consequence of (5.2) and (5.3) is this: for every sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ there exists an algebraic distribution $\mu$, uniquely determined, such that $c_n(\mu) = \alpha_n$ for all $n \in \mathbb{N}$. In other words, it is possible to define an algebraic distribution $\mu$ via a prescription of what are the monotonic cumulants $c_n(\mu)$. This can in particular be used (following a well-established pattern from other brands of non-commutative probability) in order to pinpoint the monotonic version of Poisson distribution, as follows.

*Notation* 5.3. For every $\alpha \in [0,\infty)$ we will let $\nu_\alpha : \mathbb{C}[X] \to \mathbb{C}$ be the algebraic distribution defined via the requirement that it has monotonic cumulants

$$c_n(\nu_\alpha) = \alpha, \quad \forall\, n \in \mathbb{N}. \tag{5.4}$$

The family of distributions $\big(\, \nu_\alpha \,\big)_{\alpha\in[0,\infty)}$ goes under the name of *monotonic Poisson process*.

In Belton (2006), a nice description was found for the moments of the distributions $\nu_\alpha$, in terms of a double-array of integers $J_k^{(n)}$ (not depending on $\alpha$) which satisfy a simple recursive relation. We repeat this here, and observe how the said recursive relation can be read by looking at the $NC^{(\mathrm{mton})}$-tree.

**Proposition 5.4.** (Belton, 2006, Section 2) *The moments of the monotone Poisson process are*

$$\nu_\alpha(X^n) = \sum_{k=1}^{n} J_k^{(n)} \cdot \frac{\alpha^k}{k!}, \quad \forall\, \alpha \in [0,\infty),\ n \in \mathbb{N}, \tag{5.5}$$

*with the double array* $\{J_k^{(n)} \mid n,k \in \mathbb{N},\ n \geq k\}$ *defined by initializing*

$$J_1^{(n)} = 1 \text{ and } J_n^{(n)} = n!\,, \quad \forall\, n \in \mathbb{N} \tag{5.6}$$

*(in particular* $J_1^{(1)} = 1$, $J_1^{(2)} = 1$, $J_2^{(2)} = 2$*), and then by requiring that*

$$J_k^{(n)} = J_k^{(n-1)} + nJ_{k-1}^{(n-1)}, \quad \text{for all } n \geq 3 \text{ and } 2 \leq k \leq n-1. \tag{5.7}$$

*Proof*: The uniqueness of the numbers $J_k^{(n)}$ is clearly implied by the requirements made in (5.6) + (5.7). For the existence part of the proposition, it will be convenient to define

$$J_k^{(n)} := \big|\, \{\, (\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k \,\}\, \big|, \text{ for } 1 \leq k \leq n \text{ in } \mathbb{N}. \tag{5.8}$$

Then for every $\alpha \in [0,\infty)$, the moment-cumulant formula (5.1) used for $\nu_\alpha$ gives:

$$\begin{aligned}\nu_\alpha(X^n) &= \sum_{(\pi,u)\in NC^{(\mathrm{mton})}(n)} \frac{1}{|\pi|!} \cdot \alpha^{|\pi|} \\ &= \sum_{k=1}^{n} \Big[ \sum_{\substack{(\pi,u)\in NC^{(\mathrm{mton})}(n) \\ \text{with } |\pi|=k}} \frac{\alpha^k}{k!} \Big] = \sum_{k=1}^{n} J_k^{(n)} \cdot \frac{\alpha^k}{k!},\ n \in \mathbb{N},\end{aligned}$$

showing that (5.5) holds. It is also immediate that the numbers defined by (5.8) satisfy the initial conditions (5.6), since $J_1^{(n)}$ counts the unique ordering of the partition $1_n$, while $J_n^{(n)}$ counts the $n!$ orderings of the partition $0_n := \{\, \{1\}, \{2\}, \ldots, \{n\} \,\} \in NC(n)$. We are thus left to verify that the $J_k^{(n)}$'s from (5.8) satisfy the recursion relation (5.7).

For the rest of the proof we fix $n \geq 3$ and $2 \leq k \leq n-1$, for which we will verify that (5.7) holds. Towards that end we decompose the set $\{(\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k\}$ according to whether the block $J(\pi,u)$ of $\pi$ is a singleton-block or not. We get

$$\{\, (\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k \,\} = S_0 \cup S_1 \cup \cdots \cup S_n, \text{ disjoint union},$$

where $S_0 := \{\, (\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k \text{ and } |J(\pi,u)| \geq 2\}$, and where for every $1 \leq m \leq n$ we put $S_m := \{\, (\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k \text{ and } J(\pi,u) = \{m\}\}$. We leave it as a straightforward exercise to the reader to use the description of the parent map $\mathfrak{p}$ of the $NC^{(\mathrm{mton})}$-tree (cf. Definition 2.2) in order to check that, for every $1 \leq m \leq n$, the map $\mathfrak{p}$ sends $S_m$ bijectively onto $\{\, (\rho,v) \in NC^{(\mathrm{mton})}(n-1) \mid |\rho| = k-1 \,\}$, with the consequence that $|S_m| = \big|\, \{\, (\rho,v) \in NC^{(\mathrm{mton})}(n-1) \mid |\rho| = k-1 \,\}\, \big| = J_{k-1}^{(n-1)}$. A similar exercise shows that $\mathfrak{p}$ sends $S_0$ bijectively onto $\{\, (\rho,v) \in NC^{(\mathrm{mton})}(n-1) \mid |\rho| = k \,\}$, with the consequence that $|S_0| = J_k^{(n-1)}$. By putting these things together we find that

$$J_k^{(n)} = \big|\, \{\, (\pi,u) \in NC^{(\mathrm{mton})}(n) \mid |\pi| = k\}\, \big| = |S_0| + |S_1| + \cdots + |S_n| = J_k^{(n-1)} + nJ_{k-1}^{(n-1)},$$

as required. □

In Belton (2006, Section 2) it is explained that the numbers $J_k^{(n)}$ defined by the recursion (5.7) are part of a larger family of "generalized Stirling numbers of first kind". The first few lines of the double-array $[J_k^{(n)}]_{n,k}$ are shown in Figure 4 below. It is actually possible to give an explicit formula for an individual $J_k^{(n)}$, as shown in Equation (5.10) of the next proposition. We note that in the analytic approach of Belton, the statements of Propositions 5.4 and 5.5 came in reverse order, with the formula (5.10) derived first, in the earlier paper Belton (2005).

| | $(k=1)$ | $(k=2)$ | $(k=3)$ | $(k=4)$ | $(k=5)$ | $(k=6)$ |
|---|---|---|---|---|---|---|
| $(n=1)$ | 1 | | | | | |
| $(n=2)$ | 1 | 2 | | | | |
| $(n=3)$ | 1 | 5 | 6 | | | |
| $(n=4)$ | 1 | 9 | 26 | 24 | | |
| $(n=5)$ | 1 | 14 | 71 | 154 | 120 | |
| $(n=6)$ | 1 | 20 | 155 | 580 | 1044 | 720 |

Figure 4. *The numbers $J_k^{(n)}$ for $n \leq 6$ and $k \in \{1, \ldots, n\}$.*

**Proposition 5.5.** (Belton, 2005, Section 4) *Let $n \geq 2$ be in $\mathbb{N}$, and consider the numbers $J_1^{(n)}$, $J_2^{(n)}, \ldots, J_n^{(n)}$ introduced in Proposition 5.4. One has the polynomial factorization*

$$J_1^{(n)} t + J_2^{(n)} t^2 + \cdots + J_n^{(n)} t^n = t(1+2t)\cdots(1+nt), \tag{5.9}$$

*with the consequence that one can explicitly write*

$$J_k^{(n)} = \sum_{2 \leq j_1 < \cdots < j_{k-1} \leq n} j_1 \cdots j_{k-1}, \ 2 \leq k \leq n. \tag{5.10}$$

*Proof*: It is convenient to take advantage of the framework developed in Section 4.1, which we do by using the formula for $J_k^{(n)}$ provided by Equation (5.8) from the proof of Proposition 5.4. We get:

$$\sum_{k=1}^{n} J_k^{(n)} t^k = \sum_{k=1}^{n} \Big[ \sum_{\substack{(\pi,u) \in NC^{(\mathrm{mton})}(n) \\ \text{with } |\pi| = k}} t^{|\pi|} \Big] = \sum_{(\pi,u) \in NC^{(\mathrm{mton})}(n)} t^{|\pi|} = B_n(t),$$

where $B_n$ is the polynomial introduced in Notation 4.1. Lemma 4.2 then gives us the factorization (5.9), and the formula (5.10) for $J_k^{(n)}$ follows when we extract the coefficient of $t^k$ on the two sides of (5.9). □

## 6. Laplace transforms for the $NC^{(\mathrm{mton})}$-recursion of the second kind

We now take on a development analogous to the one from Sections 3 and 4, in connection to the $NC^{(\mathrm{mton})}$-recursion property "of the second kind" introduced in Section 2.3. More precisely: we first find, in Theorem 6.1, how this $NC^{(\mathrm{mton})}$-recursion property yields a recursion for the combinatorial Laplace transforms of the random variables in question. Then we see how Theorem 6.1 can be applied in the setting of Example 2.12, in order to determine the expected number of outer blocks of a random $(\pi, u) \in NC^{(\mathrm{mton})}(n)$.

**Theorem 6.1.** *Let $\alpha, \beta, q \in \mathbb{Z}$ and let $(Z_n : NC^{(\mathrm{mton})}(n) \to \mathbb{R})_{n=1}^{\infty}$ be a sequence of random variables which satisfies a recursion of the second kind with input $(\alpha, \beta; q)$, in the sense of Definition 2.11.*

*For every $n \in \mathbb{N}$, let $L_n : (0,\infty) \to \mathbb{R}$ be the combinatorial Laplace transform of $Z_n$, defined by Equation (3.1) of Definition 3.1. Then one has the recursion:*

$$L_n(t) = \Big(q\,t^{\alpha} + ((n+1)-q)\,t^{\beta}\Big) \cdot L_{n-1}(t) + \big(t^{\alpha+1} - t^{\beta+1}\big) \cdot L'_{n-1}(t), \tag{6.1}$$

*holding for $n \geq 2$ and $t \in (0,\infty)$.*

*Proof*: Consider an $n \geq 2$ in $\mathbb{N}$ and a $t \in (0,\infty)$, for which we will verify (6.1). Following the same idea as in the proof of Theorem 3.2, we write the sum defining $L_n(t)$ as

$$L_n(t) = \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} \Big[ \sum_{(\pi,u)\in C(\rho,v)} t^{Z_n(\pi,u)} \Big], \tag{6.2}$$

where $C(\rho,v)$ denotes the set of children of $(\rho,v)$ in the $NC^{(\mathrm{mton})}$-tree.

Let us fix for the moment a $(\rho,v) \in NC^{(\mathrm{mton})}(n-1)$, and examine the inner sum corresponding to $(\rho,v)$ in (6.2). We break this sum into two by using the set $C_o(\rho,v) \subseteq C(\rho,v)$ provided by the definition of the recursion of the second kind (cf. (2.8) there), and get:

$$\begin{aligned}
\sum_{(\pi,u)\in C(\rho,v)} t^{Z_n(\pi,u)} &= \sum_{(\pi,u)\in C_o(\rho,v)} t^{Z_n(\pi,u)} + \sum_{(\pi,u)\in C(\rho,v)\setminus C_o(\rho,v)} t^{Z_n(\pi,u)} \\
&= |C_o(\rho,v)| \cdot t^{Z_{n-1}(\rho,v)+\alpha} + |C(\rho,v) \setminus C_o(\rho,v)| \cdot t^{Z_{n-1}(\rho,v)+\beta} \\
&= (Z_{n-1}(\rho,v)+q) \cdot t^{Z_{n-1}(\rho,v)+\alpha} + ((n+1) - Z_{n-1}(\rho,v) - q) \cdot t^{Z_{n-1}(\rho,v)+\beta} \\
&= t^{Z_{n-1}(\rho,v)} \Big( (t^{\alpha} - t^{\beta}) \cdot Z_{n-1}(\rho,v) + \big( qt^{\alpha} + ((n+1)-q)t^{\beta} \big) \Big).
\end{aligned}$$

Returning now to Equation (6.2), we see that the calculation shown in the preceding paragraph decomposes as $L_n(t) = \Sigma' + \Sigma''$, where

$$\begin{aligned}
\Sigma' &= \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} t^{Z_{n-1}(\rho,v)} \cdot \big(t^{\alpha} - t^{\beta}\big)\, Z_{n-1}(\rho,v) \\
&= \big(t^{\alpha+1} - t^{\beta+1}\big) \cdot \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} Z_{n-1}(\rho,v)\, t^{Z_{n-1}(\rho,v)-1} \\
&= \big(t^{\alpha+1} - t^{\beta+1}\big) \cdot L'_{n-1}(t),
\end{aligned}$$

and

$$\begin{aligned}
\Sigma'' &= \sum_{(\rho,v)\in NC^{(\mathrm{mton})}(n-1)} t^{Z_{n-1}(\rho,v)} \cdot \big(q\,t^{\alpha} + ((n+1)-q)\,t^{\beta}\big) \\
&= \big(q\,t^{\alpha} + ((n+1)-q)\,t^{\beta}\big) \cdot L_{n-1}(t).
\end{aligned}$$

The expression $\Sigma' + \Sigma''$ gives the right-hand side of the required formula (6.1). □

**Corollary 6.2.** *In the framework and notation of Theorem 6.1, the sequence of expectations $\big(E[Z_n]\big)_{n=1}^{\infty}$ satisfies the recursion*

$$E[Z_n] = \frac{(n+1)+(\alpha-\beta)}{n+1} \cdot E[Z_{n-1}] + \frac{\alpha q + \beta((n+1)-q)}{n+1}, \quad n \geq 2. \tag{6.3}$$

*Proof*: Differentiating both sides of Equation (6.1) and then evaluating them at $t = 1$ gives

$$L'_n(1) = \big(\,\alpha q + \beta((n+1)-q)\,\big) \cdot \frac{n!}{2} + \big(\,(n+1)+(\alpha-\beta)\,\big) \cdot L'_{n-1}(1). \tag{6.4}$$

We know (as noted in Remark 3.1) that in (6.4) we can substitute $L'_n(1) = E[Z_n] \cdot (n+1)!/2$ and $L'_{n-1}(1) = E[Z_{n-1}] \cdot n!/2$; doing so leads to the formula (6.3) stated in the corollary. □

**Corollary 6.3.** *Let* $\big(\,\mathrm{Out}_n : NC^{(\mathrm{mton})}(n) \to \mathbb{N}\,\big)_{n=1}^{\infty}$ *be the sequence of random variables considered in Example 2.12, where* $\mathrm{Out}_n(\pi, u)$ *counts the outer blocks of* $\pi$*, for* $(\pi, u) \in NC^{(\mathrm{mton})}(n)$*. One has that*

$$E[\mathrm{Out}_n] = \frac{2n+1}{3}, \quad n \in \mathbb{N}. \tag{6.5}$$

*Proof*: We know from Example 2.12 that the sequence of $\mathrm{Out}_n$'s satisfies a recursion of the second kind, with parameters $\alpha = 1$, $\beta = 0$ and $q = 1$. We can thus use the formula (6.3) provided by the preceding corollary. This says, for the case at hand, that we have

$$E[\mathrm{Out}_n] = \big(1 + \frac{1}{n+1}\big) \cdot E[\mathrm{Out}_{n-1}] + \frac{1}{n+1}, \quad n \geq 2. \tag{6.6}$$

From (6.6), an easy induction (initialized with $E[\mathrm{Out}_1] = 1$) gives the formula (6.5). □

## 7. The $NC_2^{(\mathrm{mton})}$-tree, and some combinatorial statistics related to it

We now turn our attention to sets of monotonically ordered non-crossing *pair-partitions*,

$$NC_2^{(\mathrm{mton})}(2n) := \Big\{ (\pi, u) \;\Big|\; \begin{array}{r} \pi \in NC_2(2n) \text{ and } u : \pi \to \{1, \ldots, n\}, \\ \text{monotonic ordering} \end{array} \Big\}, \quad n \in \mathbb{N}.$$

In this section we follow up on the discussion around these sets made in Subsection 1.3 of the Introduction, and we start with the fact that one has a natural tree structure on the union $NC_2^{(\mathrm{mton})} := \sqcup_{n=1}^{\infty} NC_2^{(\mathrm{mton})}(2n)$.

### 7.1. *Description of the* $\boldsymbol{NC_2^{(\mathrm{mton})}}$*-tree.*

*Remark* 7.1. Since $NC_2^{(\mathrm{mton})}(2n) \subseteq NC^{(\mathrm{mton})}(2n)$, we can and will continue to refer to Notation 2.1 and write "$J(\pi, u)$" for the block of a pair-partition $\pi \in NC_2(2n)$ which has the maximal label in the monotone ordering $u$ of $\pi$. The specifics of the $(\pi, u)$ considered here ensure that $|J(\pi, u)| = 2$ and $u(\,J(\pi, u)\,) = n$. Since we additionally know (cf. Remark 1.2) that $J(\pi, u)$ is a sub-interval of $\{1, \ldots, 2n\}$, it follows that $J(\pi, u) = \{m, m+1\}$ for some $1 \leq m \leq 2n-1$.

$2^o$ We will also continue to use the "restrict-and-relabel" operation $\pi \mapsto \pi_W$ introduced in Notation 2.1, making sure that we only apply it to situations where both $\pi$ and $\pi_W$ belong to $NC_2^{(\mathrm{mton})}$.

$3^o$ Similarly to the development shown for the $NC^{(\mathrm{mton})}$-tree in Section 2.1, we will describe the edges of the $NC_2^{(\mathrm{mton})}$-tree by indicating how one finds the $NC_2^{(\mathrm{mton})}$-parent of a $(\pi, u) \in NC_2^{(\mathrm{mton})}(2n)$, for $n \geq 2$. In order to distinguish the parent in the $NC_2^{(\mathrm{mton})}$-tree from the parent in the $NC^{(\mathrm{mton})}$-tree introduced in Definition 2.2, we will refer to the $NC_2^{(\mathrm{mton})}$-parent of such a $(\pi, u)$ by denoting it as $\mathtt{pp}(\pi, u)$. This is done in the next definition. The notation is consistent with the fact that, if $(\pi, u)$ was to be viewed as vertex of the $NC^{(\mathrm{mton})}$-tree, then $\mathtt{pp}(\pi, u) \in NC_2^{(\mathrm{mton})}(2n-2)$ would come out as "the grandparent" (parent of the parent) of $(\pi, u)$ in that context.

**Definition 7.2.** Let $(\pi, u)$ be a monotonically ordered pair-partition in $NC_2^{(\mathrm{mton})}(2n)$, where $n \geq 2$. Consider the pair $J(\pi, u) = \{m, m+1\} \in \pi$, and consider the restrict-and-relabel partition

$$\rho := \pi_{\{1,\ldots,2n\} \setminus \{m, m+1\}} \in NC_2(2n-2).$$

It is easily checked that the monotonic order $u : \pi \to \{1, \ldots, n\}$ induces a monotonic order $v : \rho \to \{1, \ldots, n-1\}$, defined by the formula:

$$\begin{cases} v\big(\{p,q\}\big) := u\Big(\big\{\phi(p), \phi(q)\big\}\Big) \text{ for } \{p,q\} \in \rho, \text{ where } \phi \text{ is the unique} \\ \text{increasing bijection from } \{1, \ldots, 2n-2\} \text{ onto } \{1, \ldots, 2n\} \setminus \{m, m+1\}. \end{cases} \tag{7.1}$$

The monotonically ordered pair-partition $(\rho, v) \in NC_2^{(\mathrm{mton})}(2n-2)$ obtained in this way will be called *the pair-parent* of $(\pi, u)$ and will be denoted as $\mathfrak{pp}(\pi, u)$.

**Example 7.3.** Suppose that $n = 7$ and that we look at the pair-partition

$$\pi = \big\{ \{1,6\}, \{2,3\}, \{4,5\}, \{7,8\}, \{9,14\}, \{10,13\}, \{11,12\} \big\} \in NC_2(14),$$

considered with the monotonic ordering $u : \pi \to \{1, \ldots, 7\}$ shown in the following picture:

$$(\pi, u) = \boxed{2 \ \boxed{6} \ \boxed{3}} \ \boxed{1} \ \boxed{4 \ \boxed{5} \ \boxed{7}} \in NC_2^{(\mathrm{mton})}(14).$$

Then $J(\pi, u) = \{11, 12\}$ and the pair-partition $(\rho, v) = \mathfrak{pp}(\pi, u) \in NC_2^{(\mathrm{mton})}(12)$ is depicted like this:

$$(\rho, v) = \boxed{2 \ \boxed{6} \ \boxed{3}} \ \boxed{1} \ \boxed{4 \ \boxed{5}} \in NC_2^{(\mathrm{mton})}(12).$$

*Notation and Remark* 7.4. $1^o$ Let $n \geq 2$ be in $\mathbb{N}$ and consider a $(\rho, v) \in NC_2^{(\mathrm{mton})}(2n-2)$. The set of children of $(\rho, v)$ in the $NC_2^{(\mathrm{mton})}$-tree will be denoted as $C^{(\mathrm{pair})}(\rho, v)$. Thus $C^{(\mathrm{pair})}(\rho, v) \subseteq NC_2^{(\mathrm{mton})}(2n)$, and upon examining the parent-child relation described in Definition 7.2 one easily sees that:

$$\begin{cases} \text{for every } 1 \leq m \leq 2n-1 \text{ there exists a } (\pi_m, u_m) \in C^{(\mathrm{pair})}(\rho, v), \\ \quad \text{uniquely determined, such that } J(\pi_m, u_m) = \{m, m+1\}. \end{cases} \tag{7.2}$$

As a consequence of (7.2), it follows that $|C^{(\mathrm{pair})}(\rho, v)| = 2n - 1$.

$2^o$ For the record, we note here that the homogeneity property (1.10) stated in Subsection 1.3 of the Introduction is an immediate consequence of (7.2), and same for the further consequence stated in (1.11), that $|NC_2^{(\mathrm{mton})}(2n)| = (2n-1)!!$ for every $n \in \mathbb{N}$.

### 7.2. $\boldsymbol{NC_2^{(\mathrm{mton})}}$*-analogue for the recursive pattern "of the second kind".*

For a random $(\pi, u) \in NC_2^{(\mathrm{mton})}(2n)$, the statistic which counts the blocks of $\pi$ is trivial, and the same applies to the statistics which count blocks tallied by size. It is, however, non-trivial to examine the statistic which counts the outer blocks of $\pi$. In what follows we will see how this can be done by adjusting to $NC_2^{(\mathrm{mton})}$-setting the discussion around the recursive condition of the second kind which was introduced in Subsection 2.3 in connection to the full $NC^{(\mathrm{mton})}$-tree. Let us begin with the adjusted version of that notion.

**Definition 7.5.** Let $\alpha, \beta, q \in \mathbb{Z}$ be given. We will say that a sequence of random variables $(Z_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R})_{n=1}^{\infty}$ is *pair-recursive of the second kind* with input $(\alpha, \beta; q)$ when it has the

following property: for every $n \geq 2$ and $(\rho, v) \in NC_2^{(\mathrm{mton})}(2n-2)$, the set $C^{(\mathrm{pair})}(\rho, v)$ of children of $(\rho, v)$ in the $NC_2^{(\mathrm{mton})}$-tree has a subset $C_o^{(\mathrm{pair})}(\rho, v)$ such that:

$$\begin{cases} (1) & \text{If } (\pi, u) \in C_o^{(\mathrm{pair})}(\rho, v), \text{ then } Z_n(\pi, u) = Z_{n-1}(\rho, v) + \alpha, \\ (2) & \text{If } (\pi, u) \in C^{(\mathrm{pair})}(\rho, v) \setminus C_o^{(\mathrm{pair})}(\rho, v), \text{ then } Z_n(\pi, u) = Z_{n-1}(\rho, v) + \beta, \\ \text{and} & \\ (3) & |C_o^{(\mathrm{pair})}(\rho, v)| = Z_{n-1}(\rho, v) + q. \end{cases} \tag{7.3}$$

*Remark and Notation* 7.6. In analogy with what we saw in the full $NC^{(\mathrm{mton})}$-setting, the recursion property from Definition 7.5 entails a recursion for the combinatorial Laplace transforms of the random variables in question. For any $n \in \mathbb{N}$ and $Z_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}$, the combinatorial Laplace transform of $Z_n$ is the function $L_n : (0, \infty) \to \mathbb{R}$ defined by a formula similar to the one used in the preceding sections,

$$L_n(t) = \sum_{(\pi,u) \in NC_2^{(\mathrm{mton})}(2n)} t^{Z_n(\pi,u)}, \quad t > 0. \tag{7.4}$$

When $Z_n$ is viewed as a random variable with respect to the uniform distribution on $NC_2^{(\mathrm{mton})}(2n)$, the function $L_n$ can be used towards computing moments of $Z_n$, and in particular the expectation of $Z_n$ is retrieved via the formula

$$E[Z_n] = \frac{L_n'(1)}{L_n(1)} = \frac{L_n'(1)}{(2n-1)!!} \tag{7.5}$$

(similar to the one from the parallel discussion about random variables on $NC^{(\mathrm{mton})}(n)$, but where we now use the normalization constant $L_n(1) = |NC_2^{(\mathrm{mton})}(2n)| = (2n-1)!!$).

The next theorem and corollary give the $NC_2^{(\mathrm{mton})}$-analogue for Theorem 6.1 and Corollary 6.2 of the preceding section. The proofs are mutatis mutandis repetitions of arguments already shown in Section 6, and are left to the reader.

**Theorem 7.7.** *Let $\alpha, \beta, q \in \mathbb{Z}$ and let $(Z_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R})_{n=1}^{\infty}$ be a sequence of random variables which is pair-recursive of the second kind with input $(\alpha, \beta; q)$. For every $n \in \mathbb{N}$, let $L_n : (0, \infty) \to \mathbb{R}$ be the combinatorial Laplace transform of $Z_n$. Then one has the recursion:*

$$L_n(t) = \Big(q\, t^{\alpha} + ((2n-1) - q)\, t^{\beta}\Big) \cdot L_{n-1}(t) + \big(t^{\alpha+1} - t^{\beta+1}\big) \cdot L_{n-1}'(t), \tag{7.6}$$

*holding for $n \geq 2$ and $t \in (0, \infty)$.* □

**Corollary 7.8.** *In the framework and notation of Theorem 7.7, the sequence of expectations $\big(E[Z_n]\big)_{n=1}^{\infty}$ satisfies the recursion*

$$E[Z_n] = \frac{(2n-1) + (\alpha - \beta)}{2n-1} \cdot E[Z_{n-1}] + \frac{q\,\alpha + ((2n-1) - q)\,\beta}{2n-1}, \quad n \geq 2. \tag{7.7}$$

□

7.3. *Outer pairs and interval pairs – proof of Theorem 1.10.*
In this subsection we consider the random variables $\big(\mathrm{Out}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}\big)_{n=1}^{\infty}$ and $\big(\mathrm{Int}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}\big)_{n=1}^{\infty}$ that were brought to attention in Subsection 1.3, and we give the proof of Theorem 1.10 concerning the expectations of these random variables. The latter theorem will actually come out as an easy consequence of the considerations made in Subsection 7.2, due to the following lemma.

**Lemma 7.9.**
$1^o$ *The sequence of random variables* $\big(\mathrm{Int}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}\big)_{n=1}^{\infty}$ *is pair-recursive of the second kind, with input* $(0, 1; 0)$.

$2^o$ *The sequence of random variables* $\big(\mathrm{Out}_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}\big)_{n=1}^{\infty}$ *is pair-recursive of the second kind, with input* $(1, 0; 1)$.

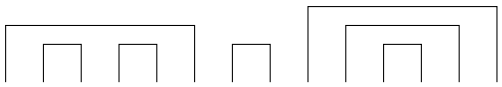
*Proof*: $1^o$ Pick an $n \geq 2$ and a $(\rho, v) \in NC_2^{(\mathrm{mton})}(2n-2)$, in connection to which we verify the conditions (1), (2), (3) stated in Definition 7.5. Let $m$ be the number of interval pairs of $\rho$, and let us explicitly denote these pairs as $\{i_1, i_1+1\}, \ldots, \{i_m, i_m+1\}$.

Now let us look at a $(\pi, u) \in C^{(\mathrm{pair})}(\rho, v) \subseteq NC_2^{(\mathrm{mton})}(2n)$ (a child of $(\rho, v)$ in the $NC_2^{(\mathrm{mton})}$-tree), and let $V := J(\pi, u) \in \pi$ be the interval pair which is appended to $\rho$ in order to go from $(\rho, v)$ to $(\pi, u)$. A moment's thought shows that we have:

$$\mathrm{Int}_n(\pi, u) = \begin{cases} \mathrm{Int}_{n-1}(\rho, v), & \text{if } V \text{ is inserted into one of the} \\ & \text{pairs } \{i_1, i_1+1\}, \ldots, \{i_m, i_m+1\} \text{ of } \rho, \\ \mathrm{Int}_{n-1}(\rho, v) + 1, & \text{otherwise.} \end{cases} \tag{7.8}$$

This verifies the conditions (1) and (2) from (7.3), with $\alpha = 0$ and $\beta = 1$. We note, moreover, that the number of possibilities counted in the first branch on the right-hand side of (7.8) is precisely equal to $\mathrm{Int}_{n-1}(\rho, v)$ (which is our $m$); this shows that the condition (7.3)(3) is also fulfilled, with $q = 0$.

In order to illustrate the discussion made in the preceding paragraph, consider an example when $n = 8$ and $(\rho, v) \in NC_2^{(\mathrm{mton})}(14)$ has underlying pair-partition

$\rho =$  $\in NC_2(14)$

(that is, $\rho = \big\{\{1,6\}, \{2,3\}, \{4,5\}, \{7,8\}, \{9,14\}, \{10,13\}, \{11,12\}\big\} \in NC_2(14)$). This $\rho$ has 4 interval pairs: $\{2,3\}$, $\{4,5\}$, $\{7,8\}$, $\{11,12\}$. The next picture shows, in dotted lines, how an interval pair "$V = J(\pi, u)$" can be inserted into one of these interval pairs of $\rho$.

In each of the 4 cases shown in dotted lines, the resulting $(\pi, u) \in NC_2^{(\mathrm{mton})}(16)$ will have $\mathrm{Int}_8(\pi, u) = \mathrm{Int}_7(\rho, v)$. (For instance if the inserted pair is $V = \{3,4\}$, then $V$ itself is an interval pair of $\pi$, but the former interval pair $\{2,3\} \in \rho$ has now become $\{2,5\} \in \pi$, and is lost from the tallying of interval pairs of $\pi$.) On the other hand, if the inserted pair $V$ is not one of those shown in dotted lines in the picture, then the resulting $(\pi, u) \in NC_2^{(\mathrm{mton})}(16)$ will have $\mathrm{Int}_8(\pi, u) = \mathrm{Int}_7(\rho, v) + 1 = 5$. This is because, in addition to $V$, the pair-partition $\pi$ will have 4 interval pairs that are inherited from $\rho$.

$2^o$ The proof of this part goes on similar lines to the one for $1^o$, and is also very similar to the arguments presented in connection to Example 2.12 earlier on. Because of that, we will omit the details and only briefly explain how the requirements made in Definition 7.5 get to be fulfilled.

So let us pick an $n \geq 2$ and a $(\rho, v) \in NC_2^{(\mathrm{mton})}(2n-2)$. Direct analysis, similar to the one from Example 2.12 shows that for $(\pi, u) \in C^{(\mathrm{pair})}(\rho, v)$ we have:

$$\mathrm{Out}_n(\pi, u) = \begin{cases} \mathrm{Out}_{n-1}(\rho, v) + 1, & \text{if } J(\pi, u) \text{ is an outer pair of } \pi, \\ \mathrm{Out}_{n-1}(\rho, v), & \text{otherwise.} \end{cases} \tag{7.9}$$

Thus the set

$$C_o^{(\mathrm{pair})}(\rho, v) := \{(\pi, u) \in C^{(\mathrm{pair})}(\rho, v) \mid J(\pi, u) \text{ is an outer pair of } \pi\}$$

will fulfill conditions (1) and (2) from Definition 7.5, with $\alpha = 1$ and $\beta = 0$. The remaining condition (3) amounts to the fact that, in (7.9), there are exactly $\mathrm{Out}_{n-1}(\rho, v) + 1$ choices of $(\pi, u)$ for which $J(\pi, u)$ is an outer pair of $\pi$. This is indeed true, because an interval pair appended to $\rho$ is outer precisely when it is inserted in between two consecutive outer pairs of $\rho$, or at one of the two ends (left or right) of the picture of $\rho$, and the number of ways to do this is equal to $\big(\mathrm{Out}_{n-1}(\rho, v) - 1\big) + 2 = \mathrm{Out}_{n-1}(\rho, v) + 1$, as desired.

For illustration, consider again the example where $n = 8$ and $(\rho, v) \in NC_2^{(\mathrm{mton})}(14)$ has the same underlying $\rho \in NC_2(14)$ used to illustrate the proof of part $1^o$ of the lemma. There are 4 possible ways to insert in $\rho$ an interval pair which is outer in the resulting $\pi \in NC_2(16)$; these 4 possibilities are indicated with dotted lines in the next picture.

As seen in the picture, the 4 dotted possibilities correspond precisely to the 2 spaces between the 3 outer pairs of $\rho$, plus the space to the left of $\{1, 6\}$ (leftmost outer pair of $\rho$) and the one to the right of $\{9, 14\}$ (rightmost outer pair of $\rho$). □

7.10. **Proof of Theorem 1.10.** $1^o$ In view of Lemma 7.9, we can invoke the recursion (7.7) found in Corollary 7.8, where we put $\alpha = 0$, $\beta = 1$, $q = 0$. The said recursion becomes

$$E[\mathrm{Int}_n] = \frac{2n-2}{2n-1} \cdot E[\mathrm{Int}_{n-1}] + 1, \quad n \geq 2; \tag{7.10}$$

from here, an easy induction verifies the claim that $E[\mathrm{Int}_n] = (2n+1)/3$ for all $n \in \mathbb{N}$.

$2^o$ In the same vein as in $1^o$ above, we now use the recursion (7.7) with specialization $\alpha = 1$, $\beta = 0$, $q = 1$, and this gives:

$$E[\mathrm{Out}_n] = \frac{2n}{2n-1} \cdot E[\mathrm{Out}_{n-1}] + \frac{1}{2n-1}, \quad n \geq 2. \tag{7.11}$$

When examining the formula (7.11), it is useful to observe that it can be rewritten as

$$E[\mathrm{Out}_n] + 1 = \frac{2n}{2n-1} \cdot \big(E[\mathrm{Out}_{n-1}] + 1\big), \quad n \geq 2;$$

from here, an immediate induction establishes the fact that for every $n \in \mathbb{N}$ we have

$$E[\mathrm{Out}_n] + 1 = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} = \frac{2^n \cdot n!}{(2n-1)!!},$$

which leads to the formula (1.13) in part $2^o$ of Theorem 1.10. The asymptotics $\mathbb{E}[\mathrm{Out}_n] \sim \sqrt{\pi n}$ for $n \to \infty$ follows from (1.13) via an easy application of Stirling's approximation formula. □

## 8. Expectation of the "area" statistic on $NC_2^{(\mathrm{mton})}(2n)$

In this section we consider the framework from Section 1.4 of the Introduction, and we give the proof of Theorem 1.11, concerning the expectation of the area-measuring random variable $A_n : NC_2^{(\mathrm{mton})}(2n) \to \mathbb{R}$. Recall that the latter random variable was introduced by putting $A_n(\pi, u) := \mathrm{area}(\pi)$, the area in between the horizontal axis and the Dyck path naturally associated to $\pi$. In the next lemma we give an alternate description for what is $\mathrm{area}(\pi)$, which does not refer to Dyck paths.

**Lemma 8.1.** *Let $n$ be in $\mathbb{N}$. One has that:*

$$(\diamondsuit_n) \qquad \mathrm{area}(\pi) = \sum_{V \in \pi} \big( \max(V) - \min(V) \big), \quad \forall\, \pi \in NC_2(2n).$$

*Proof*: We proceed by induction on $n$. The statement $(\diamondsuit_1)$ clearly holds, since the unique pair-partition $\pi = \{V_1\} \in NC_2(2)$ has $\mathrm{area}(\pi) = 1 = \max(V_1) - \min(V_1)$. For the rest of the proof we fix an $n \geq 1$, we assume that the statement $(\diamondsuit_m)$ was proved for every $1 \leq m \leq n$, and we verify that $(\diamondsuit_{n+1})$ holds as well. We divide the verification into two cases.

*Case 1. Consider a $\pi \in NC_2(2n+2)$ such that $\{1, 2n+2\}$ is a pair of $\pi$.*

In this case, we let $\pi_o \in NC_2(2n)$ be the relabeled restriction of $\pi$ to $\{2, 3, \ldots, 2n+1\}$. If we write explicitly $\pi = \{V_1, \ldots, V_n, V_{n+1}\}$ with $V_{n+1} = \{1, 2n+2\}$, then $\pi_o$ can be written as $\{V_1', \ldots, V_n'\}$ with $V_j'$ obtained by translating $V_j$ by 1 to the left (and thus having $\max(V_j') - \min(V_j') = \max(V_j) - \min(V_j)$), for $1 \leq j \leq n$. On the right-hand side of the formula $(\diamondsuit_{n+1})$ we get:

$$\sum_{V \in \pi} \big( \max(V) - \min(V) \big) = \Big[ \sum_{j=1}^{n} \big( \max(V_j) - \min(V_j) \big) \Big] + \Big[ (2n+2) - 1 \Big]$$

$$= \Big[ \sum_{j=1}^{n} \big( \max(V_j') - \min(V_j') \big) \Big] + (2n+1) = \mathrm{area}(\pi_o) + (2n+1),$$

where at the third equality sign we invoked the induction hypothesis.

In order to complete the verification of this case, we are left to check that $\mathrm{area}(\pi) = \mathrm{area}(\pi_o) + (2n+1)$. But this is immediately seen by looking at the Dyck paths associated to $\pi$ and $\pi_o$. Indeed, the path for $\pi$ can be obtained by taking the path for $\pi_o$ and placing it on top of the trapezoid with vertices at the points $(0,0)$, $(1,1)$, $(2n+1, 1)$ and $(2n+2, 0)$ of $\mathbb{Z}^2$, where the latter trapezoid has area equal to $2n+1$.

*Case 2. Consider a $\pi \in NC_2(2n+2)$ where 1 is paired with $2k$ for some $1 \leq k \leq n$.*

In this case, we let $\pi_1 \in NC_2(2k)$ and $\pi_2 \in NC_2\big(2(n+1-k)\big)$ be the relabeled restrictions of $\pi$ to $\{1, \ldots, 2k\}$ and to $\{2k+1, \ldots, 2n+2\}$, respectively. Upon looking at the pictures of the Dyck paths associated to $\pi, \pi_1$ and $\pi_2$, one immediately sees that

$$\mathrm{area}(\pi) = \mathrm{area}(\pi_1) + \mathrm{area}(\pi_2). \tag{8.1}$$

The induction hypothesis applies to both $\pi_1$ and $\pi_2$, hence (8.1) can be continued with

$$= \Big[ \sum_{V' \in \pi_1} \big( \max(V') - \min(V') \big) \Big] + \Big[ \sum_{V'' \in \pi_2} \big( \max(V'') - \min(V'') \big) \Big],$$

which is immediately seen to just be $\sum_{V \in \pi} \big( \max(V) - \min(V) \big)$, as required. □

In order to approach the area-measuring random variables $A_n$ by using the $NC_2^{(\mathrm{mton})}$-tree, we prove another lemma, as follows.

**Lemma 8.2.** *Let $n \geq 2$ be in $\mathbb{N}$, let $(\rho, v)$ be in $NC_2^{(\mathrm{mton})}(2n-2)$, and consider the set $C^{(\mathrm{pair})}(\rho, v) \subseteq NC_2^{(\mathrm{mton})}(2n)$ consisting of children of $(\rho, v)$ in the $NC_2^{(\mathrm{mton})}$-tree (cf. Notation and Remark 7.4). One has that*

$$\sum_{(\pi,u)\in C^{(\mathrm{pair})}(\rho,v)} A_n(\pi, u) = (2n-1) + (2n+1) \cdot A_{n-1}(\rho, v). \tag{8.2}$$

*Proof*: As observed in Notation and Remark 7.4, we can write

$$C^{(\mathrm{pair})}(\rho, v) = \big\{(\pi_1, u_1), \ldots, (\pi_{2n-1}, u_{2n-1})\big\},$$

where for every $1 \leq m \leq 2n-1$ the ordered partition $(\pi_m, u_m) \in NC_2^{(\mathrm{mton})}(2n)$ is uniquely determined by the requirements $\mathfrak{pp}(\pi_m, u_m) = (\rho, v)$ and $J(\pi_m, u_m) = \{m, m+1\}$. For further processing, let us write $\rho = \{V_1, \ldots, V_{n-1}\}$. Then for every $1 \leq m \leq 2n-1$ we can write $\pi_m = \big\{V_1^{(m)}, \ldots, V_{n-1}^{(m)}, \{m, m+1\}\big\}$, where $V_1^{(m)}, \ldots, V_{n-1}^{(m)}$, are the images of $V_1, \ldots, V_{n-1}$, respectively, under the unique order-preserving bijection from $\{1, \ldots, 2n-2\}$ onto $\{1, \ldots, 2n\} \setminus \{m, m+1\}$. This gives:

$$\begin{aligned}
\sum_{(\pi,u)\in C^{(\mathrm{pair})}(\rho,v)} A_n(\pi, u) &= \sum_{m=1}^{2n-1} \mathrm{area}(\pi_m) \\
&= \sum_{m=1}^{2n-1} \Big[\, 1 + \sum_{j=1}^{n-1} \big(\max(V_j^{(m)}) - \min(V_j^{(m)})\big) \Big] \quad \text{(by Lemma 8.1)} \\
&= (2n-1) + \sum_{j=1}^{n-1} \Big[ \sum_{m=1}^{2n-1} \big(\max(V_j^{(m)}) - \min(V_j^{(m)})\big) \Big].
\end{aligned} \tag{8.3}$$

Let us now fix for the moment a $j \in \{1, \ldots, n-1\}$, and let us examine the inside sum over $m$ in (8.3). For every $1 \leq m \leq 2n-1$, the relation between the pair $V_j^{(m)} \in \pi_m$ and the pair $V_j \in \rho$ forces the quantity $\max(V_j^{(m)}) - \min(V_j^{(m)})$ to be equal to

$$\begin{cases} \big[\max(V_j) - \min(V_j)\big] + 2, & \text{if } \min(V_j) \leq m-1 \text{ and } \max(V_j) \geq m, \\ \big[\max(V_j) - \min(V_j)\big], & \text{otherwise.} \end{cases} \tag{8.4}$$

Summing over $m$ in (8.4) thus gives

$$\sum_{m=1}^{2n-1} \big[\max(V_j^{(m)}) - \min(V_j^{(m)})\big] = (2n-1) \cdot \big[\max(V_j) - \min(V_j)\big] + \sum_{m=\min(V_j)+1}^{\max(V_j)} 2$$

$$= (2n-1) \cdot \big[\max(V_j) - \min(V_j)\big] + \big[\max(V_j) - \min(V_j)\big] \cdot 2 = (2n+1) \cdot \big[\max(V_j) - \min(V_j)\big],$$

where at the second equality sign we observed that the summation range for $m$, on the preceding line, has $\max(V_j) - \min(V_j)$ terms.

We now unfix the index $j$ and replace the result of the preceding calculation into the sum over $j$ from (8.3), in order to obtain the required formula (8.2):

$$\begin{aligned}
\sum_{(\pi,u)\in C^{(\mathrm{pair})}(\rho,v)} A_n(\pi, u) &= (2n-1) + \sum_{j=1}^{n-1} \Big( (2n+1) \cdot \big[\max(V_j) - \min(V_j)\big] \Big) \\
&= (2n-1) + (2n+1) \cdot A_{n-1}(\rho, v),
\end{aligned}$$

where at the latter equality sign we used Lemma 8.1 in connection to $(\rho, v)$. □

8.3. **Proof of Theorem 1.11.** For every $n \in \mathbb{N}$, let us put

$$S_n := \sum_{(\pi,u)\in NC_2^{(\mathrm{mton})}(2n)} A_n(\pi, u). \tag{8.5}$$

We claim that the numbers $S_n$ satisfy the recursion

$$S_n = (2n+1)\, S_{n-1} + (2n-1)!!\,, \;\; n \geq 2, \tag{8.6}$$

initialized with $S_1 = 1$. In order to prove (8.6), we do the usual trick of grouping the terms on the right-hand side of (8.5) according to what is the parent of $(\pi, u)$ in the $NC_2^{(\mathrm{mton})}$-tree:

$$\begin{aligned} S_n &= \sum_{(\rho,v)\in NC_2^{(\mathrm{mton})}(2n-2)} \Big( \sum_{(\pi,u)\in C^{(\mathrm{pair})}(\rho,v)} A_n(\pi,u) \Big), \\ &= \sum_{(\rho,v)\in NC_2^{(\mathrm{mton})}(2n-2)} \Big( (2n-1) + (2n+1)\cdot A_{n-1}(\rho,v) \Big) \text{ (by Lemma 8.2)} \\ &= (2n-1)\cdot |\, NC_2^{(\mathrm{mton})}(2n-2)\,| + (2n+1)\cdot \sum_{(\rho,v)\in NC_2^{(\mathrm{mton})}(2n-2)} A_{n-1}(\rho,v) \\ &= (2n-1)\cdot(2n-3)!! + (2n+1)\cdot S_{n-1}, \end{aligned}$$

which gives the right-hand side of (8.6).

From (8.6), an easy induction on $n$ yields the exact formula

$$S_n = (2n+1)!! \Big( \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} \Big), \quad n \geq 1. \tag{8.7}$$

Hence the formula for the expectation of $A_n$ becomes

$$E[\,A_n\,] = \frac{S_n}{(2n-1)!!} = \Big( (2n+1)!! \sum_{k=1}^{n} \frac{1}{2k+1} \Big) / (2n-1)!! = (2n+1) \sum_{k=1}^{n} \frac{1}{2k+1},$$

as claimed in Equation (1.15) from the statement of the theorem.

In order to verify that $E[\,A_n\,] \sim n \ln n$, we rewrite the above formula as

$$E[\,A_n\,] = \big(n + \frac{1}{2}\big) \sum_{k=1}^{n} \frac{1}{k+\frac{1}{2}}, \tag{8.8}$$

where the sum on the right-hand side is bounded above by $\sum_{k=1}^{n} 1/k = H_n$ and bounded below by $\sum_{k=1}^{n} 1/(k+1) > H_n - 1$. This leads to the inequalities

$$\frac{n\,(H_n - 1)}{n \ln n} \leq \frac{E[\,A_n\,]}{n \ln n} \leq \frac{(n+1)\,H_n}{n \ln n}, \quad n \geq 2,$$

and implies via squeeze that $\lim_{n\to\infty} E[\,A_n\,]/(n \ln n) = 1$. □

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## References


Arizmendi, O. Statistics of blocks in $k$-divisible non-crossing partitions. *Electron. J. Combin.*, **19** (2), Paper 47, 22 (2012). DOI: 10.37236/2431.

Arizmendi, O., Hasebe, T., Lehner, F., and Vargas, C. Relations between cumulants in noncommutative probability. *Adv. Math.*, **282**, 56–92 (2015). DOI: 10.1016/j.aim.2015.03.029.

Belton, A. C. R. A note on vacuum-adapted semimartingales and monotone independence. In *Quantum probability and infinite dimensional analysis*, volume 18 of *QP–PQ: Quantum Probab. White Noise Anal.*, pp. 105–114. World Sci. Publ., Hackensack, NJ (2005). ISBN 981-256-147-1. MR2211883.

Belton, A. C. R. The monotone Poisson process. In *Quantum probability*, volume 73 of *Banach Center Publ.*, pp. 99–115. Polish Acad. Sci. Inst. Math., Warsaw (2006). DOI: 10.4064/bc73-0-6.

Borodin, A. and Olshanski, G. *Representations of the infinite symmetric group*, volume 160 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2017). ISBN 978-1-107-17555-6. DOI: 10.1017/CBO9781316798577.

Hasebe, T. and Saigo, H. The monotone cumulants. *Ann. Inst. Henri Poincaré Probab. Stat.*, **47** (4), 1160–1170 (2011). DOI: 10.1214/10-AIHP379.

Kargin, V. Limit theorems for statistics of non-crossing partitions. *ArXiv Mathematics e-prints* (2019). arXiv: 1907.00632.

Merlini, D., Sprugnoli, R., and Verri, M. C. The area determined by underdiagonal lattice paths. In *Trees in algebra and programming—CAAP '96 (Linköping, 1996)*, volume 1059 of *Lecture Notes in Comput. Sci.*, pp. 59–71. Springer, Berlin (1996). ISBN 3-540-61064-2. DOI: 10.1007/3-540-61064-2_29.

Muraki, N. Monotonic independence, monotonic central limit theorem and monotonic law of small numbers. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **4** (1), 39–58 (2001). DOI: 10.1142/S0219025701000339.

Nica, A. and Speicher, R. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-85852-6; 0-521-85852-6. DOI: 10.1017/CBO9780511735127.

Ortmann, J. Large deviations for non-crossing partitions. *Electron. J. Probab.*, **17**, no. 34, 25 (2012). DOI: 10.1214/EJP.v17-2007.