ALEA, Lat. Am. J. Probab. Math. Stat. **23**, 1137–1170 (2026)
DOI: 10.30757/ALEA.v23-42

Poisson representations for tree-indexed Markov chains

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Abstract. In Forsström et al. (2026), the class of Poisson representable processes was introduced. Several well-known processes were shown not to belong to this class, with examples including both the Curie Weiss model and the Ising model on $\mathbb{Z}^2$ for certain choices of parameters. Curiously, it was also shown that all positively associated $\{0,1\}$-valued Markov chains do belong to this class. In this paper, we interpolate between Markov chains and Ising models by considering tree-indexed Markov chains. In particular, we show that for any finite tree that is not a path, whether or not the corresponding tree-indexed Markov chain is representable always depends on the parameters. Moreover, we give an example of a family of infinite trees such that the corresponding tree-indexed Markov chains are representable for some non-trivial parameters. In addition, we give alternative proofs and arguments and also strengthen several of the results in Forsström et al. (2026).

1. Introduction

In Forsström et al. (2026), Poisson representable processes were introduced. To define this family of processes, let S be a finite or countably infinite set. We think of the set S as the index set of our process. Next, let ν be a σ-finite measure on $\mathcal{P}(S)\backslash\{\emptyset\}$, where $\mathcal{P}(S)$ is the power set of S. This generates a $\{0,1\}$-valued process $X^\nu = (X_i^\nu)_{i\in S}$ defined as follows. Let Z be a Poisson process on $\mathcal{P}(S)\backslash\{\emptyset\}$ with intensity measure ν, written $Z \sim \text{Poisson}(\nu)$. Then Z is a random collection of nonempty subsets of S, and for each set $B \in \mathcal{P}(S)\backslash\{\emptyset\}$ independently, we have $B \in Z$ with probability $1 - e^{-\nu(B)}$. We define $(X_i^\nu)_{i\in S}$ by

$$X_i^\nu = X^\nu(i) = \begin{cases} 1 & \text{if } i \in \bigcup_{B\in Z} B \\ 0 & \text{otherwise.} \end{cases}$$

We let $\mathcal{R}$ denote the set of all processes $(X_i)_{i\in S}$ which are equal (in distribution) to X^ν for some measure ν on $\mathcal{P}(S)\backslash\{\emptyset\}$. In words, $X \in \mathcal{R}$ if X is the indicator function of the union of some set-valued Poisson process on its index set. The purpose of Forsström et al. (2026) was to try to

Received by the editors November 19th, 2025; accepted September 1st, 2026; published under license CC BY 4.0.
1991 *Mathematics Subject Classification.* 60J10, 60K35.
Key words and phrases. Poisson representations, Markov chains, tree-indexed Markov chains.
The author acknowledges support from the Swedish Research Council, grant number 2024-04744.

understand which processes could be represented in this way, and in this paper, we deepen the understanding of this question by studying tree-indexed Markov chains in more detail.

In Forsström et al. (2026, Theorem 3.1), it was shown that all positively associated $\{0,1\}$-valued Markov chains belong to $\mathcal{R}$. In contrast, by Forsström et al. (2026, Theorem 6.3), the Ising model on $\mathbb{Z}^d$, $d \geq 2$, does not belong to $\mathcal{R}$ if the coupling parameter is sufficiently small (see also Bethuelsen and Forsström, 2026). By a similar proof, it was also shown that tree-indexed Markov chains, defined below, are not always in $\mathcal{R}$. In particular, by Forsström et al. (2026, Theorem 6.1), a tree-indexed Markov chain X on a tree with a vertex v from which at least d distinct infinite paths emerge is not in $\mathcal{R}$ whenever $P(X_v = 0) < 1/(1 - r_2^{(d)})$, where $r_2^{(d)}$ is an explicit constant that approaches 0 as $d \to \infty$. In this paper, we refine this picture by showing that, in fact, all finite trees exhibit a phase transition, with a subset of the parameter space corresponding to processes in $\mathcal{R}$ and another subset corresponding to processes that do not belong to $\mathcal{R}$. In order to be able to present this result in more detail, we now define what we mean by tree-indexed Markov chains.

Definition 1.1. Let T be a tree with vertex set $V(T)$, edge set $E(T)$, and finite vertex degree, and let $r, p \in [0,1]$. Fix a vertex $o \in V(T)$. For each vertex $v \in V(T)$, let

$$R_v = \begin{cases} 0 & \text{w.p. } r \\ 1 & \text{w.p. } 1-r \end{cases}$$

be independent random variables. The *tree-indexed Markov chain X indexed by T with parameters* (r,p) is defined as follows.

(A) Let $X(o) = R_o$, and set $V_0 := \{o\}$. For $j = 1, 2, \ldots$, let V_j be the set of vertices $v \in V(T) \setminus V_{j-1}$ that are adjacent to some vertex $v^- \in V_{j-1}$, and for each $v \in V_j$, let

$$X(v) = \begin{cases} X(v^-) & \text{w.p. } 1-p \\ R_v & \text{w.p. } p. \end{cases}$$

Equivalently, given o and $(R_v)_{v \in V(T)}$, we can construct a random variable with the same distribution as X as follows.

(B) Let Y be a random subset of $E(T)$ obtained by removing each edge in $E(T)$, independently, with probability p. Independently for each connected component $C \subseteq V(T)$ of the resulting random subgraph of T, set $X(v) = R_{v_0}$ for all $v \in C$, where v_0 is the unique vertex in C that is closest to the origin o.

Since (A) and (B) above yield random processes with same distribution, we will use them both interchangeably throughout the paper. We note that using (A), the process $X = (X(v))_{v \in V(T)}$ can be thought of as being sampled sequentially as a reversible and positively associated Markov chain along any path of the tree. Using (B), the tree-indexed Markov chain X is instead defined as a so-called *divide-and-color process* (Steif and Tykesson, 2019), and this is equivalently the FK (Fortuin-Kasteleyn) representation corresponding to the tree-indexed-Markov chain.

The model described in Definition 1.1 is also known as a symmetric binary channel on trees and appears, e.g., when studying the broadcasting/reconstruction problems on trees (see, e.g., Mossel and Peres, 2003). Further, it can be interpreted as an Ising model on a tree with free boundary conditions, where e.g. $r = 1/2$ and $p = \sqrt{\frac{2}{1+e^{4J}}}$ corresponds to an Ising model with coupling parameter J and no external field. Further, it is well known (see, e.g., Georgii, 2011, Proposition 12.24), that the extremal states of the Ising model on trees are tree-indexed Markov chains.

Throughout the rest of this paper, for $n \geq 1$, we let $r^{(n)}$ be the largest (strictly) negative root of the polylogarithm function $\mathrm{Li}_{1-n}(z)$ of index $1-n$, defined by the power series $\sum_{k=1}^{\infty} \frac{z^k}{k^{1-n}}$. We note in particular that it is is known that $r^{(n)} \in [-1, 0)$ for all n, and that $r^{(n)} \nearrow 0$ as $n \to \infty$. Also, for $n \geq 1$, let $\tilde{B}_n$ be the nth complementary Bell number, defined by $\tilde{B}_n = \sum_{k=0}^{n} (-1)^k S(n,k)$, where

$S(n,k)$ is the Stirling number of the second kind, i.e. the number of ways to partition a set of n elements into k non-empty sets. Using the complementary Bell numbers, for integers $k, m \geq 1$, we define

$$r_0(k) := \max_{j \leq k:\ (-1)^j \tilde{B}_j > 0} \frac{\left((-1)^j \tilde{B}_j\right)^{1/(j-1)}}{1 + \left((-1)^j \tilde{B}_j\right)^{1/(j-1)}} \quad \text{and} \quad r_1(m) := 1/(1 - r^{(m)}). \tag{1.1}$$

Note that it is well known that $\lim_{n\to\infty} r_0(n) = \lim_{n\to\infty} r_1(n) = 1$.

n	$r^{(n)}$	$\tilde{B}_n$	$r_0(n)$	$r_1(n)$
3	-1	1	9	1/2
4	$-0.26795\ldots$	1	1/2	$0.78868\ldots$
5	$-0.10102\ldots$	-2	$0.54321\ldots$	$0.90825\ldots$
6	$-0.04310\ldots$	-9	$0.54321\ldots$	$0.95868\ldots$
7	$-0.01952\ldots$	-9	$0.59054\ldots$	$0.98085\ldots$
8	$-0.00915\ldots$	50	$0.63619\ldots$	$0.99093\ldots$

TABLE 1.1. The values of $r^{(n)}$, $\tilde{B}_n$, $r_0(n)$ and $r_1(n)$ for $n = 3, 4, \ldots, 8$.

By Forsström et al. (2026, Theorem 3.1), any $\{0,1\}$-valued Markov chain is in $\mathcal{R}$, and hence, equivalently, any tree-indexed Markov chain on a finite or infinite graph with no cycles where each vertex has degree at most two is Poisson representable. Our first main result shows that this result is sharp in the sense that for any finite graph with at least one vertex of degree at least three, whether or not the corresponding Markov chain is representable always depends on the parameters.

Theorem 1.2. *Let T be a finite tree, let $(r,p) \in (0,1)$, and let X be a tree-indexed Markov chain on T with parameters (r,p). Let k be the size of the boundary of T (the set of vertices of degree exactly one in T) and let m be the maximal degree of any vertex in $V(T)$. Then the following holds.*

(a) $X \in \mathcal{R}$ if $r > r_0(k)$ and p is sufficiently close to zero (depending on r).
(b) $X \notin \mathcal{R}$ if $r < r_0(k)$ and p is sufficiently close to zero (depending on r).
(c) $X \in \mathcal{R}$ if $r > r_1(m)$ and p sufficiently close to one (depending on r).
(d) $X \notin \mathcal{R}$ if $r < r_1(m)$ and p sufficiently close to one (depending on r).

A consequence of Theorem 1.2 is that the threshold in r for small p does not depend on the whole tree but only on its local structure. In particular, the threshold is the same for a large binary tree as for a tree with only four vertices and one vertex of degree three.

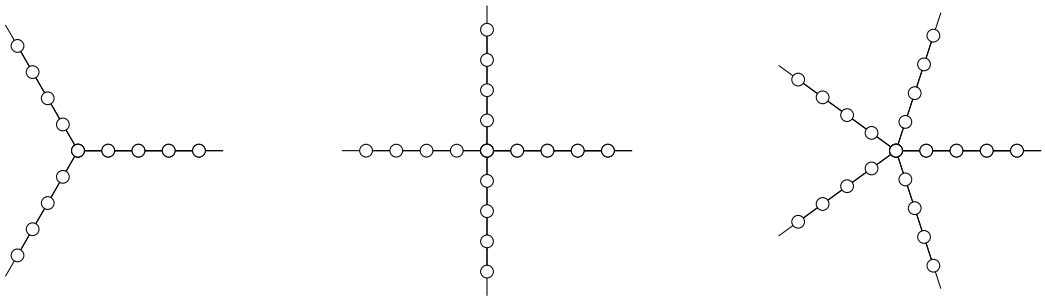
We next consider infinite trees. By extending Forsström et al. (2026, Lemma 2.14) to signed measures in Lemma 3.1, the negative results of Theorem 1.2(d) and Theorem 1.2(b) immediately extend to infinite trees. However, one might ask if there is an infinite tree T and parameters $r, p \in (0,1)$ such that the tree-indexed Markov chain on T with parameters (r,p) is in $\mathcal{R}$. This is the main content of our next theorem. In this theorem, we show that for a special family of infinite trees, which we will refer to as *octopus trees*, a positive result similar to parts (a) and (c) of Theorem 1.2 still holds.

For $m \geq 3$, let T_m denote the infinite tree consisting of m disjoint semi-infinite paths whose endpoints are joined at one vertex (see Figure 1.1). The tree T_m will be referred to as the *octopus tree of degree m*.

Theorem 1.3. *Let $m \geq 3$ and $r, p \in (0,1)$, and let X be a tree-indexed Markov chain on T_m with parameters (r,p). Then there exists a constant $r_2(m) \in (0,1)$, not depending on p, such that*

(a) $X \notin \mathcal{R}$ if $r < r_1(m)$, where $r_1(m)$ is defined in (1.1), and
(b) $X \in \mathcal{R}$ if $r \geq r_2(m)$.

Moreover,



(A) The tree T_3 (B) The tree T_4 (C) The tree T_5

FIGURE 1.1. The octopus trees T_3, T_4, and T_5.

(c) $r_2(3) = r_1(3) = 1/2$.

The first part of this theorem, Theorem 1.3(a), is an immediate consequence of Forsström et al. (2026, Theorem 6.1), which also gives a concrete lower bound for $r_2(m)$. The main contribution of Theorem 1.3 is thus part (b), which together with (a) implies that the set of Markov chains indexed by octopus trees has a phase transition, and part (c), which shows that when $m = 3$, whether or not X is in $\mathcal{R}$ is independent of p. Moreover, we note that for finite trees, whether or not $X \in \mathcal{R}$ typically depends on both r and p, see, e.g., Forsström et al. (2026, Figure 1).

The main idea of the proof of Theorem 1.3(a) is to first show that for any $m \geq 3$, there exists a number $r_1(m)$ and a finite subtree T of T_m, such that for all fixed $r \leq r_1(m)$ and all p sufficiently close to one, the restriction of X to T is not in $\mathcal{R}$. Using symmetries of the tree (see Proposition 5.2), it follows that this implies that $X \notin R$ for all $p \in (0,1)$. We elaborate on this and similar symmetry properties in Section 5, but note here that the main observation of this section is that results for p close to one or p close to zero can sometimes be extended to results for general p and the same value of r.

Our final main result, Theorem 1.4, gives versions of the previous results for general infinite trees.

Theorem 1.4. *Let T be an infinite tree, let $r, p \in (0,1)$, and let X be the tree-indexed Markov chain on T with parameters (r,p). Then, with r_0 and r_1 as in (1.1), the following holds.*

(a) Let $n_1 < \infty$ be smaller than or equal to the size of the boundary of T. Then $X \notin \mathcal{R}$ if $r < r_0(n_1)$ and p is sufficiently close to zero.

(b) Let $n_2 < \infty$ be such that there is some vertex $o \in V(T)$ with at least n_2 eventually disjoint infinite paths in $E(T)$ starting at o. Then $X \notin \mathcal{R}$ if $r < r_1(n_2)$ and p is sufficiently close to zero.

(c) Let $n_3 < \infty$ be such that there is some vertex $o \in V(T)$ with at least n_3 disjoint infinite paths in $E(T)$ starting at o. Then $X \notin \mathcal{R}$ if $r < r_1(n_3)$.

In particular, if T has an infinite boundary, then $X \notin \mathcal{R}$ for all p sufficiently close to zero.

As a part of the proofs of our main results, we extend and provide new, shorter, and more natural proofs of several of the fundamental results in Forsström et al. (2026). In particular, we mention Lemma 4.3, which substantially strengthens, but at the same time greatly simplifies the proof of Forsström et al. (2026, Theorem 3.1), Proposition 3.2, which extends Forsström et al. (2026, Proposition 3.7) to signed measures, and Proposition 7.1, which extends Forsström et al. (2026, Theorem 5.6) from trees with exactly one vertex of degree ≥ 3 to general finite trees.

Remark 1.5. Our main results all use the same parameters $r = P(Y(v) = 0)$ and p for all vertices and edges. However, many of the proofs throughout the paper allow the parameters to be different for different vertices and edges, and thus, many of the main results can, with small modifications, be proven in a more general setting. This is useful when, e.g., considering a Markov process

X on a graph G, since for any $S \subseteq V(G)$, if the subgraph of G induced by S is a tree, then $Y = X(S) \mid \{X(S^c) \equiv 0\}$ is a tree-indexed Markov chain where $P(Y(v) = 0)$ is typically not the same for all $v \in S$.

The rest of this paper will be structured as follows. In Section 2, we review the notation and some fundamental results about Poisson representable processes from Forsström et al. (2026). We also introduce the notation we will use when working with graphs and trees, give a more general definition for tree-indexed Markov chains which will simplify the proofs throughout the paper, and list a few useful properties of the polylogarithm function. Next, in Section 3, we extend Forsström et al. (2026, Proposition 3.7) to signed measures; i.e. we show that if X satisfies a Markov property and if ν is a signed measure corresponding to X, then ν has support only on connected sets. In Section 4, we use our results from Section 3 to give simpler formulas for ν in terms of X for tree-indexed Markov chains. These generalise Forsström et al. (2026, Theorem 3.1) from line graphs to trees, and in addition, give more intuitive proofs also for the simpler case of a line (when the tree-indexed Markov chain is a regular Markov chain). Next, in Section 5, we show that on self-similar trees (see Definition 5.1), we can transfer results from one set of parameters (r, p) to other parameters (r, p'). In Section 6, we consider the case $p \approx 0$, and provide proofs of Theorem 1.2(a) and Theorem 1.2(b). In Section 7, we consider the case $p \approx 1$, and prove Theorem 1.2(c) and Theorem 1.2(d). In Section 8, we give a proof of Theorem 1.3(b) and Theorem 1.3(c). Finally, in Section 9, we provide a proof of Theorem 1.4.

2. Preliminaries

In this section, we introduce notation and recall fundamental results from Forsström et al. (2026) that will be useful to us in this paper. Further, we introduce the notations we will use for graphs and trees, and also give a more general definition of tree-indexed Markov chains, which will be useful in later sections of this paper.

2.1. *Definitions and properties for Poisson representable processes.* Let a finite or countably infinite set S be given. For $i \in S$, we let

$$\mathcal{S}_i \coloneqq \big\{A \in \mathcal{P}(S) \setminus \{\emptyset\} \colon i \in A\big\},$$

and for $A \subseteq S$, we let

$$\mathcal{S}_A^{\cup} \coloneqq \bigcup_{i \in A} \mathcal{S}_i \quad \text{and} \quad \mathcal{S}_A^{\cap} \coloneqq \bigcap_{i \in A} \mathcal{S}_i.$$

In words, $\mathcal{S}_i$ the set of all subsets of S that contain i, $\mathcal{S}_A^{\cup}$ is the set of all subsets of S that have a non-empty intersection with A, and $\mathcal{S}_A^{\cap}$ is the set of all subsets of S that contain A as a subset. This notation is useful to us because given a measure ν on $\mathcal{P}(S) \setminus \{\emptyset\}$, it connects probabilities involving X^ν with the measure ν in the sense that for any set non-empty $A \subseteq S$, one has

$$P\big(X^\nu(A) \equiv 0\big) = e^{-\nu(\mathcal{S}_A^{\cup})}. \tag{2.1}$$

(This follows immediately from the definition of X^ν.) We will use this identity to go back and forth between the two random objects X^ν and $Y \sim \text{Poisson}(\nu)$. In particular, the identity (2.1) is used in the proof of the following lemma from Forsström et al. (2026), which uses the Möbius inversion theorem to obtain a formula for ν.

Lemma 2.1 (Lemma 2.12 in Forsström et al., 2026)**.** *Let S be finite, and let $X = (X_s)_{s \in S}$ be $\{0,1\}$-valued random variables such that $P\big(X(I) \equiv 0\big) > 0$ for all nonempty $I \subseteq S$. Then there is a unique signed measure ν on $\mathcal{P}(S) \setminus \{\emptyset\}$ that satisfies*

$$\nu\big(\mathcal{S}_I^{\cup}\big) = -\log P\big(X(I) \equiv 0\big), \quad I \subseteq S. \tag{2.2}$$

Moreover, ν is given by

$$\nu(K) = \sum_{I \subseteq K} (-1)^{|K|-|I|} \log P\big(X(S \setminus I) \equiv 0\big), \qquad \emptyset \neq K \subseteq S. \tag{2.3}$$

Consequently, if $\nu \geq 0$, then $X = X^\nu \in \mathcal{R}$.

Whenever S is a finite set, $X = (X_s)_{s \in S}$ is a $\{0,1\}$-valued process, and ν is given by (2.3), we say that ν *is the unique signed measure corresponding to* X.

A useful feature of the Poisson representation of a random process, if such a representation exists, is that it works well with various natural restrictions; a property that is in large inherited from the underlying Poisson process. This is the content of the following lemma.

Lemma 2.2 (Lemma 2.14 in Forsström et al., 2026)**.** *Let S be finite or countably infinite, and consider a process $X = (X_s)_{s \in S}$.*

(a) If $X = X^\nu \in \mathcal{R}$ and $B \subseteq S$, then there is a positive measure ν_B such that $X|_B = X^{\nu_B}$. Moreover, for any non-empty measurable subset $\mathcal{A} \subseteq \mathcal{P}(B)$, we have $\nu_B(\mathcal{A}) = \nu\big(\{A' \in \mathcal{P}(S) \colon A' \cap B \in \mathcal{A}\}\big)$.

(b) If $X = X^\nu \in \mathcal{R}$ and $B \subseteq S$, then there is a measure $\nu_{B,0}$ on $\mathcal{P}(B) \setminus \{\emptyset\}$ such that $X|\big\{X(B^c) \equiv 0\big\} = X^{\nu_{B,0}}$. Moreover, $\nu_{B,0} = \nu|_{\mathcal{P}(B)}$.

(c) If there exist $S_1 \subseteq S_2 \subseteq, \dots$ such that $S = \bigcup_i S_i$ and $X_{S_n} = X^{\nu_n}$ for some $\nu_n \geq 0$, then $X = X^\nu \in \mathcal{R}$ for some $\nu \geq 0$. (The projection of ν on to each S_n will simply be ν_n.)

2.2. *Notation for graphs and trees.* In this paper we will often consider processes on graphs. When we do this, we will always assume that the graphs are connected. For graphs G we will use the following notation.

- We let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G.
- We say that a set $V \subseteq V(G)$ is connected if it induces a connected subgraph of $V(G)$.
- If $v, v' \in V(G)$ are neighbors, i.e. if $(v, v') \in E(G)$, we write $v \sim v'$.
- Given a set $V \subseteq V(G)$, we let E_V denote the set of all edges $e \in E(G)$ with at least one endpoint in V. If $V = \{v\}$, we let $E_v \coloneqq E_{\{v\}}$.
- For a finite and non-empty set $V \subseteq V(G)$, we let

$$\mathcal{B}^-(V) \coloneqq \big\{v \in V \colon |\{v' \in V \colon v \sim v'\}| = 1\big\} \qquad \text{(the inner boundary)}$$

and

$$\partial V \coloneqq \mathcal{B}^+(V) \coloneqq \big\{v \in V(G) \setminus V \colon \exists v' \in V \text{ s.t. } v \sim v'\big\} \qquad \text{(the outer boundary)}.$$

For a set $V \subseteq V(G)$, we let

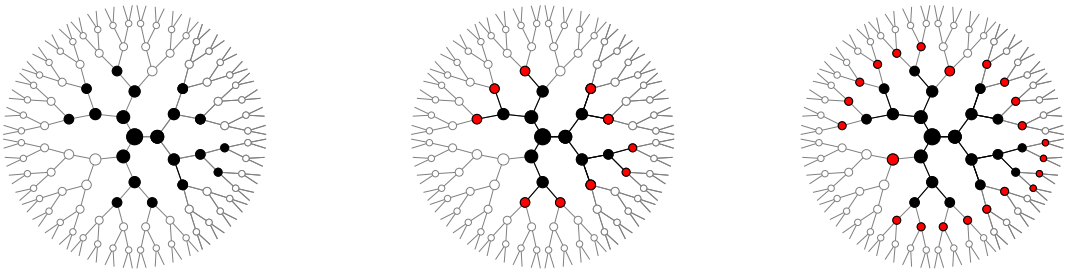
$$N(V) \coloneqq \{v' \in V(G) \setminus V \colon v' \sim v \text{ for some } v \in V\} \qquad \text{(the neighborhood of } V).$$

For $I \subseteq V$, we let

$$\partial_V I \coloneqq \{v \in \mathcal{B}^+(V) \colon \exists v' \in I \text{ s.t. } v \sim v'\} = \mathcal{B}^+(V) \cap N(I).$$

Now assume that we are given a tree T which is rooted in some vertex $o \in V(T)$.

- If V is connected, we let T_V be the subtree of T induced by V.
- Given an edge $e \in E(T)$, we let e^- and e^+ denote the endpoint of e that is closer resp. further away from o.
- Given a vertex $v \in V(T) \setminus \{o\}$, we let E_v^+ be the set of all edges $e \in E(T)$ with $e^- = v$.



(A) T in gray and V in black

(B) T_V in black and red, and $\mathcal{B}^-(V)$ in red

(C) T_V in black, and $\mathcal{B}^+(V)$ in red.

FIGURE 2.2. In the figures above, we illustrate T_V, $\mathcal{B}^-(V)$ and $\mathcal{B}^+(V)$ for a given set T and a set $V \subseteq V(T)$.

2.3. *More general tree-indexed Markov chains.* Later in the paper, it will be useful to consider tree-indexed Markov chains where the parameters r and p are allowed to be different for different vertices and edges. The following definition generalizes Definition 1.1 to this setting.

Definition 2.3. Let T be a tree with finite vertex degree, vertex set $V(T)$, and edge set $E(T)$. For each $v \in V(T)$, let $r_v \in [0,1]$, and for each $e \in E(T)$, let $p_e \in [0,1]$. Fix a vertex $o \in V(T)$. For each vertex $v \in V(T)$ independently, let

$$R_v = \begin{cases} 0 & \text{w.p. } r_v \\ 1 & \text{w.p. } 1 - r_v. \end{cases}$$

The *tree-indexed Markov chain X indexed by T with parameters* $\big((r_v)_{v\in V(T)}, (p_e)_{e\in E(T)}\big)$ is defined as follows.

(A') Let $X(o) = R_o$, let $V_0 := \{o\}$, and for $j = 1, 2, \ldots$, let $V_j := V_{j-1} \cup \mathcal{B}^+(V_{j-1})$. Sequentially, for each $j = 1, 2, \ldots$, for each $v' \in V_j$ independently, let $v \in V_{j-1}$ be such that $v \sim v'$ and set

$$X(v') = \begin{cases} X(v) & \text{w.p. } 1 - p_{(v,v')} \\ R_{v'} & \text{w.p. } p_{(v,v')}. \end{cases}$$

Equivalently, given o and $(R_v)_{v\in V(T)}$, we can construct a random variable with the same distribution as X as follows.

(B') Let Y be a random subset of $E(T)$ obtained by removing each $e \in E(T)$, independently, with probability p_e (i.e., let Y be edge percolation). Independently for each connected component $C \subseteq V(T)$ of the resulting random graph, let v' be the unique vertex in C that minimizes the distance to the origin, and set $X(v) = R_{v'}$ for all $v \in C$.

Note that both (A') and (B') above result in the same process, and their corresponding constructions reduce to (A) and (B) of Definition 1.1 in the special case $(r_v) \equiv r$ and $(p_e) \equiv p$ for some $r, p \in [0,1]$. Moreover, note that if $r_v = r$ for some $r \in [0,1]$ and all $v \in V(T)$, then the process is independent of the choice of o.

2.4. *Properties of the polylogarithm function.* Throughout this paper, the polylogarithm function will naturally appear. The polylogarithm function with index $s \in \mathbb{C}$ is defined for $z \in \mathbb{C}$ with $|z| < 1$ by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

and can be extended to the whole complex plane by analytic continuation. Some special cases include $\mathrm{Li}_1(z) = -\log(1-z)$, $\mathrm{Li}_0(z) = \frac{z}{1-z}$ and $\mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}$. When the index s is a negative integer, the polylogarithm function can also be defined using other combinatorial quantities such as Stirling numbers of the second kind and Eulerian numbers. For this reason, they naturally show up in combinatorial arguments.

For $n \geq 2$, the polylogarithm function $\mathrm{Li}_{1-n}(z)$ is well-known to satisfy the following properties (see, e.g., the paragraph before Forsström et al., 2026, Theorem 5.6).

- $\mathrm{Li}_{1-n}(0) = 0$.
- Li_{1-n} has $n-1$ real negative simple roots $r_{n-1}^{(n)} < \cdots < r_1^{(n)} < 0$ which are interlacing, in the sense that $r_{j+1}^{(n)} < r_j^{(n-1)} < r_j^{(n)}$ for all $j \in \{1, 2, \dots, n-2\}$.
- $\mathrm{Li}_{1-n}(z) < 0$ for $z \in (r_1^{(n)}, 0)$.

Using the notation of the introduction, we thus have $r^{(n)} = r_1^{(n)}$. Recalling that we defined $r_1(n) \coloneqq 1/(1-r^{(n)})$. In particular, this implies that

$$r < r_1(n) \Leftrightarrow -(1-r)/r < r^{(n)}.$$

A simple consequence of the above-listed properties of the polylogarithm function, which we will use multiple times, is the following lemma.

Lemma 2.4. *Let $n \geq 3$ and $r \in (0,1)$. Then the following holds.*

(a) If $r > r_1(n)$, then $\mathrm{Li}_{1-j}(-(1-r)/r) < 0$ for all $j \in \{1, 2, 3, \dots, n\}$.
(b) If $r < r_1(n)$, then there is $j \in \{3, \dots, n\}$ such that $\mathrm{Li}_{1-j}(-(1-r)/r) > 0$.

Proof: We first show that (a) holds. To this end, assume that $r > r_1(n)$ or equivalently, that $-(1-r)/r > r_1^{(n)}$. Since $r_1^{(n)} > r_1^{(n-1)} > \cdots > r_1^{(2)}$, it follows that $-(1-r)/r \in (r_1^{(n)}, 0)$ for all $j \in \{2, 3, \dots, n\}$. Since $\mathrm{Li}_{1-j}(z) < 0$ for all $z \in (r_1^{(n)}, 0)$, this concludes the proof of (a).

Now assume that $r < r_1(n)$ or equivalently, that $-(1-r)/r < r_1^{(n)}$. Then either $-(1-r)/r < r_1^{(2)}$ or there is $j \in \{3, 4, \dots, n\}$ such that $-(1-r)/r \in (r_2^{(j)}, r_1^{(j)})$. Since for all $j \in \{2, 3, \dots, n\}$, all roots of Li_{1-j} are simple roots and $\mathrm{Li}_{1-j}(z) < 0$ for all $z \in (r_1^{(n)}, 0)$, it follows that $\mathrm{Li}_{1-j}(z) > 0$ if either $j = 2$ and $z \in (-\infty, r_1^{(2)})$, or if $j \geq 3$ and $z \in (r_2^{(j)}, r_1^{(j)})$. Combining the above observations, we obtain (b). □

3. Properties of the signed representation

In this section, we state and prove two results that will be very useful in subsequent sections. Both of these results are more general versions of results that first appeared in Forsström et al. (2026).

The first result of this section, Lemma 3.1 below, extends Lemma 2.2 to signed measures in a finite setting.

Lemma 3.1. *Let S be a finite set, let $X = (X_s)_{s \in S}$ be a $\{0,1\}$-valued process, and let ν be the unique signed measure which satisfies (2.2). Let $B \subseteq S$. Then the following holds.*

(a) For every measurable subset $\mathcal{A} \subseteq \mathcal{P}(B) \setminus \{\emptyset\}$, let $\nu_B(\mathcal{A}) \coloneqq \nu\big(\{A' \in \mathcal{P}(S) \colon A' \cap B \in \mathcal{A}\}\big)$. Then ν_B is the unique signed measure corresponding to $X|_B$.
(b) For every measurable subset $\mathcal{A} \subseteq \mathcal{P}(B) \setminus \{\emptyset\}$, let $\nu_{B,0}(\mathcal{A}) \coloneqq \nu|_{\mathcal{P}(B)}(\mathcal{A})$. Then $\nu_{B,0}$ is the unique signed measure corresponding to $X|\{X(B^c) \equiv 0\}$.

Proof: For $I \subseteq B$, let $\mathcal{S}_I^{\cup,B} \coloneqq \mathcal{S}_I^{\cup} \cap \mathcal{P}(B)$.

We now show that (a) holds. To this end, let ν_0 be the unique signed measure which corresponds to $X|_B$, and let $I \subseteq B$. Then, by Lemma 2.1, we have

$$\nu_0(\mathcal{S}_I^{\cup}) = -\log P\big(X|_B(I) \equiv 0\big) = -\log P\big(X(I) \equiv 0\big) = \nu(\mathcal{S}_I^{\cup}).$$

At the same time, by definition, we have

$$\nu_B(\mathcal{S}_I^{\cup,B}) = \nu\big(\{A' \in \mathcal{P}(S)\colon A' \cap B \in \mathcal{S}_I^{\cup,B}\}\big) = \nu\big(\{A' \in \mathcal{P}(S)\colon A' \in \mathcal{S}_I^{\cup}\}\big) = \nu(\mathcal{S}_I^{\cup}).$$

Again using Lemma 2.1, it follows that $\nu_0 = \nu_B$. This concludes the proof of (a).

We now show that (b) holds. To this end, let ν_0 be the unique signed measure which corresponds to $X|\{X(B^c) \equiv 0\}$. Let $I \subseteq B$. Then

$$\nu_0\big(\mathcal{S}_I^{\cup,B}\big) = -\log P\big(X(I) \equiv 0 | X(B^c) \equiv 0\big)$$

and

$$\begin{aligned}\nu_{B,0}\big(\mathcal{S}_I^{\cup,B}\big) &= \nu|_{\mathcal{P}(B)}\big(\mathcal{S}_I^{\cup,B}\big) = \nu\big(\mathcal{S}_I^{\cup,B}\big) = \nu\big(\mathcal{S}_I^{\cup} \setminus \mathcal{S}_{B^c}^{\cup}\big) = \nu\big(\mathcal{S}_{I\cup B^c}^{\cup}\big) - \nu\big(\mathcal{S}_{B^c}^{\cup}\big)\\ &= -\log P\big(X(I \cup B^c) \equiv 0\big) - \Big(-\log P\big(X(B^c) \equiv 0\big)\Big)\\ &= -\log P\big(X(I) \equiv 0 \mid X(B^c) \equiv 0\big).\end{aligned}$$

Using Lemma 2.1, it follows that $\nu_0 = \nu_{B,0}$. This concludes the proof of (b). □

In this paper, we mostly consider tree-indexed Markov chains, which naturally satisfy a Markov property. More generally, one could consider a Markov field on a graph. Assuming the graph is finite, let ν be the unique signed measure which satisfies (2.2). By Forsström et al. (2026, Proposition 3.7), if $\nu \geq 0$, then $\nu(S) = 0$ whenever $S \subseteq V(G)$ is a disconnected set. The following result shows that this holds even if we drop the assumption that $\nu \geq 0$.

Proposition 3.2. *Let X be a $\{0,1\}$-valued process on a finite connected graph G that satisfies the Markov property, and assume that for all finite $A \subseteq V(G)$ we have*

$$P\big(X(A) \equiv 0 \mid X(\mathcal{B}^+(A)) \equiv 0\big) > 0.$$

Let ν be the unique signed measure that satisfies (2.2), and let D be any disconnected subset of $V(G)$. Then $\nu(D) = 0$.

Proof: We will first show that for any non-empty sets $A_1 \subseteq V(G)$ and $A_2 \subseteq V(G) \setminus (A_1 \cup \mathcal{B}^+(A_1))$, we have

$$\nu\big((\mathcal{S}_{A_1}^{\cup} \cap \mathcal{S}_{A_2}^{\cup}) \setminus \mathcal{S}_{V(G)\setminus(A_1\cup A_2)}^{\cup}\big) = 0. \tag{3.1}$$

To this end, let $A_1 \subseteq V(G)$ and $A_2 \subseteq V(G) \setminus (A_1 \cup \mathcal{B}^+(A_1))$ be two non-empty sets, and let $B := (A_1 \cup A_2)^c = V(G) \setminus (A_1 \cup A_2)$. Since X satisfies the Markov property, we have

$$P\big(X(A_1) \equiv 0 \mid X(\mathcal{B}^+(A_1)) \equiv 0\big) = P\big(X(A_1) \equiv 0 \mid X(A_1^c) \equiv 0\big) = P\big(X(A_1) \equiv 0 \mid X(B) \equiv 0\big).$$

Since

$$P\big(X(A_1) \equiv 0 \mid X(B) \equiv 0\big) = e^{-\nu(\mathcal{S}_{A_1}^{\cup} \setminus \mathcal{S}_B^{\cup})}$$

and

$$P\big(X(A_1) \equiv 0 \mid X(\mathcal{B}^+(A_1)) \equiv 0\big) = e^{-\nu(\mathcal{S}_{A_1}^{\cup} \setminus \mathcal{S}_{A_1^c}^{\cup})},$$

and, by assumption, we have

$$0 < P\big(X(A_1) \equiv 0 \mid X(\mathcal{B}^+(A_1)) \equiv 0\big) \leq 1,$$

it follows that

$$0 < e^{-\nu(\mathcal{S}_{A_1}^{\cup} \setminus \mathcal{S}_B^{\cup})} = e^{-\nu(\mathcal{S}_{A_1}^{\cup} \setminus \mathcal{S}_{A_1^c}^{\cup})} \leq 1,$$

and hence

$$0 \leq \nu(\mathcal{S}^{\cup}_{A_1} \smallsetminus \mathcal{S}^{\cup}_{B}) = \nu(\mathcal{S}^{\cup}_{A_1} \smallsetminus \mathcal{S}^{\cup}_{A_1{}^c}) < \infty.$$

Since $\partial A_1 \subseteq B$, this yields

$$\nu\big((\mathcal{S}^{\cup}_{A_1} \cap \mathcal{S}^{\cup}_{A_2}) \smallsetminus \mathcal{S}^{\cup}_{B}\big) = \nu\big((\mathcal{S}^{\cup}_{A_1} \cap \mathcal{S}^{\cup}_{A_1^c}) \smallsetminus \mathcal{S}^{\cup}_{B}\big) = 0$$

and thus completes the proof of (3.1).

We now complete the proof of the proposition by showing that for any non-empty sets $A_1 \subseteq V(G)$ and $A_2 \subseteq V(G) \smallsetminus (A_1 \cup \mathcal{B}^+(A_1))$, we have $\nu(A_1 \cup A_2) = 0$. To this end, note first that if $|A_1| = |A_2| = 1$, then, by (3.1), we have

$$\nu(A_1 \cup A_2) = \nu\big((\mathcal{S}^{\cup}_{A_1} \cap \mathcal{S}^{\cup}_{A_2}) \smallsetminus \mathcal{S}^{\cup}_{B}\big) = 0.$$

For the more general statement, we note that

$$\begin{aligned} 0 = \nu\big((\mathcal{S}^{\cup}_{A_1} \cap \mathcal{S}^{\cup}_{A_2}) \smallsetminus \mathcal{S}^{\cup}_{B}\big) &= \sum_{S_1 \subseteq A_1,\, S_2 \subseteq A_2 \colon S_1, S_2 \neq \emptyset} \nu(S_1 \cup S_2) \\ &= \nu(A_1 \cup A_2) + \sum_{\substack{S_1 \subseteq A_1,\, S_2 \subseteq A_2 \colon S_1, S_2 \neq \emptyset, \\ |S_1|+|S_2| < |A_1|+|A_2|}} \nu(S_1 \cup S_2). \end{aligned}$$

Using induction, the desired conclusion immediately follows. □

4. Formulas for tree-indexed Markov chains

By Forsström et al. (2026, Theorem 3.1), the formula in (2.3) can be substantially simplified when the process X is a Markov chain. The proof relied on using the corresponding probability measure and several algebraic manipulations. The following lemma extends this result from Markov chains to Markov processes on graphs. When the graph is a line, the corresponding process is a regular Markov chain, and in this case we recover Forsström et al. (2026, Theorem 3.1). However, even in this special case, the proof presented below is substantially easier than the corresponding proof in Forsström et al. (2026).

Lemma 4.1. *Let X be a $\{0,1\}$-valued process on a finite graph G that satisfies the Markov property, and assume that for all sets $A \subseteq V(G)$ we have*

$$P\big(X(A) \equiv 0 \mid X(\mathcal{B}^+(A)) \equiv 0\big) > 0.$$

Let ν be the unique signed measure corresponding to X, and let $S \subseteq V(G)$ be non-empty. Then

$$\nu(S) = \begin{cases} \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\big(X\big(J \cup \mathcal{B}^+(S)\big) \equiv 0\big) & \text{if } S \text{ is connected} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Remark 4.2. The formula in (4.1) is valid also for finite subsets of the vertex set of infinite graphs by restriction (using Lemma 2.2). However, we stress that in this case, we cannot conclude that the formula in (4.1) holds for infinite sets.

The main motivation for obtaining a lemma such as Lemma 4.1 is that for a graph, the number of vertices in the boundary of a connected subset is sometimes substantially smaller than the number of vertices in the set, and in this case the number of terms in (4.2) is much lower than the number of terms in (2.3). This is, in particular, true when the graph is a line graph, in which case the boundary of any connected set has a cardinality of at most two.

Proof of Lemma 4.1: If S is not connected, then, by Proposition 3.2, we have $\nu(S) = 0$. Now assume that S is connected. Then, since ν has support only on connected sets by Proposition 3.2, we have

$$\nu(S) = \nu(\mathcal{S}^{\cap}_{\mathcal{B}^-(S)} \smallsetminus \mathcal{S}^{\cup}_{\mathcal{B}^+(S)}).$$

Let $\nu_{\mathcal{B}^+(S)\cup\mathcal{B}^-(S)}$ be the unique signed measure corresponding to $X|_{\mathcal{B}^+(S)\cup\mathcal{B}^-(S)}$. Then, by Lemma 3.1(a), we have

$$\nu(\mathcal{S}^{\cap}_{\mathcal{B}^-(S)} \setminus \mathcal{S}^{\cup}_{\mathcal{B}^+(S)}) = \nu_{\mathcal{B}^+(S)\cup\mathcal{B}^-(S)}(\mathcal{S}^{\cap}_{\mathcal{B}^-(S)} \setminus \mathcal{S}^{\cup}_{\mathcal{B}^+(S)}) = \nu_{\mathcal{B}^+(S)\cup\mathcal{B}^-(S)}(\mathcal{B}^-(S)).$$

Combining the previous equations and applying Lemma 2.1, we obtain

$$\begin{aligned}\nu(S) = \nu_{\mathcal{B}^+(S)\cup\mathcal{B}^-(S)}\big(\mathcal{B}^-(S)\big) &= \sum_{I\subseteq\mathcal{B}^-(S)} (-1)^{|\mathcal{B}^-(S)|-|I|} \log P\big(X((\mathcal{B}^-(S)\cup\mathcal{B}^+(S))\setminus I)\equiv 0\big)\\ &= \sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|} \log P\big(X(J\cup\mathcal{B}^+(S))\equiv 0\big),\end{aligned}$$

which is the desired conclusion. □

In the case of tree-indexed Markov chains, the formula in (4.1) can be simplified further, and this is the content of the next lemma.

Lemma 4.3. *Let T be a finite tree, and let X be a tree-indexed Markov chain on T with parameters $((r_v),(p_e))$. Let ν be the unique signed measure corresponding to X in the sense of Lemma 2.1. Then, for any non-empty $S\subseteq V(T)$, we have*

$$\nu(S) = \begin{cases} \sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\log P\big(X\big(J\cup(\mathcal{B}^+(S)\setminus\partial_S J)\big)\equiv 0\big) & \text{if } S \text{ is connected and } |S|\geq 2\\ -\log P(X(S\cup\mathcal{B}^+(S))\equiv 0 \mid X\big(\mathcal{B}^+(S)\big)\equiv 0) & \text{if } S=\{v\}\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

Proof: Let S be a non-empty subset of $V(T)$. The desired conclusion will follow from Lemma 4.1 if we can show that for tree-indexed Markov chains, the formula in (4.1) simplifies to the formula in (4.2). To this end, note first that if S is not connected, then the two formulas trivially agree. Next, assume that $S=\{v\}$ for some $v\in V(T)$. Then $\mathcal{B}^-(S)=S=\{v\}$, and hence, by Lemma 4.1, we have

$$\nu(S) = \log P\big(X\big(\mathcal{B}^+(S)\big)\equiv 0\big) - \log P\big(X\big(S\cup\mathcal{B}^+(S)\big)\equiv 0\big)$$

which immediately implies (4.2) in this case. Now assume that S is connected and $|S|\geq 2$. Then, for any $J\subseteq\mathcal{B}^-(S)$, we have

$$P\big(X\big(J\cup\mathcal{B}^+(S)\big)\equiv 0\big) = P\big(X(J\cup(\mathcal{B}^+(S)\setminus\partial_S J))\equiv 0\big)\prod_{v\in J,\,v'\in\partial_S v}(1-p_{(v,v')}+p_{(v,v')}r_{v'}),$$

and hence

$$\begin{aligned}&\sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\log P\big(X\big(J\cup\mathcal{B}^+(S)\big)\equiv 0\big)\\ &\quad= \sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\Big(\log P\big(X\big(J\cup(\mathcal{B}^+(S)\setminus\partial_S J)\big)\equiv 0\big) + \sum_{v\in J,\,v'\in\partial_S v}\log(1-p_{(v,v')}+p_{(v,v')}r_{v'})\Big).\end{aligned}$$

Now note that

$$\begin{aligned}&\sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\sum_{v\in J,\,v'\in\partial_S v}\log(1-p_{(v,v')}+p_{(v,v')}r_{v'})\\ &\quad= \sum_{v\in\mathcal{B}^-(S),\,v'\in\partial_S v}\log(1-p_{(v,v')}+p_{(v,v')}r_{v'})\sum_{J\subseteq\mathcal{B}^-(S):\,v\in J}(-1)^{|J|}.\end{aligned}$$

Since $|S|\geq 2$ implies that $|\mathcal{B}^-(S)|\geq 2$, we have $\sum_{J\subseteq\mathcal{B}^-(S):\,v\in J}(-1)^{|J|}=0$, and hence

$$\sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\sum_{v\in J,\,v'\in\partial_S v}\log(1-p_{(v,v')}+p_{(v,v')}r_{v'}) = 0,$$

implying in particular that (4.2) holds in this case. This concludes the proof. □

Remark 4.4. Forsström et al. (2026, Theorem 3.1), which in particular gives a simple formula for ν when X is a Markov chain, holds not only for Markov chains but also for renewal processes. Since it is not obvious how to define renewal processes on trees, Lemma 4.3 does not mention renewal processes, even though in the special case that the tree is a line, the proof of Lemma 4.3 gives the more general result of Forsström et al. (2026, Theorem 3.1).

5. Scaling properties of trees

In this section, we first define what we mean by a tree being self-similar, and then show how this property relates to a Markov process indexed by such a tree being Poisson representable.

Definition 5.1. We say that an infinite tree T is *self-similar* if there are arbitrarily large $k \geq 2$ such that if we let T^k be the tree obtained by replacing each edge of T with a path of length k, then T^k is a subgraph of T.

We note that with this definition, both line graphs, octopus trees, and regular trees are self-similar.

The main result in this section is the following proposition.

Proposition 5.2. *Let T be a self-similar infinite tree, and for $r, p \in (0,1)$, let $X_{r,p}$ be a tree-indexed Markov chain on T with parameters (r,p). Then the following holds.*

(a) If there is $\varepsilon, r \in (0,1)$ such that $X_{r,p} \notin \mathcal{R}$ for all $p \in (1-\varepsilon, 1)$, then $X_{r,p} \notin \mathcal{R}$ for all $p \in (0,1)$.

(b) If there is $\varepsilon, r \in (0,1)$ such that $X_{r,p} \in \mathcal{R}$ for all $p \in (0,\varepsilon)$, then $X_{r,p} \in \mathcal{R}$ for all $p \in (0,1)$.

Proof: Let $r, p \in (0,1)$ and let $X = X_{r,p}$.

For $k \geq 1$, let T_k be as in Definition 5.1, and let $V_0(T^k)$ denote the vertices in T^k that are the endpoints of the added paths in Definition 5.1. Note that if $k \geq 1$ is such that T^k is a subgraph of T, then $X(V_0(T^k))$ is equal in distribution to a tree-indexed Markov chain $X^{(k)}$ on T with parameters $(r, 1-(1-p)^k)$. Using Lemma 3.1(a), it follows that if $X \in \mathcal{R}$, then $X^{(k)} \in \mathcal{R}$, and if $X^{(k)} \notin \mathcal{R}$, then $X \notin \mathcal{R}$.

We now show that (a) holds. To this end, assume that r and $\varepsilon \in (0,1)$ are such that such that $X_{r,p'} \notin \mathcal{R}$ for all $p' \in (1-\varepsilon, 1)$. Further, let $k \geq 1$ be such that T^k is a subgraph of T and $1-(1-p)^k > 1-\varepsilon$. Then, by assumption, we have $X^{(k)} \notin \mathcal{R}$. Since $X^{(k)}$ is equal in distribution to $X(V_0(T^k))$ which is a restriction of X, using Lemma 3.1(a), it follows that $X \notin \mathcal{R}$.

We now show that (b) holds. To this end, instead assume that r and $\varepsilon \in (0,1)$ are such that $X_{r,p'} \in \mathcal{R}$ for all $p' \in (0,\varepsilon)$. Further, let $k \geq 1$ be such that T^k is a subgraph of T and $1-(1-p)^{1/k} < \varepsilon$. Let $\hat{X}$ be a tree-indexed Markov chain on T with parameters $(r, 1-(1-p)^{1/k})$. By the choice of k, we have $\hat{X} \in \mathcal{R}$. Since X is equal in distribution to $\hat{X}^{(k)}$, using Lemma 3.1(a), it follows that $X \in \mathcal{R}$. □

6. Finite trees and p close to zero

In this section, we provide proofs of parts (a) and (b) of Theorem 1.2. The main tool in these proofs is the following Proposition 6.1 below. In this proposition, and throughout the rest of this paper, we will use the following notation. For a multiset $E \subseteq E(T)$, where $e \in E$ has multiplicity $\mu(e)$, we write $\frac{d}{dp_E}$ to denote the partial derivative $\prod_{e \in E} \frac{d^{\mu(e)}}{dp_e^{\mu(e)}}$. Analogously, for a multiset $V \subseteq V(T)$, where $v \in V$ has multiplicity $\mu(v)$, we write $\frac{d}{dr_V}$ to denote the partial derivative $\prod_{v \in V} \frac{d^{\mu(v)}}{dp_v^{\mu(v)}}$.

Proposition 6.1. *Let T be a finite tree, and let $S \subseteq V(T)$ be connected and have a cardinality of at least two. Let $r \in (0,1)$, and for $e \in E(T)$, let $p_e \in (0,1)$. Further, let X be the $(r,(p_e))$-tree indexed Markov chain on T, and let ν be the signed measure corresponding to X in the sense of Lemma 2.1. Then, the following holds.*

(a) Let $E \subseteq E(T)$ be a multiset with $|E| < |\mathcal{B}^+(S)|$. Then $\left[\frac{d}{dp_E}\nu(S)\right]_{(p_e)\equiv 0} = 0$.

(b) Let $E \subseteq E(T)$ be a multiset with $|E| = |\mathcal{B}^+(S)|$. Let E_S be the set of edges between S and S^c. Then

$$\left[\frac{d}{dp_E}\nu(S)\right]_{(p_e)\equiv 0} = \begin{cases} (1-r)r^{|\mathcal{B}^+(S)|-1} - (-1)^{|\mathcal{B}^+(S)|}\tilde{B}_{|\mathcal{B}^+(S)|}(1-r)^{|\mathcal{B}^+(S)|} & \text{if } E = E_S \\ 0 & \text{otherwise,} \end{cases}$$

where we recall that for $n \geq 0$, $\tilde{B}_n$ is the nth complementary Bell number.

Before we provide a proof of Proposition 6.1, we show how it implies (a) and (b) of Theorem 1.2.

Proof of Theorem 1.2(a) and (b): For $j \geq 2$ let

$$f_j(r) := (1-r)r^{j-1} - (-1)^j \tilde{B}_j (1-r)^j.$$

Then $f_j(1) = 0$, $f_j(r) > 0$ for all $r \in (0,1)$ sufficiently close to one, and f_j has at most two real roots. Moreover, if f_j has exactly two roots, then both are simple, with one root at $r = 1$ and a second root at

$$\tilde{r}^{(j)} = \frac{\left((-1)^j \tilde{B}_k\right)^{1/(j-1)}}{1 + \left((-1)^j \tilde{B}_j\right)^{1/(j-1)}}.$$

Let ν be the unique signed measure corresponding to X. For $p = 0$, one easily verifies that $\nu(S) = -\log r$ if $S = V(T)$ and $\nu_T(S) = 0$ else. Moreover, if S is not connected, then $\nu(S) = 0$ by Proposition 3.2. Next, let $S \subseteq V(T)$ be connected with $S \neq V(T)$. Then, using a Taylor expansion, it follows from Proposition 6.1 that if p is sufficiently small, then

$$\operatorname{sign}\nu(S) = \operatorname{sign}\left((1-r)r^{|\mathcal{B}^+(S)|-1} - (-1)^{|\mathcal{B}^+(S)|}\tilde{B}_{|\mathcal{B}^+(S)|}(1-r)^{|\mathcal{B}^+(S)|}\right) = \operatorname{sign} f_{|\mathcal{B}^+(S)|}(r), \tag{6.1}$$

i.e. then $\nu(S)$ and $f_{|\mathcal{B}^+(S)|}(r)$ have the same sign. Since k is the size of the boundary of T, we have $|\mathcal{B}^+(S)| \leq k$, and hence, if $r > r_0(k)$, it follows that $\nu(S) > 0$ for sufficienty small p. Since this holds for all connected sets $S \subseteq V(T)$, this concludes the proof of part (a).

To see that part (a) holds, assume that $r < r_0(k)$. Then there is $j \leq k$ such that $f_j(r) < 0$ and a connected set $S \subseteq V(T)$ such that $\mathcal{B}^+(S) = j$. Using (6.1), it follows that $\nu(S) < 0$ for sufficiently small p, and hence for such p, we must have $X \notin \mathcal{R}$, thus concluding the proof of (b). □

We now proceed to the proof of Proposition 6.1.

Proof of Proposition 6.1: Let E_S be the set of edges between S and S^c.

For $J \subseteq \mathcal{B}^-(S)$, consider the event

$$\mathcal{E}_J := \left\{X\left(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\right) \equiv 0\right\}.$$

Since S is a connected set and $|S| \geq 2$, by Lemma 4.3, we have

$$\nu(S) = \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P(\mathcal{E}_J). \tag{6.2}$$

For any fixed $e' \in E_S$, let v denote its endpoint in $\mathcal{B}^-(S)$. Then, if $E \subseteq E(T) \setminus \{e'\}$ is a multiset, for any $J \subseteq \mathcal{B}^-(S)$ we have

$$\left[\frac{d}{dp_E}\log P(\mathcal{E}_J)\right]_{p_{e'}=0} = \left[\frac{d}{dp_E}\log P(\mathcal{E}_{J\Delta\{v\}})\right]_{p_{e'}=0}.$$

Hence, for any multi-set $E \subseteq E(T)$ with $E_S \not\subseteq E$, we have

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 0} = 0.$$

This completes the proof of (a), and also the proof of (b) in the case $E \neq E_S$.

We now prove part (b) in the case $E = E_S$. To this end, note first that for any non-empty $J \subseteq \mathcal{B}^-(S)$, the event $\mathcal{E}_J$ does not depend on all edges in E_S, and hence

$$\frac{d}{dp_{E_S}} \log P(\mathcal{E}_J) = 0.$$

Hence,

$$\frac{d}{dp_{E_S}}\nu(S) = \frac{d}{dp_{E_S}} \log P(\mathcal{E}_\emptyset) = \frac{d}{dp_{E_S}} \log P\big(X\big(\mathcal{B}^+(S)\big) \equiv 0\big).$$

Let $\mathcal{P}(E_S)$ denote the set of all partitions of E_S into non-empty sets. Then

$$\begin{aligned}&\Big[\frac{d}{dp_{E_S}} \log P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)\equiv 0}\\ &\quad= \sum_{(E_1,\dots,E_k)\in\mathcal{P}(E_S)} \frac{\big[(-1)^{k-1}\prod_{j=1}^k \frac{d}{dp_{E_j}} P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\big]_{(p_e)\equiv 0}}{\big[P\big(X(\mathcal{B}^+(S)) \equiv 0\big)^k\big]_{(p_e)\equiv 0}}.\end{aligned}$$

We now make a few additional observations about the quantities on the right-hand side of the previous equation. First, note that

$$\Big|P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)\equiv 0} = r.$$

Next, note that for any multi-set $E' \subset E(T)$ that contains two copies of some edge, we have

$$\frac{d}{dp_{E'}} P\big(X(\mathcal{B}^+(S)) \equiv 0\big) = 0.$$

We now consider the case that $E' \subseteq E_S$ is not a multiset. For this case, note first that since $E(T_S)$ and E_S are disjoint, we have

$$\Big[\frac{d}{dp_{E'}} P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)\equiv 0} = \Big[\frac{d}{dp_{E'}}\Big[P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)_{e\in E(T_S)}\equiv 0}\Big]_{(p_e)\equiv 0}.$$

Moreover, we have

$$\Big[P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)_{e\in E(T_S)}\equiv 0} = r\prod_{e\in E_S}(1-p_e+p_e r) + (1-r)\prod_{e\in E_S} p_e r$$

Using this observation, it follows that if $E' \subset E_S$ is not a multiset, then

$$\Big[\frac{d}{dp_{E'}} P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)\equiv 0} = r(-1+r)^{|E'|}.$$

and similarly, if $E' = E_S$, then

$$\Big[\frac{d}{dp_{E'}} P\big(X(\mathcal{B}^+(S)) \equiv 0\big)\Big]_{(p_e)\equiv 0} = r(-1+r)^{|E_S|} + (1-r)r^{|E_S|}.$$

Combining the above equations and simplifying, it follows that

$$\Big[\frac{d}{dp_{E_S}}\nu(S)\Big]_{(p_e)\equiv 0} = -(-1+r)^{|\mathcal{B}^+(S)|}\sum_{(E_1,\dots,E_k)\in\mathcal{P}(E_S)}(-1)^k + (1-r)r^{|\mathcal{B}^+(S)|-1}.$$

This completes the proof of (b). □

7. Finite trees and p close to one

The main purpose of this section is to provide proofs of Theorem 1.2(c) and Theorem 1.2(d). The main tool of the proofs will be the following proposition.

Proposition 7.1. *Let T be a finite tree, and let S be a connected subset of $V(T)$ with a cardinality of at least two. Further, let $r \in (0,1)$, for $e \in E(T)$, let $p_e \in (0,1)$, and let X be the tree indexed Markov chain on T with parameters $(r,(p_e))$. Further, let ν be the signed measure corresponding to X in the sense of Lemma 2.1. Then the following holds.*

(a) Let $E \subseteq E(T)$ be a non-empty multiset with $E(T_S) \not\subseteq E$. Then $\big[\frac{d}{dp_E}\nu(S)\big]_{(p_e)\equiv 1} = 0$.

(b) Let $E \subseteq E(T)$ be a non-empty multiset with $E = E(T_S)$. Then

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{p_{E(T)}\equiv 1} = (-1)^{|\mathcal{B}^-(S)|+1}\Big(\frac{r}{1-r}\Big)^{|V(T_S)\setminus\mathcal{B}^-(S)|-1}\prod_{j=2}^{\infty}\mathrm{Li}_{1-j}\big(-r^{-1}(1-r)\big)^{k_j}, \tag{7.1}$$

where k_j is the number of vertices of degree j in T_S.

Before we provide a proof of Proposition 7.1, we show how it implies (c) and (d) of Theorem 1.2.

Proof of Theorem 1.2(c) and Theorem 1.2(d): Let $S \subseteq V(T)$, and let ν be the signed measure which corresponds to X as in Lemma 2.1.

When $(p_e) \equiv 1$, $X(v)$ and $X(v')$ are independent for distinct $v, v' \in V(T)$, implying in particular that $X \in \mathcal{R}$ with

$$\nu(S) = \begin{cases} -\log r & \text{if } |S| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For general (p_e), we know from Lemma 4.3 that we have $\nu(S) \geq 0$ if $|S| = 1$ or if S is not connected.

Now assume that $|S| > 1$ is a connected set. Then, by Proposition 7.1 and a Taylor expansion, it follows that for all p sufficiently close to one, $\nu(S)$ has the same sign as

$$(-1)^{|E(T_S)|}(-1)^{|\mathcal{B}^-(S)|+1}\prod_{j=2}^{\infty}\Big(\mathrm{Li}_{1-j}(-(1-r)/r)\Big)^{k_j} = \prod_{j=2}^{\infty}\Big(-\mathrm{Li}_{1-j}\big(-(1-r)/r\big)\Big)^{k_j}, \tag{7.2}$$

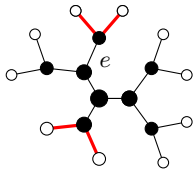
where for $j \geq 2$, k_j is the number of vertices in T_S with degree j. Since by Lemma 2.4(a), $-\mathrm{Li}_{1-j}(-(1-r)/r) > 0$ for all $j \in \{2,3\ldots,m\}$ if $r > r_1(m)$, this concludes the proof of (c).

To see that part (d) holds, we recall from Lemma 2.4(b) that if $r < r_1(m)$, then there is $m' \leq m$ such that $-\mathrm{Li}_{1-m'}(-(1-r)/r) < 0$. Since m is the maximal degree of any vertex in T, there is a set $S \subseteq V(T)$ such that T_S is connected, T_S has exactly one vertex with degree m', and all other vertices have degree 1. For this set, since $-(1-r)/r \in (r_2^{(m')}, r_1^{(m')})$, it follows that (7.2) is strictly negative, and hence the restriction of X to S is not in $\mathcal{R}$ for (p_e) sufficiently close to one. Using Lemma 2.2(a), it follows that $X \notin \mathcal{R}$ for (p_e) sufficiently close to one, thus completing the proof of part (d). □

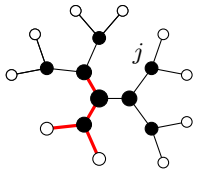
We now move to the proof of Proposition 7.1, which will use a number of lemmas that we first state and prove. The first of these lemmas, Lemma 7.2 below, shows that in the setting of Proposition 7.1, if the set E is not connected, then the derivative $\frac{d}{dp_E}$ of the terms appearing on the righ-hand side of (4.2) of Lemma 4.3 is zero.

Lemma 7.2. *In the setting of Proposition 7.1, let $E \subseteq E(T)$ be a non-empty multi-set that is not connected (see Figure 7.3a). Then, for any $J \subseteq \mathcal{B}^-(S)$, we have*

$$\Big[\frac{d}{dp_E}\log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = 0.$$



(A) The setting of Lemma 7.2



(B) The setting of Lemma 7.4

FIGURE 7.3. The figures above illustrate the settings of Lemma 7.2 and Lemma 7.4. In both figures, we have drawn a finite tree T, a set S in black, the corresponding set $\mathcal{B}^+(S)$ in white, and a set E in red.

Proof: Let $J \subseteq \mathcal{B}^-(S)$, and let Y be random edge set obtained from the percolation process corresponding to X.

Since E is not connected, there is an edge in $e \in E(T_S) \setminus E$ that separates $J \cup (\mathcal{B}^+(S) \setminus \partial_S J)$ into two non-empty sets A and B. On the event $e \notin Y$ the events $X(A) \equiv 0$ and $X(B) \equiv 0$ are independent. The implies that

$$\begin{aligned}
&\Big[\frac{d}{dp_E} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{p_e=1} \\
&\quad = \frac{d}{dp_E} \log P\big(X(A \sqcup B) \equiv 0 \mid e \notin Y\big) \\
&\quad = \frac{d}{dp_E} \log\Big(P\big(X(A) \equiv 0 \mid e \notin Y\big) P\big(X(B) \equiv 0 \mid e \notin Y\big)\Big) \\
&\quad = \frac{d}{dp_E} \Big(\log P\big(X(A) \equiv 0 \mid e \notin Y\big) + \log P\big(X(B) \equiv 0 \mid e \notin Y\big)\Big).
\end{aligned}$$

(Here and below, $\sqcup$ is used for a disjoint union.) Since E intersects both $E(T_A)$ and $E(T_B)$ and these sets are disjoint, we have

$$\frac{d}{dp_{E_A}} P\big(X(B) \equiv 0 \mid e \notin Y\big) = \frac{d}{dp_{E_B}} P\big(X(A) \equiv 0 \mid e \notin Y\big) = 0,$$

and hence

$$\Big[\frac{d}{dp_E} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = 0.$$

This completes the proof. □

The first step of the proof of Proposition 7.1 is the following lemma, which reduces the calculation of the derivative in (7.1) to the calculation of the derivative of the logarithm of a probability of a particular event.

Lemma 7.3. *In the setting of Proposition 7.1, for any set $E \subseteq E(T)$, we have*

$$\Big[\frac{d}{dp_E} \nu(S)\Big]_{(p_e)\equiv 1} = \begin{cases} 0 & \text{if } E(T_S) \not\subseteq E \\ (-1)^{|\mathcal{B}^-(S)|} \Big[\frac{d}{dp_E} \log P\Big(X\big(\mathcal{B}^-(S)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} & \text{if } E = E(T_S). \end{cases}$$

Proof: To simplify notation, let E_S be the set of edges between S and S^c.

Since $|S| \geq 2$ and S is connected, by Lemma 4.3, we have

$$\nu(S) = \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big). \tag{7.3}$$

By Lemma 7.2, for any multiset $E \subseteq E(T)$ with $E(T_S) \nsubseteq E$, we have

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} = 0.$$

Next, assume $E = E(T_S)$. By (7.3), we have

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} = \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus \partial_S J)\big)\equiv 0\Big)\Big]_{(p_e)\equiv 1}.$$

Let Y be the random edge set corresponding to the percolation process corresponding to X (see (B') of Definition 2.3). Then, since $E = E(T_S)$, we have

$$\begin{aligned}
&\Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus \partial_S J)\big)\equiv 0\Big)\Big]_{(p_e)\equiv 1}\\
&\quad= \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus \partial_S J)\big)\equiv 0 \mid Y\cap E_S = \emptyset\Big)\Big]_{(p_e)\equiv 1}\\
&\quad= \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log\Big[P\big(X(J)\equiv 0\big)\prod_{j\in\mathcal{B}^-(S)\setminus J} r^{|\partial_S\{j\}|}\Big]\Big]_{(p_e)\equiv 1},
\end{aligned}$$

where the last equality follows since on the event $Y\cap E_S = \emptyset$, the random vectors $X(S)$ and $X(\mathcal{B}^+(S))$ are independent. Combining the previous equations, we obtain

$$\begin{aligned}
\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} &= \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log\Big[P\big(X(J)\equiv 0\big)\prod_{j\in\mathcal{B}^-(S)\setminus J} r^{|\partial_S\{j\}|}\Big]\Big]_{(p_e)\equiv 1}\\
&= \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log P\big(X(J)\equiv 0\big)\Big]_{(p_e)\equiv 1}\\
&\quad+\Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log\prod_{j\in\mathcal{B}^-(S)\setminus J} r^{|\partial_S\{j\}|}\Big]_{(p_e)\equiv 1}\\
&= \Big[\frac{d}{dp_E}\sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|}\log P\big(X(J)\equiv 0\big)\Big]_{(p_e)\equiv 1}.
\end{aligned}$$

If $J\neq \mathcal{B}^-(S)$, then

$$\Big[\frac{d}{dp_E}\log P\big(X(J)\equiv 0\big)\Big]_{(p_e)\equiv 1} = 0$$

since the event $X(J)\equiv 0$ does not depend on all edges in $E = E(T_S)$. From this the desired conclusion immediately follows. □

The following lemma, Lemma 7.4 below, shows that in the setting of Proposition 7.1, if the set E is not adjacent to some vertex $v\in\mathcal{B}^-(S)$, then the derivative $\frac{d}{dp_E}$ of the terms appearing on the righ-hand side of (4.2) of Lemma 4.3 is zero.

Lemma 7.4. *In the setting of Proposition 7.1, let $E\subseteq E(T)$ be a non-empty multi-set. Assume that $j\in\mathcal{B}^-(S)$ is such that $E_j\cap E = \emptyset$ (see Figure 7.3b). In other words, assume there is no edge in E with one endpoint in j. Then*

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} = 0.$$

Proof: For $I\subseteq\mathcal{B}^-(S)$, consider the event

$$\mathcal{E}_I := \big\{X\big(I\cup(\mathcal{B}^+(S)\setminus\partial_S I\big)\equiv 0\big\}.$$

Then, by Lemma 4.3, we have

$$\nu(S) = \sum_{I \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P(\mathcal{E}_I).$$

Let $J \subseteq \mathcal{B}^-(S) \setminus \{j\}$. Then, since $E_j \cap E = \emptyset$,

$$\begin{aligned}\Big[\frac{d}{dp_E} \log P(\mathcal{E}_J)\Big]_{(p_e)\equiv 1} &= \Big[\frac{d}{dp_E} \log\big(P(\mathcal{E}_{J\sqcup\{j\}}) r^{|\partial_S\{j\}|-1}\big)\Big]_{(p_e)\equiv 1} \\ &= \Big[\frac{d}{dp_E} \log P(\mathcal{E}_{J\sqcup\{j\}})\Big]_{(p_e)\equiv 1} + \frac{d}{dp_E} \log\big(r^{|\partial_S\{j\}|-1}\big) = \Big[\frac{d}{dp_E} \log P(\mathcal{E}_{J\sqcup\{j\}})\Big]_{(p_e)\equiv 1}.\end{aligned}$$

Since $(-1)^{|J\sqcup\{j\}|} = -(-1)^{|J|}$, it follows that

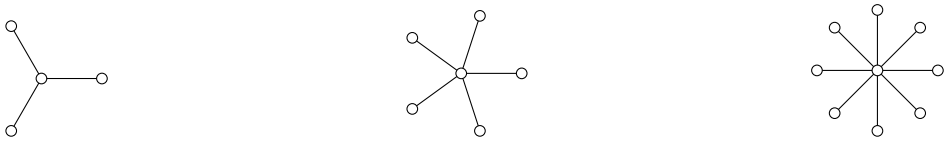
$$\Big[\frac{d}{dp_E} \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = 0.$$

From this, the desired conclusion immediately follows. □

The following lemma collects a useful observation obtained by combining the previous result, Lemma 7.3, with Forsström et al. (2026, Theorem 5.6).

Lemma 7.5. *Let $n \geq 2$, and let T be a tree be a n-star graph, consisting of one vertex, called o, of degree n, and n leaves (see Figure 7.4), and let $S \subseteq \mathcal{B}^-(S)$. Further, let X be a tree indexed Markov chain on T with parameters $(r, (p_e))$, and let ν be the corresponding signed measure. Then*

$$\Big[\frac{d}{dp_{E(T_S)}} \log P\Big(X\big(\mathcal{B}^-(S)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = -\operatorname{Li}_{1-|\mathcal{B}^-(S)|}(-r^{-1}(1-r)).$$



(A) $n = 3$ (B) $n = 5$ (C) $n = 8$

FIGURE 7.4. The three figures above show examples of the graphs considered in Lemma 7.5.

Proof: Let X' be the restriction of X to $\mathcal{B}^-(S)$. Let ν' be the signed measure corresponding to X'. Then, by Proposition 3.2, we have $\nu(S) = \nu'(\mathcal{B}^-(S))$. Moreover, we note

$$X' = r\Pi_{p(1-r)} + (1-r)\Pi_{1-p+p(1-r)},$$

where for $a \in [0,1]$, Π_a denotes a product distribution on $\{0,1\}^n$ with $\Pi_a(X(1) = 1) = a$.

Now let $\alpha_1 := r$, $\alpha_2 := 1-r$, $x_1 := 1$, $x_2 := pr/(1-p(1-r))$, and $q := 1-p(1-r)$. With this notation, we have

$$X' \sim \alpha_1 \Pi_{1-qx_1} + \alpha_2 \Pi_{1-qx_2}.$$

Then, by the proof of Forsström et al. (2026, Theorem 5.6) we have

$$\Big[\frac{d^\ell}{dx_2^\ell} \nu'(\mathcal{B}^-(S))\Big]_{x_2=1} = \begin{cases} 0 & \text{if } \ell < |\mathcal{B}^-(S)| \\ \ell!(-1)^{\ell+1} \operatorname{Li}_{1-\ell}(-\alpha_1^{-1}(1-\alpha_1)) & \text{if } \ell = |\mathcal{B}^-(S)|. \end{cases}$$

Since $x_2 = 1 \Leftrightarrow p = 1$, it follows that

$$\Big[\frac{d^\ell}{dp^\ell}\nu(S)\Big]_{p=1} = \begin{cases} 0 & \text{if } \ell < |\mathcal{B}^-(S)| \\ \ell!(-1)^{\ell+1}\operatorname{Li}_{1-\ell}(-r^{-1}(1-r)) & \text{if } \ell = |\mathcal{B}^-(S)|. \end{cases}$$

At the same time, by Lemma 7.3, for any $E \subseteq E(T)$, we have

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} = \begin{cases} 0 & \text{if } E(T_S) \nsubseteq E \\ (-1)^{|\mathcal{B}^-(S)|}\Big[\frac{d}{dp_E}\log P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0\big)\Big]_{(p_e)\equiv 1} & \text{if } E = E(T_S). \end{cases}$$

Combining these two equations, it follows that if $E = E(T_S)$, then

$$\begin{aligned}\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} &= (-1)^{|\mathcal{B}^-(S)|}\Big[\frac{d}{dp_E}\log P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0\big)\Big]_{(p_e)\equiv 1} \\ &= (-1)^{|E(T_S)|+1}\operatorname{Li}_{1-|E(T_S)|}(-r^{-1}(1-r))\end{aligned}$$

Noting that for the specific tree considered here, we have $|E(T_S)| = |\mathcal{B}^-(S)|$, this concludes the proof. □

The next main result of this section, which will be used in the proof of Proposition 7.1 is the following proposition, which gives a concrete expression for derivatives of the terms appearing on the right-hand side of (4.2) in Lemma 4.3 in terms of polylogarithm functions.

Proposition 7.6. *Let T be a tree, let $S \subseteq V(T)$ be connected and have cardinality at least two. For $v \in V(T)$ and $e \in E(T)$, let $r_v \in (0,1]$ and $p_e \in [0,1]$, and let X be the tree-indexed Markov chain on T with parameters $((r_v),(p_e))$. Then*

$$\Big[\frac{d}{dp_{E(T_S)}}\log P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0\big)\Big]_{(p_e)\equiv 1} = -\frac{1-r}{r}\prod_{v\in S\setminus L} f_{\deg v}(r)$$

where for $j \geq 2$, k_j is the number of vertices of degree j in T_S, $[j] = \{1,2,\dots,j\}$, and

$$f_j(r) \coloneqq \frac{r}{1-r}\sum_{\substack{(F_1,\dots,F_m)\in\mathcal{P}([j]):\\ |F_1|,\dots,|F_m|\geq 2}} (r-1)^m \prod_{i=1}^m \Big(1-((r-1)/r)^{|F_i|-1}\Big). \tag{7.4}$$

Before we prove Proposition 7.6, we will state and prove a number of shorter lemmas that will be useful in its proof.

Lemma 7.7. *In the setting of Proposition 7.6, let $E \subseteq E(T_S)$ and assume that there is a vertex $v \in V(T_S)\setminus\mathcal{B}^-(S)$ that is the end-point of exactly one edge in $e \in E$. Then*

$$\Big[\frac{d}{dp_E}P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = 0.$$

Proof: Let Y be the random edge set corresponding to the percolation process corresponding to X (see (B') of Definition 2.3). Note that

$$\Big[\frac{d}{dp_E}P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = \Big[\frac{d}{dp_E}P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0 \mid (E(T_S)\setminus E)\cap Y = \emptyset\big)\Big]_{(p_e)\equiv 1}.$$

Let $v \in V(T_S)\setminus\mathcal{B}^-(S)$ be the end-point of exactly one edge $e \in E$. Then, since $(r_v) \equiv r$ and we obtain X using (B') of Definition 2.3, conditioned on the event $(E(T_S)\setminus E)\cap Y = \emptyset$ the two events $X(\mathcal{B}^-(S)) \equiv 0$ and $e \in Y$ are independent, and hence

$$\Big[\frac{d}{dp_E}P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0\big|(E(T_S)\setminus E)\cap Y = \emptyset\big)\Big]_{(p_e)\equiv 1} = 0.$$

Combining the previous equations, we obtain the desired conclusion. □

Lemma 7.8. *In the setting of Proposition 7.6, let $v \in S \setminus \mathcal{B}^-(S)$ be such that $L \coloneqq N(v) \cap \mathcal{B}^-(S)$ is nonempty. Let $E \subseteq E_L$ satisfy $|E| \geq 2$, and let L_E be the set of all vertices in L that are the endpoint of some edge in E. Then*

$$\Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|\mathcal{B}^-(S)\setminus L_E|}\big(r(r-1)^{|E|} + (1-r)r^{|E|}\big).$$

Proof: We have

$$\begin{aligned}\Big[\frac{d}{dp_E} P\big(X(L_E) \equiv 0\big)\Big]_{(p_e)\equiv 1} &= \Big[\frac{d}{dp_E}\Big(r\prod_{e\in E}(1-p_e+p_e r) + (1-r)\prod_{e\in E}(p_e r)\Big)\Big]_{(p_e)\equiv 1} \\ &= r\prod_{e\in E}(-1+r) + (1-r)\prod_{e\in E} r = r(-1+r)^{|E|} + (1-r)r^{|E|}.\end{aligned}$$

This concludes the proof. □

Lemma 7.9. *In the setting of Proposition 7.6, let $v \in S \setminus \mathcal{B}^-(S)$ be such that $L \coloneqq N(v) \cap \mathcal{B}^-(S)$ is nonempty, and let $E \subseteq E(T_S) \setminus E_L$. Then*

$$\Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|L|-1}\Big[\frac{d}{dp_E} P\big(X\big(\mathcal{B}^-(S\setminus L)\big) \equiv 0\big)\Big]_{(p_e)\equiv 1}.$$

Proof: Recall that $E_v \subseteq E(T_S)$ is the set of all edges adjacent to v, and note that on the event $E_v \cap Y = \emptyset$, the random variables $X(\mathcal{B}^-(S)\setminus L)$, $X(v)$, and $X(v')$ for $v' \in L$ are independent. Using this observation, and noting that $v \in \mathcal{B}^-(S\setminus L)$, it follows that

$$\begin{aligned}\Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} &= \Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0 \mid E_v \cap Y = \emptyset\big)\Big]_{(p_e)\equiv 1} \\ = \Big[r^{|L|}\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)\setminus L) \equiv 0\big)\Big]_{(p_e)\equiv 1} &= \Big[r^{|L|-1}\frac{d}{dp_E} P\big(X\big(\mathcal{B}^-(S\setminus L)\big) \equiv 0\big)\Big]_{(p_e)\equiv 1}.\end{aligned}$$

This concludes the proof. □

Lemma 7.10. *In the setting of Proposition 7.6, let $v \in S \setminus \mathcal{B}^-(S)$ be such that $L \coloneqq N(v) \cap \mathcal{B}^-(S)$ is nonempty, and let e_v be the unique edge connecting v to $E(T_S) \setminus E(T_{L\cup\{v\}})$. Let $E \subseteq E(T_S)$ and assume that $e_v \in E$ and that $E \cap E_L \neq \emptyset$. Let L_E be the set of all vertices in L that are the endpoint of some edge in E. Then*

$$\begin{aligned}&\Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} \\ &\quad = r^{|L\setminus L_E|}\big((r-1)^{|E\cap E_L|} - r^{|E\cap E_L|}\big)\Big[\frac{d}{dp_{E\setminus E_L}} P\big(X\big(\mathcal{B}^-(S)\setminus L\big) \equiv 0\big)\Big)\Big]_{(p_e)\equiv 1}.\end{aligned}$$

Proof: Recall that Y is the set of retained edges in the edge percolation process used to construct X. If we sum over all possiblities for $Y \cap (E \cap E_L)$, we get

$$P\big(X(\mathcal{B}^-(S)) \equiv 0\big) = \sum_{E_0 \subseteq E\cap E_L} P\big(X(\mathcal{B}^-(S)) \equiv 0 \text{ and } Y \cap (E\cap E_L) = E_0\big).$$

If $E_0 \subseteq E \cap E_L$ is empty, then the corresponding probability in the above sum does not depend on p_{e_v}, and hence

$$\frac{d}{dp_E} P\big(X\big(\mathcal{B}^-(S)\big) \equiv 0 \text{ and } Y \cap (E\cap E_L) = E_0\big) = 0.$$

Now, instead assume that $E_0 \subseteq E \cap E_L$ is non-empty. In this case, noting that $v \in \mathcal{B}^-(S)\setminus L$, we have

$$\begin{aligned}&P\big(X(\mathcal{B}^-(S)) \equiv 0 \text{ and } Y \cap (E\cap E_L) = E_0\big) \\ &\quad = r^{-1}P\big(X(L) \equiv 1 \text{ and } Y \cap (E\cap E_L) = E_0\big)P\big(X\big(\mathcal{B}^-(S)\setminus L\big) \equiv 0\big).\end{aligned}$$

Combining the two cases, we get

$$\begin{aligned}
&\frac{d}{dp_E}\sum_{E_0\subseteq E\cap E_L} P\big(X(\mathcal{B}^-(S))\equiv 0 \text{ and } Y\cap(E\cap E_L)=E_0\big)\\
&\quad=\frac{d}{dp_E}\Big(r^{-1}P\big(X(L)\equiv 0 \text{ and } Y\cap(E\cap E_L)\neq\emptyset\big)P\Big(X\big(\mathcal{B}^-(S)\setminus L\big)\equiv 0\Big)\Big)\\
&\quad=r^{-1}\frac{d}{dp_{E\cap E_L}}P\big(X(L)\equiv 0 \text{ and } Y\cap(E\cap E_L)\neq\emptyset\big)\frac{d}{dp_{E\setminus E_L}}P\Big(X\big(\mathcal{B}^-(S)\setminus L\big)\equiv 0\Big)\Big).
\end{aligned}$$

Now note that for the first of these terms, we have

$$\begin{aligned}
&\Big[\frac{d}{dp_{E\cap E_L}}P\big(X(L)\equiv 0 \text{ and } Y\cap(E\cap E_L)\neq\emptyset\big)\Big]_{p_e\equiv 1}\\
&\quad=r^{|L\setminus L_E|}\Big[\frac{d}{dp_{E\cap E_L}}P\big(X(L_E)\equiv 0 \text{ and } Y\cap(E\cap E_L)\neq\emptyset\big)\Big]_{p_e\equiv 1}\\
&\quad=r^{|L\setminus L_E|}\Big[\frac{d}{dp_{E\cap E_L}}P\big(X(L_E)\equiv 0\big)-P\big(X(L_E)\equiv 0 \text{ and } Y\cap(E\cap E_L)=\emptyset\big)\Big]_{p_e\equiv 1}.
\end{aligned}$$

Since

$$\frac{d}{dp_{E\cap E_L}}P\big(X(L_E)\equiv 0 \text{ and } Y\cap(E\cap E_L)=\emptyset\big)=\frac{d}{dp_{E\cap E_L}}\prod_{e\in E\cap E_L} rp_e=r^{|E\cap E_L|}$$

and

$$\begin{aligned}
\frac{d}{dp_{E\cap E_L}}P\big(X(L_E)\equiv 0\big)&=\frac{d}{dp_{E\cap E_L}}\Big(r\prod_{e\in E\cap E_L}\big((1-p_e)+rp_e\big)+(1-r)\prod_{e\in E\cap E_L}\big(rp_e\big)\Big)\\
&=r(-1+r)^{|E\cap E_L|}+(1-r)r^{|E\cap E_L|},
\end{aligned}$$

the desired conclusion follows. □

We now proceed to the proof of Proposition 7.6

Proof of Proposition 7.6: To simplify the notation in the proof, for a connected set $S_0\subseteq S$, we let

$$A(S_0):=\Big[\frac{d}{dp_{E(T_{S_0})}}\log P\Big(X\big(\mathcal{B}^-(S_0)\big)\equiv 0\Big)\Big]_{(p_e)\equiv 1}$$

and note that with this notation, we need to prove that

$$A(S)=-\frac{1-r}{r}\prod_{j=2}^{\infty}\Big(\frac{\mathrm{Li}_{1-j}(-(1-r)/r)}{r^{j-1}(1-r)}\Big)^{k_j}.$$

Fix any $o\in V(T_S)\setminus\mathcal{B}^-(S)$. Let Y be the random set corresponding to the edge percolation process corresponding to X.

Let $\mathcal{P}(E(T_S))$ denote the set of all partitions of $E(T_S)$ into non-empty sets. Then

$$\begin{aligned}
A(S)&=\Big[\frac{d}{dp_{E(T_S)}}\log P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1}\\
&=\sum_{(E_1,\dots,E_k)\in\mathcal{P}(E(T_S))}\frac{\big[(-1)^{k-1}\prod_{j=1}^{k}\frac{d}{dp_{E_j}}P\big(X(\mathcal{B}_T^-(S))\equiv 0\big)\big]_{(p_e)\equiv 1}}{\big[P\big(X(\mathcal{B}^-(S))\equiv 0\big)^k\big]_{(p_e)\equiv 1}},\qquad(7.5)\\
&=\sum_{(E_1,\dots,E_k)\in\mathcal{P}(E(T_S))}(-1)^{k-1}r^{-k|\mathcal{B}^-(S)|}\prod_{j=1}^{k}\Big[\frac{d}{dp_{E_j}}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1},
\end{aligned}$$

If $(E_1, \dots, E_k) \in \mathcal{P}(E(T_S))$ ontains either a partition element that is not connected, or a partition element E_j such that a vertex $v \in V(T_S) \setminus \mathcal{B}^-(S)$ is the endpoint of exactly one edge in E_j, then, by Lemma 7.2 applied with $J = \mathcal{B}^-(S)$, and Lemma 7.7, we have

$$\prod_{j=1}^{k} \Big[\frac{d}{dp_{E_j}} P\big(X(\mathcal{B}^-(S)) \equiv 0\big) \Big]_{(p_e)\equiv 1} = 0.$$

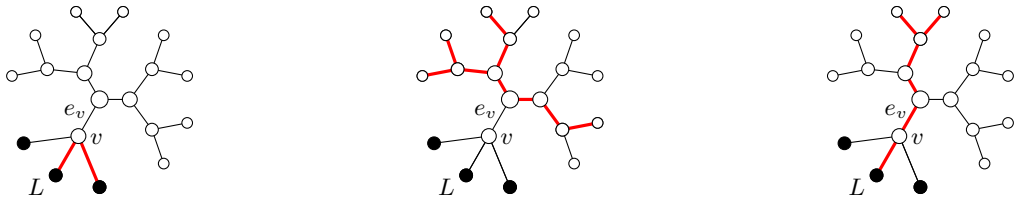
Hence, in (7.5) we only need to sum over partitions $(E_1, \dots, E_k)$ of $E(T_S)$ such that for each $j \in [k]$, the following holds.

(i) E_j is connected, and
(ii) no vertex in $V(T_S) \setminus \mathcal{B}^-(S)$ is the endpoint of exactly one edge in E_j.

Let $\hat{\mathcal{P}}(E(T_S))$ be the set of all such partitions. We note that if $((E_1, \dots, E_k) \in \hat{\mathcal{P}}(E(T_S))$, then for all $j \in [k]$,

(iii) E_j is connected to some vertex in $\mathcal{B}^-(S)$, and

Now let $v \in S \setminus \mathcal{B}^-(S)$ be such that $L := N(v) \cap \mathcal{B}^-(S)$ is nonempty. Let e_v be the unique edge connecting v to $E(T_S) \setminus E(T_{L\cup\{v\}})$. Assume that $E \subseteq E(T_S)$ satisfies (iii) and (ii), let L_E be the set of all vertices in L that are the endpoint of some edge in E, and recall that E_L is the set of all edges in $E(T_S)$ which have one endpoint in L. Note that since we can assume E to be connected, if it doesn't contain the edge e_v, it either contains only edges from E_L or no edge from E_L. We thus have three cases; (1) $E \subseteq E_L$, (2) $E \cap E_L = \emptyset$, and (3) $e_v \in E$ (see Figure 7.5).



(A) The setting of (1) in the proof of Proposition 7.6

(B) The setting of (2) in the proof of Proposition 7.6

(C) The setting of (3) in the proof of Proposition 7.6

FIGURE 7.5. The settings of (1), (2), and (3) in the proof of Proposition 7.6. In the figures, the set L is drawn in black, and a set E in red.

(1) Assume that $E \subseteq E_L$. Equivalently, assume that each edge in E is connected to some vertex in L. Note that since E satisfies (ii), we have $|E| \geq 2$. Consequently, by Lemma 7.8, we have

$$\begin{aligned}\Big[\frac{d}{dp_E} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} &= r^{|\mathcal{B}^-(S)\setminus L_E|}\big(r(r-1)^{|E|} + (1-r)r^{|E|}\big) \\ &= r^{|L\setminus L_E|-1}\big(r(r-1)^{|E|} + (1-r)r^{|E|}\big)\Big[\frac{d}{dp_{E\setminus E_L}} P\Big(X\big(\mathcal{B}^-(S\setminus L)\big)\Big) \equiv 0\Big)\Big]_{(p_e)\equiv 1}\end{aligned}$$

where the last equality follows from the observation that if $E \subseteq E_L$, then $E \setminus E_L = \emptyset$ and $L_E = L$, and hence

$$\begin{aligned}\Big[\frac{d}{dp_{E\setminus E_L}} P\Big(X\big(\mathcal{B}^-(S\setminus L)\big)\Big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} &= \Big[P\Big(X\big(\mathcal{B}^-(S\setminus L)\big)\Big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} \\ &= r^{|\mathcal{B}^-(S)\setminus L_E|+1}.\end{aligned}$$

(2) Assume that $E \cap E_L = \emptyset$. Equivalently, since E satisfies (ii) by assumption, assume that E is not connected to $L \cup \{v\}$. Noting that in this case, we have $E = E \setminus E_L$ and $L_E = \emptyset$, it

follows from Lemma 7.9 that

$$\Big[\frac{d}{dp_E}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|L\setminus L_E|-1}\Big[\frac{d}{dp_{E\setminus E_L}}P\big(X\big(\mathcal{B}^-(S\setminus L)\big)\big)\equiv 0\big)\Big]_{(p_e)\equiv 1}.$$

(3) Assume that $e_v \in E$. Then, since E satisfies (ii) by assumption, we have $E\cap E_L \neq \emptyset$. Consequently, by Lemma 7.10, we have

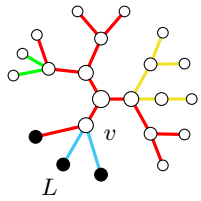
$$\Big[\frac{d}{dp_E}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|L\setminus L_E|-1}r\big((r-1)^{|E\cap E_L|}-r^{|E\cap E_L|}\big)\Big[\frac{d}{dp_{E\setminus E_L}}P\big(X\big(\mathcal{B}^-(S\setminus L)\big)\equiv 0\big)\Big]_{(p_e)\equiv 1}.$$

Combining these cases, we obtain

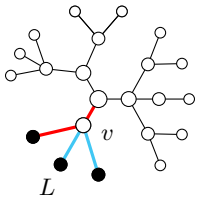
$$\Big[\frac{d}{dp_E}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|L\setminus L_E|-1}\Big[\frac{d}{dp_{E\setminus E_L}}P\big(X\big(\mathcal{B}^-(S\setminus L)\big)\equiv 0\big)\Big]_{(p_e)\equiv 1}\begin{cases}1 & \text{if } E\cap E_L=\emptyset\\ r(r-1)^{|E|}+(1-r)r^{|E|} & \text{if } E\subseteq E_L\\ r(r-1)^{|E\cap E_L|}-r^{|E\cap E_L|}r & \text{otherwise.}\end{cases} \tag{7.6}$$

We now return to the sum in (7.5). To this end, note that there is a natural bijection from the set of partitions $\hat{\mathcal{P}}(E(T_S))$ to the direct product of $\hat{\mathcal{P}}(E(T_{S\setminus L}))$ and $\hat{\mathcal{P}}(E_v)$ (see Figure 7.6). For $(F_1,\dots,F_\ell)\in\hat{\mathcal{P}}(E(T_{S\setminus L}))$ and $(G_1,\dots,G_m)\in\hat{P}(E_v)$, we will without loss of generality assume that $e_v\in F_1\cap G_1$. Using this bijection together with (7.6), it follows that

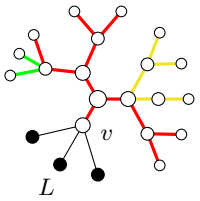
$$\begin{aligned}A(S) &= \sum_{(E_1,\dots,E_k)\in\hat{P}(E(T_S))}(-1)^{k-1}r^{-k|\mathcal{B}^-(S)|}\prod_{j=1}^{k}\Big[\frac{d}{dp_{E_j}}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1}\\ &= \sum_{\substack{(F_1,\dots,F_\ell)\in\hat{P}(E(T_{S\setminus L}))\\(G_1,\dots,G_m)\in\hat{P}(E_v)}}(-1)^{(\ell+m-1)-1}r^{-(\ell+m-1)|\mathcal{B}^-(S)|}\Big[\frac{d}{dp_{F_1\cup G_1}}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1}\\ &\qquad\cdot\prod_{i=2}^{\ell}\Big[\frac{d}{dp_{F_i}}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1}\prod_{j=2}^{m}\Big[\frac{d}{dp_{G_i}}P\big(X(\mathcal{B}^-(S))\equiv 0\big)\Big]_{(p_e)\equiv 1}.\end{aligned}$$



(A) A tree T_S with a partition $(E_1,E_2,E_3,E_4)\in\hat{P}(E(T_S))$ (red, blue, green, and orange).



(B) A tree T_S and a partition $(E_1,E_2)\in\hat{P}(E_v)$ (red and blue).



(C) A tree T_S with a partition $(E_1,E_2,E_3)\in\hat{P}(E(T_{S\setminus L}))$ (red, green, and orange).

FIGURE 7.6. The setting of the end of the proof of Proposition 7.6, illustrating the bijection between $\hat{P}(E(T_S))$ and the direct product of $\hat{P}(E(T_{S\setminus L}))$ and $\hat{P}(E_v)$.

For $i = 2, 3, \dots, \ell$, using the first case of (7.6), we have

$$\Big[\frac{d}{dp_{F_i}} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|L|-1}\Big[\frac{d}{dp_{F_i}} P\big(X(\mathcal{B}^-(S \smallsetminus L)) \equiv 0\big)\Big]_{(p_e)\equiv 1},$$

for $j = 2, 3, \dots, m$, using the second case of (7.6), we have

$$\Big[\frac{d}{dp_{G_j}} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1} = r^{|\mathcal{B}^-(S)|-|L_{G_j}|}\Big(r(-1+r)^{|G_j|} + (1-r)r^{|G_j|}\Big),$$

and finally, using the third case of (7.6), since $L_{F_1\cup G_1} = L_{G_1}$, $(F_1 \cup G_1) \cap E_L = G_1 \smallsetminus \{e_v\}$, and $(F_1 \cup G_1) \smallsetminus E_L = F_1$, we have

$$\begin{aligned}&\Big[\frac{d}{dp_{F_1\cup G_1}} P\big(X(\mathcal{B}^-(S)) \equiv 0\big)\Big]_{(p_e)\equiv 1}\\ &\qquad = r^{|L|-|L_{G_1}|}\Big((r-1)^{|G_1|-1} - r^{|G_1|-1}\Big)\Big[\frac{d}{dp_{F_1}} P\Big(X\big(\mathcal{B}^-(S \smallsetminus L)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1}.\end{aligned}$$

Combining these equations and noting that $\sum_{j=1}^m |L_{G_j}| = |L|$, $(\sum_{j=1}^m |G_j|) - 1 = |L|$, and $|\mathcal{B}^-(S)| - |L| + 1) = |\mathcal{B}^-(S \smallsetminus L)|$, we obtain

$$\begin{aligned}A(S) = &\sum_{\substack{(F_1,\dots,F_\ell)\in\hat{P}(E(T_{S\smallsetminus L}))\\(G_1,\dots,G_m)\in\hat{P}(E_v)}} (-1)^{(\ell+m-1)-1} r^{-\ell|\mathcal{B}^-(S)|+(\ell-1)(|L|-1)} \cdot \Big((r-1)^{|G_1|-1} - r^{|G_1|-1}\Big)\\ &\qquad \cdot \prod_{i=1}^{\ell}\Big[\frac{d}{dp_{F_i}} P\big(X(\mathcal{B}^-(S \smallsetminus L)) \equiv 0\big)\Big]_{(p_e)\equiv 1} \cdot \prod_{j=2}^{m}\Big(r(r-1)^{|G_j|} - (r-1)r^{|G_j|}\Big)\\ = &\sum_{\substack{(F_1,\dots,F_\ell)\in\hat{P}(E(T_{S\smallsetminus L}))\\(G_1,\dots,G_m)\in\hat{P}(E_v)}} (-1)^{\ell-1} r^{-\ell|\mathcal{B}^-(S\smallsetminus L)|} \cdot \prod_{i=1}^{\ell}\Big[\frac{d}{dp_{F_i}} P\big(X(\mathcal{B}^-(S \smallsetminus L)) \equiv 0\big)\Big]_{(p_e)\equiv 1}\\ &\qquad \cdot (-r)(r-1)^{m-1} \prod_{j=1}^{m}\Big(1 - ((r-1)/r)^{|G_j|-1}\Big)\\ = A(S \smallsetminus L) \cdot &\frac{r}{1-r} \sum_{(G_1,\dots,G_m)\in\hat{P}(E_v)} (r-1)^m \prod_{j=1}^{m}\Big(1 - \big((r-1)/r\big)^{|G_j|-1}\Big).\end{aligned}$$

Consequently, if we for $j \geq 2$, let

$$f_j(r) \coloneqq \frac{r}{1-r} \sum_{\substack{(F_1,\dots,F_m)\in\mathcal{P}([j]):\\|F_1|,\dots,|F_m|\geq 2}} (r-1)^m \prod_{i=1}^{m}\Big(1 - ((r-1)/r)^{|F_i|-1}\Big),$$

then we have shown that

$$A(S) = A(S \smallsetminus L) f_{\deg v}(r). \tag{7.7}$$

Now note that if S_0 is a connected set with $|S_0| = 2$, then

$$A(S_0) = \Big[\frac{d}{dp_{E(T_{S_0})}} \log P\big(X(S_0) \equiv 0\big)\Big]_{(p_e)\equiv 1} = \frac{r^2 - r}{r^2} = -\frac{1-r}{r}.$$

Combining this observation with (7.7), we obtain

$$A(S) = -\frac{1-r}{r} \prod_{v\in S\smallsetminus L} f_{\deg v}(r).$$

This concludes the proof. □

We are now finally ready to give a proof of Proposition 7.1.

Proof of Proposition 7.1: To simplify notation, let E_S be the set of edges between S and S^c, and note that since $|S| \geq 2$ and S is connected, by Lemma 4.3, we have

$$\nu(S) = \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big).$$

Now note that $|\partial_S\{j\}|$ is equal to the number of children in $\mathcal{B}^+(S)$ of $j \in \mathcal{B}^-(T_S)$, Thus, by definition, for any $J \subseteq \mathcal{B}^-(S)$ we have

$$\Big[P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = r^{|J| + \sum_{j\in J} |\partial_S\{j\}|}.$$

Using Lemma 4.3, we thus obtain

$$\begin{aligned}
\big[\nu(S)\big]_{(p_e)\equiv 1} &= \Big[\sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} \\
&= \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log\big(r^{|J| + \sum_{j\in J} |\partial_S\{j\}|}\big) \\
&= \log r \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \big(|J| + \sum_{j\in J} |\partial_S\{j\}|\big) \\
&= \log r \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} |J| + \log r \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \sum_{j\in J} |\partial_S\{j\}| \\
&= \log r \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} |J| + \log r \sum_{j \in \mathcal{B}^-(S)} |\partial_S\{j\}| \sum_{J \subseteq \mathcal{B}^-(S):\, j\in J} (-1)^{|J|} = 0,
\end{aligned}$$

since both sums in the last sum of the previous equation are equal to zero if $|\mathcal{B}^-(S)| \geq 2$. This completes the proof of (a) in the case $E = \emptyset$.

Next, note that by Lemma 7.3, for any $E \subseteq E(T)$, we have

$$\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} = \begin{cases} 0 & \text{if } E(T_S) \nsubseteq E \\ (-1)^{|\mathcal{B}^-(S)|}\Big[\frac{d}{dp_E} \log P\Big(X\big(\mathcal{B}^-(S)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} & \text{if } E = E(T_S) \end{cases}$$

By Proposition 7.6, we have

$$\Big[\frac{d}{dp_{E(T_S)}} \log P\Big(X\big(\mathcal{B}^-(S)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = -\frac{1-r}{r} \prod_{v \in S \setminus L} f_{\deg v}(r),$$

where f_j is defined in (7.4). Now note that for a tree T_0 with exactly one vertex v with $\deg v > 1$ and $S_0 = V(T_0)$, we have

$$\Big[\frac{d}{dp_{E(T_S)}} \log P\Big(X\big(\mathcal{B}^-(S)\big) \equiv 0\Big)\Big]_{(p_e)\equiv 1} = -\frac{1-r}{r} f_{\deg v}(r).$$

Using Lemma 7.5 and noting that $\deg v = |\mathcal{B}^-(S_0)|$, it thus follows that

$$f_{\deg v}(r) = \frac{r}{1-r} \operatorname{Li}_{1-|\mathcal{B}^-(S)|}(-r^{-1}(1-r)),$$

where for $j \geq 1$, Li_{1-j} is the polylogarithm function with index $1-j$. Combining the previous equations, it follows that if $E = E(T_S)$, then

$$\begin{aligned}\Big[\frac{d}{dp_E}\nu(S)\Big]_{(p_e)\equiv 1} &= (-1)^{|\mathcal{B}^-(S)|}\Big[\frac{d}{dp_E}\log P\big(X\big(\mathcal{B}^-(S)\big)\equiv 0\big)\Big]_{(p_e)\equiv 1}\\ &= (-1)^{|\mathcal{B}^-(S)|+1}\frac{1-r}{r}\prod_{v\in S\setminus L} f_{\deg v}(r)\\ &= (-1)^{|\mathcal{B}^-(S)|+1}\frac{1-r}{r}\prod_{v\in S\setminus L}\Big(\frac{r}{1-r}\,\mathrm{Li}_{1-\deg v}(-r^{-1}(1-r))\Big)\\ &= (-1)^{|\mathcal{B}^-(S)|+1}\Big(\frac{r}{1-r}\Big)^{|V(T_S)|-|\mathcal{B}^-(S)|-1}\prod_{j=2}^{\infty}\mathrm{Li}_{1-j}\big(-r^{-1}(1-r)\big)^{k_j},\end{aligned}$$

where for $j \geq 2$, k_j is the number of vertices of degree j in T_S, This concludes the proof. □

8. Octopus trees

In this section, the main goal will be to provide a proof of Theorem 1.3.

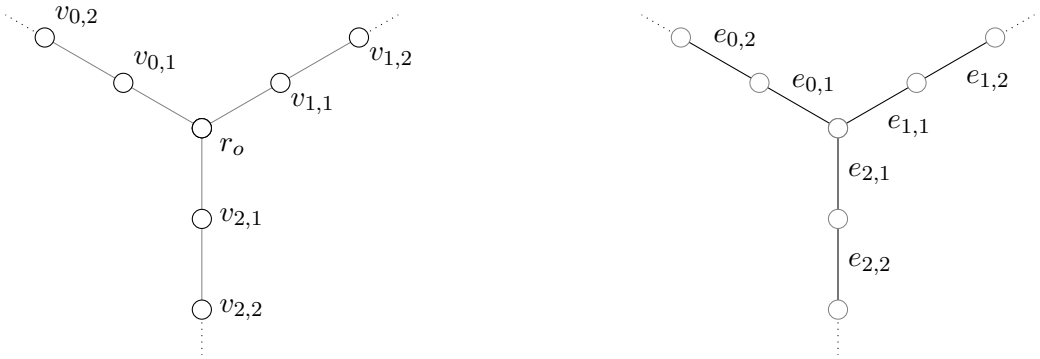


FIGURE 8.7. The tree T_3 and the notation of Proposition 8.2 and Proposition 8.3.

8.1. *r close to zero.* The main tool in the proof of Theorem 1.3(a) is Proposition 8.1, which we now state and prove.

Proposition 8.1. *Let $m \geq 3$ and $r, p \in (0,1)$, and let X be a tree-indexed Markov chain on T_m with parameters (r,p). Then, if $r < r_1(m) \in (0,1)$, and $(p_e)_{e\in E(T)}$ is sufficiently close to 1, we have $X \notin \mathcal{R}$.*

We note that Proposition 8.1 is equivalent to Forsström et al. (2026, Theorem 6.1), but since the proof below is different from the corresponding proof in Forsström et al. (2026), we include the argument here anyway for completeness.

Proof of Proposition 8.1: Let $m' \in \{3,4,\dots,m\}$ be such that $\mathrm{Li}_{1-m'}(-(1-r)/r) > 0$. Since $r < r_1(m)$, Lemma 2.4(b) guarantees that such m' exists

Let $S = \{o\} \cup \mathcal{N}(o)$, let T be the restriction of T_m to $V(T_{m'}) \cap S \cup \mathcal{B}^+(S)$, and let ν_T be the unique signed measure corresponding to $X(V(T))$.

When $(p_e) \equiv 1$, $(X(v))_{v\in V(T)}$ are independent random variables, and hence we have

$$\big[\nu_T(\{v\})\big]_{p=1} = -\log P(X(v)=0) = -\log r > 0 \quad \forall v \in S,$$

and moreover, $\big[\nu_T(S)\big]_{p=1} = 0$ for all $S \in \mathcal{P}(S) \setminus \{\emptyset\}$ with $|S| \neq 1$. Consequently, by Proposition 7.1 and a Taylor expansion, for $(p_e)_{e\in E(T)}$ sufficiently close to one, $\nu_T(S)$ has the same sign as

$$(-1)^{|E(T_S)|}(-1)^{|\mathcal{B}^-(S)|+1}\operatorname{Li}_{1-m'}\big(-r^{-1}(1-r)\big) = -\operatorname{Li}_{1-m'}\big(-r^{-1}(1-r)\big).$$

Since $-\operatorname{Li}_{1-m'}\big(-r^{-1}(1-r) < 0$, we must have $X|_{V(T)} \notin \mathcal{R}$. Using Lemma 2.2(a), it follows that $X \notin \mathcal{R}$. This concludes the proof. □

Proof of Theorem 1.3(a): Since T_m is self-similar, the desired conclusion follows by combining Proposition 8.1 and Proposition 5.2 that $X \notin \mathcal{R}$ for any $p \in (0,1)$. This concludes the proof. □

8.2. *r close to one.* In this section, we provide a proof of Theorem 1.3(b). The main tool in this proof is the following proposition.

Proposition 8.2. *Let $m \geq 3$ and let $T = T_m$. For $v \in V(T)$ and $e \in E(T)$, let $r_v \in (0,1)$ and $p_e \in (0,1)$. Let X be the tree-indexed Markov chain on T with parameters $((r_v)_{v\in V}, (p_e)_{e\in E})$. Let ν be the corresponding signed measure in the sense of Lemma 2.1, and let $S \coloneqq \{o\} \cup N(\{o\})$. Let the vertices and edges of T be labeled as in Figure 8.7. Then the following holds.*

(a) $\big[\nu(S)\big]_{r_o=1} = 0$.

(b) For any multiset $K \subseteq V(T) \setminus \{o\}$, we have $\big[\frac{d}{dr_K}\nu(S)\big]_{(r_v)\equiv 1} = 0$.

(c) $\big[\frac{d}{dr_o}\nu(S)\big]_{(r_v)\equiv 1} = -\prod_{j=1}^m (1-p_{e_{j,1}})p_{e_{j,2}}$.

(d) There is a constant $C_m > 0$ that depends only on m such that for all $v, v' \in V(T)$, we have

$$\Big|\frac{d^2}{dr_v dr_{v'}}\nu(S)\Big| \leq \frac{C_m}{\prod_{j=1}^m r_{v_{j,1}}^{2m+1}} \cdot \prod_{j=1}^m (1-p_{e_{j,1}})p_{e_{j,2}}.$$

Before we provide a proof of Proposition 8.2, we show how we use it to prove Theorem 1.3(b).

Proof of Theorem 1.3(b): Let S be any finite subset of $V(T_m)$. Let T be any finite subtree of T_m so that $S \cup \mathcal{B}^+(S) \subseteq V(T)$. Let ν_T be the unique signed measure corresponding to $X(T)$ in the sense of Lemma 2.1.

We first make the following observations.

(i) If $|S| = 1$ then $\nu_T(S) \geq 0$ by Lemma 4.3.

(ii) If S is not connected, then $\nu_T(S) = 0$ by Proposition 3.2.

(iii) If S is a connected set that does not contain the origin o, then S is an interval of length $j \coloneqq |S|$. By Lemma 4.3, we have

$$\begin{aligned}\nu_T(S) &= \sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\log P\Big(X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\Big)\\ &= \log\big((1-p)^{j+1}r + (1-(1-p)^{j+1})r^2\big) + \log\big((1-p)^{j-1}r + (1-(1-p)^{j-1})r^2\big)\\ &\quad - 2\log\big((1-p)^j r + (1-(1-p)^j)r^2\big)\\ &= \log\Big(1 + \frac{p^2(1-p)^{j-1}r(1-r)}{(r+(1-p)^j-(1-p)^j r)^2}\Big) > 0.\end{aligned}$$

Now assume that $|S| > 1$ and $o \in S$. Let $V_1, \ldots, V_m \subseteq V(T_m)$ denote the vertex sets corresponding to the connected components of the restriction of T_m to $V(T_m) \setminus \{o\}$. By Proposition 8.2 applied with $p_{e_{j,1}} = 1-(1-p)^{|S\cap V_j|}$ and $p_{e_{j,2}} = p$ for all $j \in [m]$, there exists a constant $r_2(m) \in (0,1)$, independent of S, T, and p, such that, by Taylor's theorem, $\nu_T(S) > 0$ for all $r \geq r_2(m)$.

To finish the proof, note that by Lemma 4.3, $\nu_T(S)$ is independent of the choice of T. For $n \geq 1$, let $T_m^{\leq n}$ be the subtree of T_m which contains all vertices at distance at most n from the origin. Applying Lemma 2.2(c) with $S_n = V(T_m) \setminus V(T_m^{\leq n})$, the desired conclusion immediately follows. □

Proof of Proposition 8.2: For $j \in [m]$, let $V_j := \{v_{j,1}, v_{j,2}\}$, and note that $S = \{o\} \cup \{v_{j,1},\, j \in [m]\}$. For $J \subseteq \mathcal{B}^-(S)$ consider the event

$$\mathcal{E}_J := \big\{X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\big\},$$

and for $J \subseteq \mathcal{B}^-(S)$ and $j \in [m]$, consider the event

$$\mathcal{E}_J^j := \begin{cases} X(v_{j,1}) = 0 & \text{if } J \cap V_j \neq \emptyset \\ X(v_{j,2}) = 0 & \text{otherwise.} \end{cases}$$

Since $|S| \geq 2$, by Lemma 4.3, we have

$$\nu(S) = \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P(\mathcal{E}_J).$$

Let $(R_v)_{v \in T}$ be as in A' and B' of Section 2.3. Then

$$\big[\nu(S)\big]_{r_o=1} = \Big[\sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P(\mathcal{E}_J)\Big]_{r_o=1} = \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\big(\mathcal{E}_J \mid R_o = 0\big).$$

Conditioned on R_o, the random vectors $X(V_j)$, $j \in [m]$ are independent. Using this observation, it follows that

$$\begin{aligned}
\sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P\big(\mathcal{E}_J \mid R_o = 0\big) &= \sum_{J \subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log \prod_{j=1}^m P\big(\mathcal{E}_J^j \mid R_o = 0\big) \\
&= \sum_{J_1 \subseteq \mathcal{B}^-(S) \cap V_1} \cdots \sum_{J_m \subseteq \mathcal{B}^-(S) \cap V_m} (-1)^{|J_1|+|J_2|+\cdots+|J_m|} \log \prod_{j=1}^m P(\mathcal{E}_{J_j}^j \mid R_o = 0) \\
&= \sum_{J_1 \subseteq \mathcal{B}^-(S) \cap V_1} \cdots \sum_{J_m \subseteq \mathcal{B}^-(S) \cap V_m} (-1)^{|J_1|+|J_2|+\cdots+|J_m|} \sum_{j=1}^m \log P(\mathcal{E}_{J_j}^j \mid R_o = 0) \\
&= \sum_{j=1}^m \Bigg(\sum_{J_j \subseteq \mathcal{B}^-(S) \cap V_j} (-1)^{|J_j|} \log P(\mathcal{E}_{J_j}^j \mid R_o = 0) \prod_{k \in [m] \setminus \{j\}} \sum_{J_k \subseteq \mathcal{B}^-(S) \cap V_k} (-1)^{|J_k|} \Bigg).
\end{aligned}$$

Now note that, for any $j \in [m]$ and $k \in [m] \setminus \{j\}$, the set $\mathcal{B}^-(S) \cap V_k$ is non-empty (for any $j \in [m]$, such k exist since $m \geq 2$). This implies in particular that for any $j \in [m]$ and $k \in [m] \setminus \{j\}$, we have

$$\sum_{J_k \subseteq \mathcal{B}^-(S) \cap V_k} (-1)^{|J_k|} = 0.$$

Combining the previous equations, we obtain

$$\big[\nu(S)\big]_{r_o=1} = 0.$$

This completes the proof of (a). Since differentiation is a linear operator, the same conclusion holds for any combination of derivatives of $\nu(S)$ that do not include $\frac{d}{dr_o}$. In particular, for any multiset $K \subseteq V(T) \setminus \{o\}$, we have

$$\Big[\frac{d}{dr_K} \nu(S)\Big]_{r_o=1} = 0.$$

This completes the proof of (b).

We now calculate $\frac{d}{dr_o}\nu(S)|_{(r_v)\equiv 1}$. To this end, note first that for any $J \subseteq \mathcal{B}^-(S)$, we have

$$\frac{d}{dr_o} P(\mathcal{E}_J) = P(\mathcal{E}_J \mid R_o = 0) - P(\mathcal{E}_J \mid R_o = 1).$$

Using this observation, it follows that

$$\frac{d}{dr_o}\nu(S) = \frac{d}{dr_o}\sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\log P(\mathcal{E}_J) = \sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{\frac{d}{dr_o}P(\mathcal{E}_J)}{P(\mathcal{E}_J)}$$
$$= \sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{P(\mathcal{E}_J \mid R_o=0) - P(\mathcal{E}_J \mid R_o=1)}{P(\mathcal{E}_J)},$$

and hence

$$\Big[\frac{d}{dr_o}\nu(S)\Big]_{r_o=1} = \sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\left[1-\frac{P(\mathcal{E}_J \mid R_o=1)}{P(\mathcal{E}_J \mid R_o=0)}\right] = -\sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{P(\mathcal{E}_J \mid R_o=1)}{P(\mathcal{E}_J \mid R_o=0)}.$$

Now recall that given r_o, the random vectors $X(V_1), \ldots, X(V_m)$ are independent. From this it follows that

$$\sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{P(\mathcal{E}_J \mid R_o=1)}{P(\mathcal{E}_J \mid R_o=0)} = \sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\prod_{j=0}^{m}\frac{P(\mathcal{E}^j_{J\cap V_j} \mid R_o=1)}{P(\mathcal{E}^j_{J\cap V_j} \mid R_o=0)}$$
$$\sum_{J_1\subseteq\mathcal{B}^-(S)\cap V_1}\cdots\sum_{J_m\subseteq\mathcal{B}^-(S)\cap V_m}\prod_{j=1}^{m}(-1)^{|J_j|}\frac{P(\mathcal{E}^j_{J_j} \mid R_o=1)}{P(\mathcal{E}^j_{J_j} \mid R_o=0)}.$$

Changing the order of multiplication and summation, we can rewrite the previous expression as

$$\prod_{j=1}^{m}\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}(-1)^{|J_j|}\frac{P(\mathcal{E}^j_{J_j} \mid R_o=1)}{P(\mathcal{E}^j_{J_j} \mid R_o=0)}.$$

Now note that for any $j\in[m]$, we have

$$\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}(-1)^{|J_j|}\frac{P(\mathcal{E}^j_{J_j} \mid R_o=1)}{P(\mathcal{E}^j_{J_j} \mid R_o=0)}$$
$$= \frac{P\big(X(v_{j,2})=0 \mid R_o=1\big)}{P\big(X(v_{j,2})=0 \mid R_o=0\big)} - \frac{P\big(X(v_{j,1})=0 \mid R_o=1\big)}{P\big(X(v_{j,1})=0 \mid R_o=0\big)}$$
$$= \frac{p_{e_{j,2}}r_{v_{j,2}} + p_{e_{j,1}}(1-p_{e_{j,2}})r_{v_{j,2}}}{p_{e_{j,2}}r_{v_{j,2}} + p_{e_{j,1}}(1-p_{e_{j,2}})r_{v_{j,1}} + (1-p_{e_{j,1}})(1-p_{e_{j,2}})} - \frac{p_j r_{v_{j,1}}}{p_{e_{j,1}}r_{v_{j,1}} + (1-p_{e_{j,1}})}$$
$$= \frac{(1-p_{e_{j,1}})p_{e_{j,2}}r_{v_{j,2}}}{(p_{e_{j,2}}r_{v_{j,2}} + p_{e_{j,1}}(1-p_{e_{j,2}})r_{v_{j,1}} + (1-p_{e_{j,1}})(1-p_{e_{j,2}}))(p_{e_{j,1}}r_{v_{j,1}} + (1-p_{e_{j,1}}))}.$$

Combining the previous equations, it follows that

$$\frac{d}{dr_o}\nu(S)\big|_{(r_v)\equiv 1} = -\prod_{j=1}^{m}(1-p_{e_{j,1}})p_{e_{j,2}},$$

thus completing the proof of (c).

It remains to show that (d) holds. To this end, let $K\subseteq\{o\}\cup\{v_{j,i}\colon i\in[2], j\in[m]\}$ be a non-empty multiset. Then, for any $j\in[m]$ and $J\subseteq\mathcal{B}^-(S)$, we have

$$\big[P(\mathcal{E}_J)\big]_{p_{e_{j,2}}=0} = \big[P(\mathcal{E}_{J\Delta\{v_{j,1}\}})\big]_{p_{e_{j,2}}=0}$$

and hence

$$\begin{aligned}\Big[\frac{d}{dr_K}\nu(S)\Big]_{p_{e_{j,2}}=0} &= \Big[\sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{d}{dr_K}\log P(\mathcal{E}_J)\Big]_{p_{e_{j,2}}=0}\\ &= \Big[\sum_{J\subseteq\mathcal{B}^-(S)\setminus\{v_{j,1}\}}(-1)^{|J|}\frac{d}{dr_K}\log P(\mathcal{E}_J)-\log P(\mathcal{E}_{J\cup\{v_{j,1}\}})\Big]_{p_{e_{j,2}}=0}\\ &= \Big[\sum_{J\subseteq\mathcal{B}^-(S)\setminus\{v_{j,1}\}}(-1)^{|J|}\frac{d}{dr_K}\log P(\mathcal{E}_J)-\log P(\mathcal{E}_J)\Big]_{p_{e_{j,2}}=0}=0.\end{aligned}$$

Next, note that if $p_{e_{j,1}}=1$ for some $j\in[m]$, then $X(V_j)$ and $X(S\setminus V_j)$ are independent. Consequently, we have

$$\begin{aligned}&\Big[\frac{d}{dr_K}\nu(S)\Big]_{p_{e_{j,1}}=1} = \Big[\sum_{J\subseteq\mathcal{B}^-(S)}(-1)^{|J|}\frac{d}{dr_K}\log P(\mathcal{E}_J)\Big]_{p_{e_{j,1}}=1}\\ &= \Big[\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J_j|+|J|}\frac{d}{dr_K}\log P(\mathcal{E}^j_{J_j})P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus(V_j\cup\partial_S J))\big)\equiv 0\Big)\Big]_{p_{e_{j,1}}=1}\\ &= \Big[\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J_j|+|J|}\frac{d}{dr_K}\log P(\mathcal{E}^i_{J_i})\Big]_{p_{e_{j,1}}=1}\\ &\quad+\Big[\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J_j|+|J|}\frac{d}{dr_K}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus(V_j\cup\partial_S J))\big)\equiv 0\Big)\Big]_{p_{e_{j,1}}=1}.\end{aligned}$$

Noting that, since $\mathcal{B}^-(S)\setminus V_j\neq\emptyset$ and $\mathcal{B}^-(S)\cap V_j\neq\emptyset$, we have

$$\begin{aligned}&\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J_j|+|J|}\frac{d}{dr_K}\log P(\mathcal{E}^i_{J_i})\\ &\quad=\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}(-1)^{|J_j|}\frac{d}{dr_K}\log P(\mathcal{E}^i_{J_i})\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J|}=0\end{aligned}$$

and

$$\begin{aligned}&\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J_j|+|J|}\frac{d}{dr_K}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus(V_j\cup\partial_S J))\big)\equiv 0\Big)\\ &\quad=\sum_{J\subseteq\mathcal{B}^-(S)\setminus V_j}(-1)^{|J|}\frac{d}{dr_K}\log P\Big(X\big(J\cup(\mathcal{B}^+(S)\setminus(V_j\cup\partial_S J))\big)\equiv 0\Big)\sum_{J_j\subseteq\mathcal{B}^-(S)\cap V_j}(-1)^{|J_j|}=0\end{aligned}$$

it follows that $\big[\frac{d}{dr_K}\nu(S)\big]_{p_{e_{j,1}}=1}=0$. To sum up, we have shown that

$$\big[\frac{d}{dr_K}\nu(S)\big]_{p_{e_{j,2}}=0}=\big[\frac{d}{dr_K}\nu(S)\big]_{p_{e_{j,1}}=1}=0,\quad\forall j\in[m].$$

Now note that for any distinct $v, v' \in V(T)$ we have

$$\begin{aligned}\frac{d^2}{dr_v dr_{v'}}\nu(S) &= \frac{d^2}{dr_v dr_{v'}} \sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|} \log P(\mathcal{E}_J) \\ &= \sum_{J\subseteq \mathcal{B}^-(S)} (-1)^{|J|} \frac{P(\mathcal{E}_J)\frac{d^2}{dr_v dr_{v'}}P(\mathcal{E}_J) - \frac{d}{dr_v}P(\mathcal{E}_J)\frac{d}{dr_{v'}}P(\mathcal{E}_J)}{P(\mathcal{E}_J)^2} \\ &= \frac{\sum_{J\subseteq \mathcal{B}^-(S)}(-1)^{|J|}\big(P(\mathcal{E}_J)\frac{d^2}{dr_v dr_{v'}}P(\mathcal{E}_J) - \frac{d}{dr_v}P(\mathcal{E}_J)\frac{d}{dr_{v'}}P(\mathcal{E}_J)\big)\prod_{I\subseteq \mathcal{B}^-(S), I\neq J} P(\mathcal{E}_I)^2}{\prod_{J\subseteq \mathcal{B}^-(S)} P(\mathcal{E}_J)^2} \end{aligned} \tag{8.1}$$

Note that both the numerator and the denominator are polynomials in $(r_v)_{v\in V(T)}$, and $(p_e)_{e\in E(T)}$. Further, note that by positive association, we have $P(\mathcal{E}_J) \geq \prod_{j=1}^m r_{v_{j,1}}$ for all $J \subseteq \mathcal{B}^-(S)$. Since $\big[\frac{d^2}{dr_v dr_{v'}}\nu(S)\big]_{p_{e_{j,2}}=0} = 0$ and $\big[\frac{d^2}{dr_v dr_{v'}}\nu(S)\big]_{p_{e_{j,1}}=1} = 0$ for all $j \in [m]$, it follows that the numerator of (8.1) must have $\prod_{j=1}^m p_{e_{j,2}}(1-p_{e_{j,1}})$ as a factor. Since each $P(\mathcal{E}_J)$ is a sum of 2^{2m+1} monomials of degree at most one in each variable, we obtain (d) as desired. This completes the proof. □

8.3. $m = 3$. In this section, we provide a proof of Theorem 1.3(c). The main additional tool we will need in this proof is the following proposition.

Proposition 8.3. *Let T be the restriction of T_3 to the set of vertices at distance at most two from the root, let the vertices and edges be labeled as in Figure 8.7, and let $S := \{o\} \cup \mathcal{B}^+(o)$. For $j \in [3]$ and $k \in [2]$, let $r_o = r_{v_{j,k}} = r \in [\frac{1}{2}, 1]$ and $p_{e_{j,k}} \in [0,1]$. Further, let X be the tree-indexed Markov chain on T with parameters $((r_v),(p_e))$, and let ν be the corresponding signed measure as in Lemma 2.1. Then $\nu(S) \geq 0$.*

Before we give a proof of Proposition 8.3, we show how it implies Theorem 1.3(c).

Proof of Theorem 1.3(c): Using Theorem 1.3(a), it suffices to show that $X \in \mathcal{R}$ if $r \geq 1/2$.

Let $S \subseteq V(T_3)$ be finite and let T be a finite subtree of T_3 such that $S \cup \mathcal{B}^+(S) \subseteq V(T)$. Let ν_T be the signed measure corresponding to $X\big(V(T)\big)$ in the sense of Lemma 2.1. We now show that $\nu_T(S) \geq 0$. To this, end, note first that as in the proof of Theorem 1.3(b), we know that $\nu_T(S) \geq 0$ if (i) $|S| = 1$, (ii) S is not connected, or (iii) $o \notin S$.

Now assume that S is a connected set that contains the origin. Let V_1, V_2, and V_3, be the sets of vertices on each arm of T_3, so that $V(T_3) = \{o\} \sqcup V_1 \sqcup V_2 \sqcup V_3$, and the restriction of T_3 to V_j is an infinite path for each $j \in [3]$. Let T' be the restriction of T_3 to the set of vertices on distance at most two from the root, and let X' be the corresponding tree-indexed Markov chain with parameters $(r,(p_e))$, where for $j \in [3]$, we let $p_{e_{j,1}} = 1-(1-p)^{|S\cap V_j|}$ and $p_{e_{j,2}} = p$. Then, by Lemma 4.3, we have

$$\nu_T(S) = \nu_{T'}(N(o)) \tag{8.2}$$

and hence, by Proposition 8.3, we have $\nu_T(S) \geq 0$.

Finally, note that by Lemma 8.2, $\nu_T(S)$ is independent of the choice of T whenever $S \cup \mathcal{B}^+(S) \subseteq T$. Using Lemma 2.2(c), the desired conclusion immediately follows. □

Proof of Proposition 8.3: To simplify notation, for $J \subseteq \mathcal{B}^-(S)$, consider the event

$$\mathcal{E}_J := \big\{X\big(J \cup (\mathcal{B}^+(S) \setminus \partial_S J)\big) \equiv 0\big\}.$$

By Lemma 4.3, we have

$$\begin{aligned}\nu(S) &= \sum_{J\subseteq\mathcal{B}^-(S)} (-1)^{|J|}\log P(\mathcal{E}_J) = \log \prod_{J\subseteq\mathcal{B}^-(S)} P(\mathcal{E}_J)^{(-1)^{|J|}} \\ &= \log\left(1+\frac{\prod_{\substack{J\subseteq\mathcal{B}^-(S):\\ |J| \text{ even}}} P(\mathcal{E}_J) - \prod_{\substack{J\subseteq\mathcal{B}^-(S):\\ |J| \text{ odd}}} P(\mathcal{E}_J)}{\prod_{\substack{J\subseteq\mathcal{B}^-(S):\\ |J| \text{ odd}}} P(\mathcal{E}_J)}\right).\end{aligned}$$

Hence $\nu(S)\geq 0$ if and only if

$$\prod_{\substack{J\subseteq\mathcal{B}^-(S):\\ |J| \text{ even}}} P(\mathcal{E}_J) - \prod_{\substack{J\subseteq\mathcal{B}^-(S):\\ |J| \text{ odd}}} P(\mathcal{E}_J) \geq 0. \tag{8.3}$$

To simplify notation, let

$$\begin{cases} q^0_{j,1} \coloneqq P(X_{v_{j,1}}=0 \mid X_o=0) = rp_{e_{j,1}} + (1-p_{e_{j,1}}) \\ q^1_{j,1} \coloneqq P(X_{v_{j,1}}=0 \mid X_o=1) = rp_{e_{j,1}} \\ q^0_{j,2} \coloneqq P(X(v_{j,2})=0 \mid X_0=0) = rp_{e_{j,2}} + rp_{e_{j,1}}(1-p_{e_{j,2}}) + (1-p_{e_{j,1}})(1-p_{e_{j,2}}) \\ q^1_{j,2} \coloneqq P(X(v_{j,2})=0 \mid X_0=1) = rp_{e_{j,2}} + rp_{e_{j,1}}(1-p_{e_{j,2}}). \end{cases}$$

Then, for the left-hand side of (8.3), we have

$$\begin{aligned}&\prod_{J\subseteq\mathcal{B}^-(S):\,|J| \text{ even}} P(\mathcal{E}_J) - \prod_{J\subseteq\mathcal{B}^-(S):\,|J| \text{ odd}} P(\mathcal{E}_J) \\ &\quad= \Big[r\prod_{j\in[3]} q^0_{j,2} + (1-r)\prod_{j\in[3]} q^1_{j,2}\Big]\cdot\prod_{k\in[3]}\Big[rq^0_{k,2}\prod_{j\in[3]\setminus\{k\}} q^0_{j,1} + (1-r)q^1_{k,2}\prod_{j\in[3]\setminus\{k\}} q^1_{j,1}\Big] \\ &\qquad - \Big(r\prod_{j\in[3]} q^0_{j,1} + (1-r)\prod_{j\in[3]} q^1_{j,1}\Big)\cdot\prod_{k\in[3]}\Big[rq^0_{k,1}\prod_{j\in[3]\setminus\{k\}} q^0_{j,2} + (1-r)q^1_{k,1}\prod_{j\in[3]\setminus\{k\}} q^1_{j,2}\Big] \\ &\quad= r(1-r)\prod_{j\in[3]}(q^0_{j,1}q^1_{j,2} - q^1_{j,1}q^0_{j,2})\Big(r^2\prod_{j\in[3],k\in[2]} q^0_{j,k} - (1-r)^2\prod_{j\in[3],k\in[2]} q^1_{j,k}\Big).\end{aligned}$$

Now note that for any $j\in[3]$, we have $q^0_{j,1}q^1_{j,2} - q^1_{j,1}q^0_{j,2} = r(1-p_{e_{j,1}})p_{e_{j,2}} > 0$, that for all $j\in[3]$ and $k\in[2]$, by positive association, we have $q^0_{j,k}\geq q^1_{j,k}$, and that since $r\geq 1/2$, we have $r^2\geq(1-r)^2$. Combining these observations with the above equations, the desired conclusion follows. □

9. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. To this end, we first note that (a) and (c) of Theorem 1.4 are essentially an immediate consequence of Theorem 1.2(b) and Theorem 1.3(a) respectively.

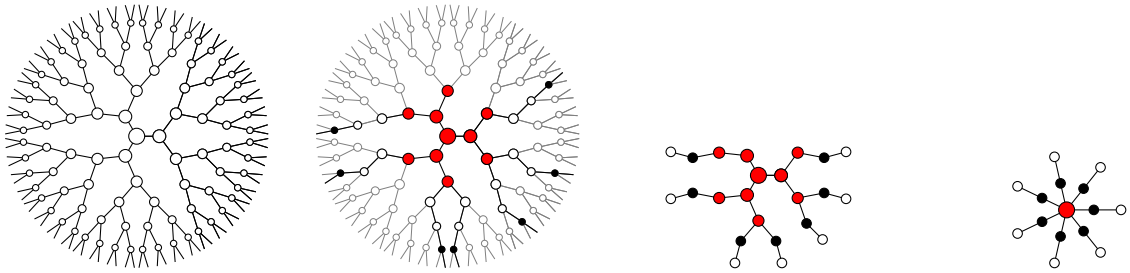
Proof of Theorem 1.4(a): Using Lemma 2.2(b), Theorem 1.4(a) is an immediate consequence of Theorem 1.2(b). This concludes the proof. □

Proof of Theorem 1.4(c): By assumption, T_{n_3} is a subtree of T. By Theorem 1.3(a), $X(T_{n_3})\notin\mathcal{R}$, and hence, by Lemma 2.2(b), $X\notin\mathcal{R}$. This concludes the proof. □

We now give a proof of Theorem 1.4(b). The main idea of the proof is that for any $m\geq 3$, if p is very small and T is a tree which contains m eventually disjoint infinite paths starting from o, then T contains a subtree X' such that $X(T')$ with parameters (p,r) is very close to $X(T_m)$ with parameters (p',r), except possible very close to the root vertex o. Since these edges are only finitely many, by continuity, the result essentially follows from Theorem 1.3(a).

Proof of Theorem 1.4(b): Let $m \geq 1$ be such that there are m distinct infinite paths $P_1, \dots, P_m$ in T starting from o, and such that $-(1-r)/r \in (r_2^{(m)}, r_1^{(m)})$. Since $r < r_1(n_2)$ implies $-(1-r)/r < r_1^{(n_2)}$, such m exists, and moreover, for such m, we have $\mathrm{Li}_{1-m}(-(1-r)/r) > 0$.

For $j \geq 1$, let T^j be the subgraph of T which corresponds to all vertices at distance at most j from the origin. Let $k = k(m)$ be the smallest $k \geq 0$ such that the sets $P_1 \setminus V(T^k)$, …, and $P_m \setminus V(T^k)$ are disjoint. Note that $E(T^k)$ is finite. Let $V_k := (P_1 \cup \cdots \cup P_m) \cap V(T^k)$. For $j \geq k$, let $V_{k,j}$ be the union of V_k and the $(k+j)$th and $(k+2j)$th vertices on each of the paths $P_1, \dots, P_m$ (see Figure 9.8c).



(A) The tree T.

(B) T^2 in red, paths $P_1, \dots, P_7$, and on each path, a vertex at distance 3 from T^2 in black.

(C) The subgraph of T induced by $V_{2,1}$, and the set S_1 in black and red.

(D) T_7 and the set S_0 in black and red.

FIGURE 9.8. In the figures above, for a tree T, we draw the related trees and paths that appear in the proof of Theorem 1.4. Note that in this case we have $k(7) = 2$.

Let $T_m^{\leq 2}$ denote the subgraph of T_m induced by all vertices at distance at most two from the origin. Let $p^0 \in (0,1)$, and let X^0 be the tree-indexed Markov chain on $T_m^{\leq 2}$ with parameters (r, p^0) (see Figure 9.8d), and let ν_0 be the corresponding signed measure. Further, let $S_0 := \{o\} \cup N(\{o\})$. Since $\mathrm{Li}_{1-m}(-(1-r)/r) > 0$, using a Taylor expansion, it follows from Proposition 7.1 implies that that $\nu_0(S_0) < 0$ whenever p^0 is sufficiently close to one. By continuity, for all sufficiently small $\varepsilon > 0$ there is a non-empty interval $I_0 \subseteq (0,1)$ such that $\nu_0(S_0) < -\varepsilon$ whenever $p^0 \in I_0$.

Next, let $T^{k,1}$ be the subgraph of T induced by $V_{k,1}$. Let $p^1 \in [0,1)$, and let X^1 be the tree-indexed Markov chain on $T^{k,1}$ with parameters $(r, (p_e))$, where $p_e = p^1$ for all $e \in E(T^k)$ and $p_e = p^0$ for all $e \in E(T^{k,1}) \setminus E(T^k)$. Let ν_1 be the corresponding signed measure. Let $S_1 := V_k \cup \mathcal{B}^+_{T_{k,1}}(V_k)$. Note that if $p^1 = 0$ then $X^1(\{o\} \cup (V_{k,1} \setminus V_k))$ is equal in distribution to X^0, and hence, by Lemma 2.2, we have $\nu_1(S_1) = \nu_0(S_0) < 0$ when $p^1 = 0$. Since $\nu_1(S_1)$ is a continuous function in p^0 and p^1 and V_k is finite, it follows that there is $\delta > 0$ and a non-empty interval $I_1 \subseteq I_0$ such that $\nu_1(S_1) < -\varepsilon/2$ whenever $p^1 \in (0, \delta)$ and $p^0 \in I_1$. Hence, for such p^0 and p^1, we have $X^1 \notin \mathcal{R}$.

Now let X be a tree-indexed Markov chain on T with parameters (r, p). Assume that p is small enough to ensure $p = p_1 \in (0, \delta)$ and that $p_0 = 1 - (1-p)^j \in I_1$ for some $j \geq 1$. Fix any $j \geq 1$ such that this holds. Then $X(V_{k,j})$ is equal in distribution to $X^1(S_1)$. Since $X^1 \notin \mathcal{R}$, using Lemma 2.2, it follows that $X \notin \mathcal{R}$. This concludes the proof. □

Acknowledgements

The author would like to thank an anonymous referee for pointing out several mistakes in an earlier version of this paper.

References

Bethuelsen, S. A. and Forsström, M. P. Mixing for Poisson representable processes and consequences for the Ising model and the contact process. *Stochastic Process. Appl.*, **192**, Paper No. 104831, 16 (2026). DOI: 10.1016/j.spa.2025.104831.

Forsström, M. P., Gantert, N., and Steif, J. E. Poisson representable processes. *Probab. Theory Related Fields*, **195** (1-2), 741–780 (2026). DOI: 10.1007/s00440-025-01391-8.

Georgii, H.-O. *Gibbs measures and phase transitions*, volume 9 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition (2011). ISBN 978-3-11-025029-9. DOI: 10.1515/9783110250329.

Mossel, E. and Peres, Y. Information flow on trees. *Ann. Appl. Probab.*, **13** (3), 817–844 (2003). DOI: 10.1214/aoap/1060202828.

Steif, J. E. and Tykesson, J. Generalized divide and color models. *ALEA Lat. Am. J. Probab. Math. Stat.*, **16** (2), 899–955 (2019). DOI: 10.30757/alea.v16-33.