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Limit theorems for the Wiener process with resetting

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Abstract. We establish a large deviation principle for the trajectories of Wiener processes subject to random resets to the origin occurring according to a Poisson process. In addition to the pathwise large deviation principle, we identify the rate function and establish a large deviation principle for the supremum of the process over long time intervals.

1. Introduction

In this paper, we continue our investigation into large deviations for Markov processes with catastrophes of Logachov et al. (2020, 2025, 2024a). Within this framework, stochastic processes with resettings can be viewed as a very special – yet remarkably prominent – subclass of processes with catastrophes. Although technically a special case, reset-type dynamics stand out due to their rich structure and broad applicability in search processes, queueing systems, biology, and even in quantum physics. The volume of literature on resetting is substantial, and a comprehensive review is beyond our scope. We restrict ourselves to mentioning a few representative and survey-style works.

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The importance of resetting in modern stochastic modeling is reflected in the rapidly growing literature and, in particular, in the special issue "Stochastic Resetting: Theory and Applications" (Kundu and Reuveni, 2024; see their preface), dedicated to the 10th anniversary of the foundational paper "Diffusion with Stochastic Resetting" of Evans and Majumdar (2011). Among the contributions to this issue, we refer to Nagar and Gupta (2023) for a review of resetting-type processes in particle systems, and to Zamparo (2022) for a comprehensive overview of rigorous mathematical results on resetting. Before this special issue, a valuable review of stochastic processes with resetting is provided in Evans et al. (2020); see also the brief review Gupta and Jayannavar (2022) for a concise introduction to the topic. For readers interested in reset-type processes in queueing systems, we refer to Roy et al. (2024).

Large deviations for stochastic processes with resetting remain a relatively underexplored area of research. Notable exceptions include the paper Meylahn et al. (2015) (see also Zamparo, 2022, the letter Harris and Touchette, 2017, and the recent lecture notes Evans and Sunil, 2024), where large deviation principles are established for integral (additive) functionals of diffusion processes with resetting. In Coghi and Olsen (2025), large deviation techniques are employed to design stochastic resetting protocols that increase the probability of reaching desired configurations, such as promoting homogeneity in systems with network heterogeneity. It is also worth mentioning the paper Monthus (2021), where large deviations were studied for the empirical measure, flow, and trajectories between successive resettings. We would particularly like to mention the paper Hartmann et al. (2024), where the authors derive the large deviation rate function for the maximum of Brownian motion with resetting. Observe that the resulting rate function coincides with the one established in Theorem 2.6 below.

We are interested in large deviations at the level of process trajectories. In Logachov et al. (2020), we studied the Wiener process with catastrophes, where, at each Poissonian resetting time, a new position is chosen randomly according to a conditional distribution that depends on the state of the process just before the reset. We focused on a class of such distributions referred to as *quasi-uniform catastrophes*, where the new state is selected (quasi-)uniformly between the previous state and zero. While our original goal was to establish a full large deviation principle, the results of that work were limited to a local large deviation principle in the functional space of cádlág functions, from which we were nevertheless able to derive the explicit form of the rate functional.

Here, we prove the (full) large deviation principle for the Wiener processes with resetting at the origin. In the literature on processes with catastrophes, the resetting at the origin corresponds to the so-called total catastrophes. We prove the principle in the metric space $\mathbb{L}[0,1]$, the space of all real-valued functions defined on the interval $[0,1]$ that are integrable in the sense of Lebesgue, with distance given by the integral of the absolute difference. Moreover, we prove the principle also for the extreme statistics.

It appears that for stochastic processes with catastrophes, one could also hope to establish a large deviation principle in the metric space $\mathbb{L}[0,1]$. However, obtaining such results is considerably more difficult. For example, in Logachov et al. (2020) the lower bound in the local large deviation principle becomes finite for certain continuous functions that are not absolutely continuous. This already indicates that the form of the rate functional in the large deviations setting for the Wiener process with catastrophes is significantly more involved, and that the proof itself becomes much more delicate.

Finally, it is worth noting that the classical approach developed by Feng and Kurtz (2006) for proving large deviation principles in Markov processes relies on certain uniform integrability conditions on the jump sizes, often referred to as the Cramér condition. Intuitively, this condition requires that the process does not make very large jumps, regardless of its current state. In the case of a Wiener process with resettings to the origin, the size of each jump depends on the current position of the process, which can become arbitrarily large over time. Although the expected jump size is finite at each point, it is not uniformly bounded across all states. As a result, the Cramér condition

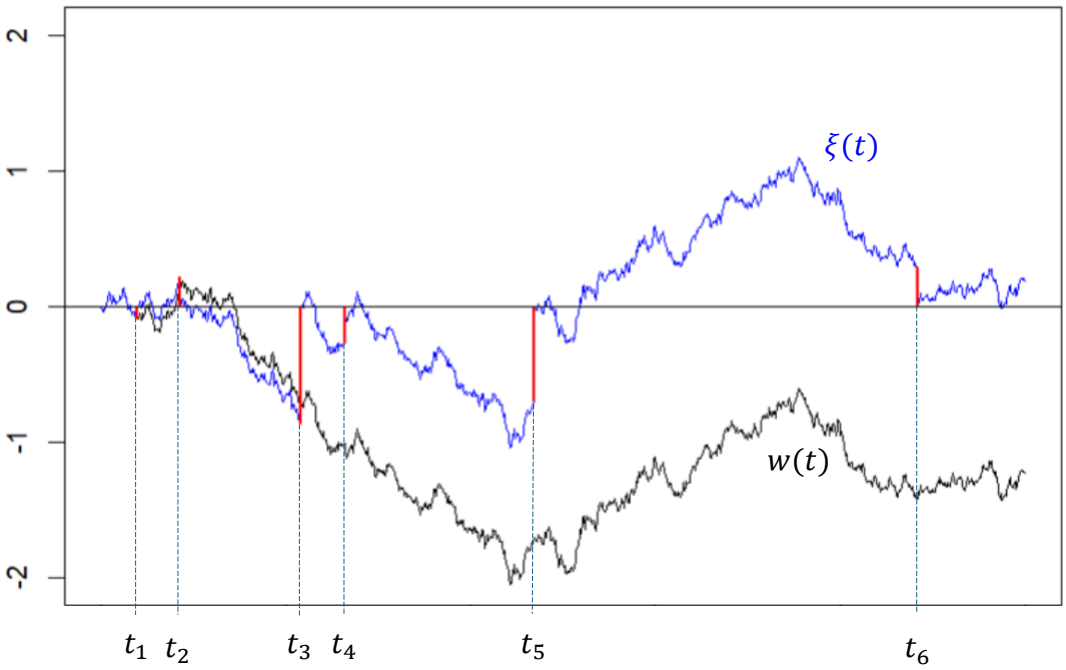

FIGURE 2.1. The black curve represents the original trajectory of the Wiener process w on the non-negative semi-axis $[0,\infty)$; the blue curve shows the trajectory of the process ξ, with red vertical segments indicating the resetting jumps.

fails, and the standard large deviation framework of Feng and Kurtz does not directly apply. This motivates a more tailored, hands-on approach to establishing the large deviation principle in this setting.

The paper is organized as follows. In the next section, Section 2, we recall some definitions and state the main results, Theorem 2.4 and Theorem 2.6. The proofs of these theorems are given in Section 3. Finally, in Section 4, we collect some auxiliary lemmas.

2. Definitions and main results

Let $(\Omega,\mathfrak{F},\mathbf{P})$ be a probability space supporting a Wiener process $w(t)$ and a Poisson process $\nu(t)$ independent of it, with rate λ, $\mathbf{E}\nu(1)=\lambda>0$.

The Wiener process with resetting is constructed as a stochastic process defined recursively as follows:

$$\xi(t) := w(t) - \sum_{k=1}^{\nu(t)} \xi(t_k-),$$

where $t_1, t_2, \dots, t_k, \dots$ are the jump times of the Poisson process ν; here and throughout, we assume that $\sum_{k=1}^{0}=0$ and $t_0:=0$. Observe that the stochastic process ξ can be written as

$$\xi(t) = w(t) - w(t_{\nu(t)}), \quad t\ge 0. \tag{2.1}$$

See Figure 2.1 for the illustration of two representations of the process ξ.

We are interested in the large deviation behavior of the sequence

$$\xi_n(t) := \frac{\xi(nt)}{n} = w_n(t) - \frac{1}{n}\sum_{k=1}^{\nu(nt)} \xi(t_k-), \quad t\in[0,1],$$

where $w_n(t) := \frac{w(nt)}{n}$. Representation (2.1) yields the following expression for the stochastic process ξ_n:

$$\xi_n(t) = w_n(t) - \frac{1}{n} w(t_{\nu(nt)}) = w_n(t) - w_n\Big(\frac{t_{\nu(nt)}}{n}\Big), \quad t \in [0,1]. \tag{2.2}$$

We consider the trajectories of the sequence of stochastic processes ξ_n in the metric space $(\mathbb{L}[0,1], \rho_L)$, where

$$\mathbb{L}[0,1] := \Big\{ f : \int_0^1 |f(t)|\, dt < \infty \Big\}, \quad \rho_L(f,g) := \int_0^1 |f(t) - g(t)|\, dt, \quad f, g \in \mathbb{L}[0,1].$$

We will also sometimes use the uniform metric as an auxiliary metric:

$$\rho(f,g) := \sup_{t \in [0,1]} |f(t) - g(t)|.$$

For a set $A \subseteq \mathbb{L}[0,1]$, we denote

$$\rho_L(f,A) := \inf_{g \in A} \rho_L(f,g), \qquad \rho(f,A) := \inf_{g \in A} \rho(f,g).$$

Recall that in the metric space $(\mathbb{L}[0,1], \rho_L)$, functions that differ only on a set of Lebesgue measure zero (i.e., elements of the same equivalence class) are identified as a single element: the distance between any such functions is zero. Accordingly, throughout the paper we work with arbitrary representatives of the corresponding equivalence classes, without specifying them explicitly.

For instance, when we say that a function $f \in \mathbb{L}[0,1]$ is continuous, we mean that there exists a continuous function $\hat{f}$ such that $\rho_L(f, \hat{f}) = 0$. Similarly, the statement that a function f has bounded total variation on $[0,1]$ is understood as the existence of a function $\hat{f}$ with bounded total variation such that $\rho_L(f, \hat{f}) = 0$, and analogous conventions will be used throughout.

For the reader's convenience, we recall the necessary definitions.

Definition 2.1. A family of random elements ξ_n satisfies the local large deviations principle (LLDP) in the metric space $(\mathbb{X}, \rho_X)$ with a rate function $I = I(x) : \mathbb{X} \to [0, \infty]$ and a normalizing function $\psi(n)$ such that $\lim\limits_{n\to\infty} \psi(n) = \infty$, if the following equality holds for any $x \in \mathbb{X}$:

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\rho_X(\xi_n, x) < \varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\rho_X(\xi_n, x) < \varepsilon) = -I(x).$$

Definition 2.2. A family of random variables ξ_n is said to be exponentially tight (ET) in the metric space $(\mathbb{X}, \rho_X)$ if, for any $N > 0$, there exists a compact set $K_N \subset \mathbb{X}$ such that

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}\big(\xi_n \in (K_N)^c\big) \le -N,$$

where $(K_N)^c$ denotes the complement of the set K_N.

We denote the closure and interior of the set B by $[B]$ and (B), respectively.

Definition 2.3. A family of random variables ξ_n satisfies the LDP on the metric space $(\mathbb{X}, \rho_X)$ with a *good* rate function $I = I(f) : \mathbb{X} \to [0, \infty]$ and normalizing function $\psi(n)$, with $\lim\limits_{n\to\infty} \psi(n) = \infty$, if for any $c \ge 0$, the set $\{x \in \mathbb{X} : I(x) \le c\}$ is compact, and for every Borel set $B \in \mathfrak{B}(\mathbb{X})$ the following inequalities hold:

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\xi_n \in B) \le -I([B]),$$

$$\liminf_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\xi_n \in B) \ge -I((B)),$$

where $\mathfrak{B}(\mathbb{X})$ is the Borel σ-algebra on $\mathbb{X}$, $I(B) = \inf\limits_{x \in B} I(x)$, and $I(\emptyset) = \infty$.

In what follows, we will also use the following notation:

- $\mathbb{D}[0,1]$ is the space of càdlàg functions (i.e., right-continuous with left limits on the interval $[0,1]$);
- $\mathbb{C}[0,1]$ is the space of continuous functions on $[0,1]$;
- $\mathbb{AC}_0[0,1]$ is the space of absolutely continuous functions on $[0,1]$ satisfying $f(0)=0$;
- $\widetilde{\mathbb{AC}}_0[0,1]$ is the set of functions $f \in \mathbb{D}[0,1]$ satisfying:
  (1) $f(0)=0$;
  (2) If t^* is a point of discontinuity of f, then $f(t^*)=0$;
  (3) If $t^* < t^{**}$ are two consecutive points of discontinuity (i.e., there are no other discontinuities between them), then f is absolutely continuous on the interval $[t^*, t^{**})$;
- $\mathbb{V}[0,1]$ is the space functions with a finite variation.

For a function $f \in \mathbb{L}[0,1]$, define the set of points where it is nonzero:

$$M_f^{\pm} := \{t \in [0,1] : f(t) \neq 0\}.$$

The following two theorems, which establish LDPs for ξ_n and for $\sup_{t\in[0,1]} |\xi_n(t)|$, are the main results of this work.

Theorem 2.4. *The sequence ξ_n satisfies the LDP on the metric space $(\mathbb{L}[0,1], \rho_L)$ with normalizing function $\psi(n)=n$ and rate function*

$$I(f) := \begin{cases} \lambda \mathfrak{m}(M_f^{\pm}) + \frac{1}{2}\int_{M_f^{\pm}} \dot{f}^2(t)\,dt, & \text{if } f \in \widetilde{\mathbb{AC}}_0[0,1], \\ \infty, & \text{if } f \notin \widetilde{\mathbb{AC}}_0[0,1], \end{cases} \tag{2.3}$$

where $\mathfrak{m}$ denotes the Lebesgue measure and $\dot{f}$ is the derivative of f in the sense of absolute continuity.

Remark 2.5. Note that the result of Theorem 2.4 remains valid if we replace the metric space $(\mathbb{L}[0,1], \rho_L)$ by $(\mathbb{L}_p[0,1], \rho_{L_p})$, where $p \in (1,\infty)$,

$$\mathbb{L}_p[0,1] := \Big\{ f : \int_0^1 |f(t)|^p dt < \infty \Big\}, \quad \rho_{L_p}(f,g) := \Big(\int_0^1 |f(t)-g(t)|^p dt \Big)^{1/p}, \quad f,g \in \mathbb{L}_p[0,1].$$

In addition, we establish a large deviation principle (LDP) for the maximum absolute value of the process $\xi_n(t)$ over the interval $[0,1]$, denoted as $\sup_{t\in[0,1]} |\xi_n(t)|$.

Before formulating Theorem 2.6, we would like to mention the paper Hartmann et al. (2024), in which the authors study large deviations of the maximum of Brownian motion with stochastic resetting in the presence of a fixed target. In the limit as the target tends to infinity, the corresponding rate function (see formula (15.33) on p. 368 of Hartmann et al., 2024) coincides with the one established in Theorem 2.6; see (2.4). Their analysis is based on the asymptotic behavior of the target survival probability (or, equivalently, the distribution of the maximum) and relies on a steepest-descent analysis of an exact Bromwich integral representation for the survival probability. Their primary objective is to derive the asymptotic behavior of the distribution rather than to establish a large deviation principle in the probabilistic sense. By contrast, our probabilistic approach establishes a full large deviation principle.

We also recall the dynamical phase-transition interpretation discussed in Hartmann et al. (2024). More generally, as emphasized in Harris and Touchette (2017), dynamical phase transitions in reset processes may arise from a competition between two mechanisms for producing a rare fluctuation: one involving resetting during a positive fraction of the observation time and another involving a macroscopically long reset-free trajectory. In the present setting, this competition can be described explicitly. If $s \in (0,1]$ denotes the duration of the final reset-free excursion, the cost of reaching the level x during this excursion is $\lambda s + \frac{x^2}{2s}$. The first term is the cost of suppressing resets over an interval of length s, whereas the second is the Wiener cost of reaching x during that interval. The unconstrained minimizer is $s_x = x/\sqrt{2\lambda}$, which gives $t_x = 1 - s_x = 1 - x/\sqrt{2\lambda}$, see Figure 2.2.

Consequently, if $x < \sqrt{2\lambda}$, the optimal trajectory undergoes resetting up to the macroscopic time t_x and then follows a linear trajectory without resetting, with slope $\sqrt{2\lambda}$. Substituting the optimal value $s_x = x/\sqrt{2\lambda}$ into the cost yields $\lambda s_x + x^2/(2s_x) = x\sqrt{2\lambda}$, which explains why the rate function is linear in x in this regime. If $x > \sqrt{2\lambda}$, the unconstrained minimizer would satisfy $s_x > 1$ and is therefore inadmissible. The minimum is then attained at the boundary $s = 1$: the optimal trajectory has no resets over the entire observation interval, and the quadratic Wiener cost, together with the cost of suppressing all resets, is recovered.

At the critical value $x = \sqrt{2\lambda}$, the macroscopic fraction of time during which resetting occurs vanishes continuously. Correspondingly, the first derivative of the rate function in (2.4) remains continuous, whereas its second derivative is discontinuous. This change in the optimal fluctuation mechanism is interpreted as a continuous, or second-order, dynamical phase transition.

Theorem 2.6. *For any $x \geq 0$, the following equality holds:*

$$I_{\sup}(x) := \lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\sup_{t\in[0,1]} |\xi_n(t)| \geq x \Big) = \begin{cases} \infty, & \text{if } x < 0, \\ \sqrt{2\lambda}x, & \text{if } x \in [0, \sqrt{2\lambda}], \\ \lambda + \frac{x^2}{2}, & \text{if } x > \sqrt{2\lambda}. \end{cases} \tag{2.4}$$

The proof of this theorem is based on the optimization of the rate function established by Theorem 2.4, which provides the optimal trajectories, as shown in Figure 2.2 (A). The trajectories belong to the special class of monotone trajectories (3.28). The rate function on the phase space $\mathbb{R}_+$ is represented in Figure 2.2 (B), for the case $\lambda = 1$. We have presented the graph in a slightly unconventional manner, placing the domain of the function along the vertical axis (ordinate) and the values of the rate function along the horizontal axis (abscissa). This choice allows us to connect two figures: (A) the graph of optimal trajectories reaching a given level x (now represented along the horizontal axis) in the phase space, and (B) the corresponding value of the rate function. We hope that this alternative presentation will not cause discomfort to the reader and will serve as a meaningful illustration of the result obtained.

3. Proofs of the Main Results

3.1. *Proof of Theorem 2.4.* The proof of Theorem 2.4 follows the standard implication (see, for example, Dembo and Zeitouni, 1998, Lemma 4.1.23)

$$\text{ET and LLDP} \Rightarrow \text{LDP}.$$

In the next section, we establish exponential tightness. As usual, the proof of the local large deviation principle (LLDP) is then divided into the upper and lower bound estimates.

As the process ξ_n can be regarded as a simple (random) transformation of the Wiener process w_n, see (2.2), the rate function corresponding to the Wiener process naturally appears as a key element in the proofs. The following proposition characterizes the rate function, extending the result to the space $(\mathbb{L}[0,1], \rho_L)$.

Proposition 3.1. *The sequence of processes w_n satisfies the Large Deviation Principle (LDP) in the spaces $(\mathbb{C}[0,1], \rho)$ and $(\mathbb{L}[0,1], \rho_L)$ with the normalization function $\psi(n) = n$ and the "good" rate function $\hat{I}$*

$$\hat{I}(f) := \begin{cases} \frac{1}{2}\int_0^1 \dot{f}^2(t)dt, & \text{if } f \in \mathbb{AC}_0[0,1], \\ \infty, & \text{if } f \notin \mathbb{AC}_0[0,1]. \end{cases} \tag{3.1}$$

The proof of the proposition follows from Freidlin and Wentzell (2012, Chapter 3, Theorems 2.1, 2.2) and auxiliary Lemma 4.1 in Section 4.

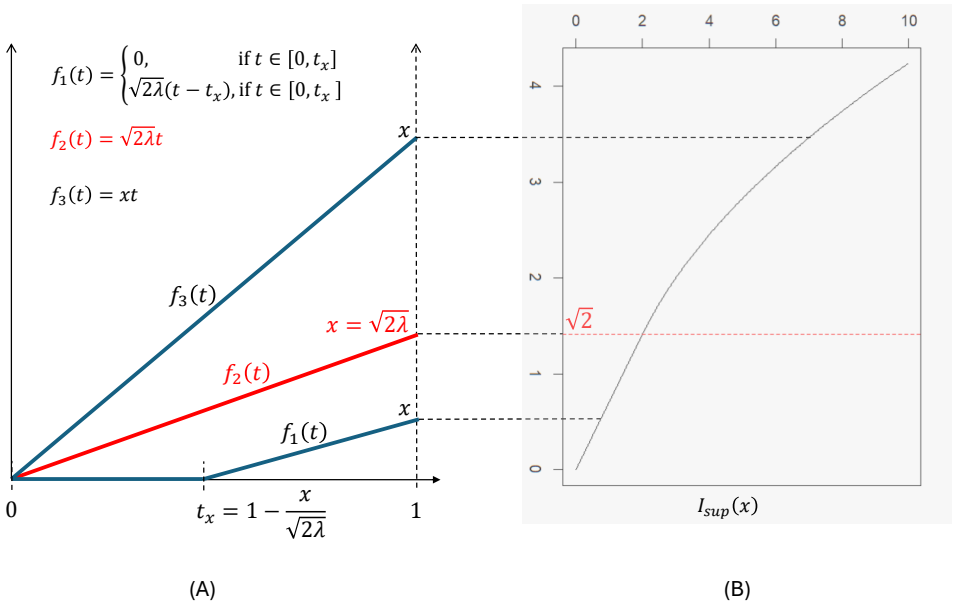

FIGURE 2.2. **Figure (A)** shows the optimal trajectories that bring the supremum of the process ξ_n to the level x during the time interval $[0, 1]$. **Figure (B)** shows the rate function (2.4) (with values on the horizontal axis) corresponding to each level x in the phase space (vertical axis), for $\lambda = 1$.

Remark 3.2. We note that, in particular, the "good" rate function of the sequence w_n satisfies the Exponential Tightness (ET) property in both spaces $(\mathbb{C}[0,1], \rho)$ and $(\mathbb{L}[0,1], \rho_L)$ (see, for example, Feng and Kurtz, 2006, Lemma 3.5).

Let us start by establishing exponential tightness.

Exponential tightness. Here we prove that the sequence ξ_n is exponentially tight (ET) in the space $(\mathbb{L}[0,1], \rho_L)$.

For any positive real number $N > 0$, let us define the following two sets

$$C_N := \{f \in \mathbb{L}[0,1] : \hat{I}(f) \le N\}, \quad C'_N := \{f \in \mathbb{L}[0,1] : \mathrm{Var}_{[0,1]} f \le 2\sqrt{2N},\ f(0) = 0\},$$

where $\mathrm{Var}_{[0,1]} f$ denotes the total variation of the function f on the interval $[0,1]$, and $\hat{I}$ is the rate function (3.1).

Note that the space $(\mathbb{L}[0,1], \rho_L)$ is complete and separable, and any probability measure on it is tight. In this setting, where a sequence of tight measures is considered, an equivalent condition for exponential tightness is known; see Feng and Kurtz (2006, Lemma 3.3). According to this lemma, we construct a suitable expanded event to demonstrate exponential tightness by taking the ε-neighborhood of the set C_N in the space $(\mathbb{L}[0,1], \rho_L)$.

Observe that the set C_N contains only the absolutely continuous functions, $C_N \subset \mathbb{AC}_0[0,1]$, and the set C'_N contains the functions with bounded variation, $C'_N \subset \mathbb{V}[0,1]$. Moreover, both sets are compact. Indeed, since $\hat{I}$ is a "good" rate function according to Proposition 3.1, then the set C_N is a compact set. The compactness of the set C'_N in $(\mathbb{L}[0,1], \rho_L)$ follows from Logachov et al. (2024b, Lemma A.1), which states that the set of functions f with bounded variation and for some $c \ge 0$ satisfying the condition $f(0) \in [0, c]$ is compact in $(\mathbb{L}[0,1], \rho_L)$.

For any function $f \in C_N$, using the Cauchy–Schwarz inequality, we obtain

$$\mathrm{Var}_{[0,1]} f = \int_0^1 |\dot{f}(t)| dt \le \sqrt{2\hat{I}(f)} \le \sqrt{2N}. \tag{3.2}$$

Therefore, for any $\varepsilon > 0$, the inclusion $C_N \subset C'_N$ holds.

If a trajectory of the Wiener process w_n remains uniformly within distance ε of a function f, then the corresponding trajectory of ξ_n, obtained from w_n via the transformation (2.2), remains uniformly within distance 2ε of the function

$$\widehat{f}(t) := f(t) - f\Big(\frac{t_{\nu(nt)}}{n}\Big), \qquad t \in [0,1].$$

Indeed, by the triangle inequality,

$$\begin{aligned}\sup_{t\in[0,1]} \big|\xi_n(t) - \widehat{f}(t)\big| &= \sup_{t\in[0,1]} \Big| w_n(t) - w_n\Big(\frac{t_{\nu(nt)}}{n}\Big) - \Big(f(t) - f\Big(\frac{t_{\nu(nt)}}{n}\Big)\Big)\Big| \\ &\le \sup_{t\in[0,1]} \big|w_n(t) - f(t)\big| + \sup_{t\in[0,1]} \Big| w_n\Big(\frac{t_{\nu(nt)}}{n}\Big) - f\Big(\frac{t_{\nu(nt)}}{n}\Big)\Big| \le 2\rho(w_n, f).\end{aligned}$$

and consequently

$$\{\omega : \rho(w_n, f) < \varepsilon\} \subseteq \{\omega : \rho(\xi_n, \hat{f}) < 2\varepsilon\}. \tag{3.3}$$

Here and further, ω is an element of the set Ω, and the inclusions and equalities of events are understood up to events of zero probability.

It is easy to see that

$$f(0) = \hat{f}(0) \text{ a.s.} \quad \text{and} \quad \mathrm{Var}_{[0,1]} \hat{f} \le 2\mathrm{Var}_{[0,1]} f \text{ a.s.}$$

Therefore, from formulas (3.2), (3.3) it follows that for any $\varepsilon > 0$

$$\big\{\omega : w_n \in C_N^{(\varepsilon)}\big\} \subseteq \big\{\omega : \xi_n \in C_N^{\prime(\varepsilon)}\big\}, \tag{3.4}$$

where

$$C_N^{(\varepsilon)} := \big\{f \in \mathbb{L}[0,1] : \rho(f, C_N) < \varepsilon\big\}, \quad C_N^{\prime(\varepsilon)} := \big\{f \in \mathbb{L}[0,1] : \rho(f, C'_N) < 2\varepsilon\big\}.$$

Since $\rho(f,g) \ge \rho_L(f,g)$, we have

$$C_N^{\prime(\varepsilon)} \subseteq C_{N,L}^{\prime(\varepsilon)} := \big\{f \in \mathbb{L}[0,1] : \rho_L(f, C'_N) < 2\varepsilon\big\}. \tag{3.5}$$

Combining formulas (3.4) and (3.5), the large deviation principle for w_n in the metric space $(\mathbb{C}[0,1], \rho)$ (see Proposition 3.1), the almost sure continuity of the trajectories of the process w_n, and the fact that the set $C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]$ is open in $(\mathbb{C}[0,1], \rho)$ for any $\varepsilon > 0$ and $N > 0$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\xi_n \in \big(C_{N,L}^{\prime(\varepsilon)}\big)^c\Big) &\le \limsup_{n\to\infty} \frac{1}{n} \ln\Big(1 - \mathbf{P}\Big(\xi_n \in C_N^{\prime(\varepsilon)}\Big)\Big) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \ln\Big(1 - \mathbf{P}\Big(w_n \in C_N^{(\varepsilon)}\Big)\Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} \ln\Big(1 - \mathbf{P}\Big(w_n \in C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]\Big)\Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(w_n \in \big(C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]\big)^c_{(\mathbb{C}[0,1],\rho)}\Big) \\ &\le - \inf_{f \in \big(C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]\big)^c_{(\mathbb{C}[0,1],\rho)}} \hat{I}(f) \le -N,\end{aligned} \tag{3.6}$$

where $\big(C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]\big)^c_{(\mathbb{C}[0,1],\rho)}$ denotes the complement of the set $C_N^{(\varepsilon)} \cap \mathbb{C}[0,1]$ in the space $(\mathbb{C}[0,1], \rho)$.

The exponential tightness of the sequence ξ_n follows from (3.6) and Feng and Kurtz (2006, Lemma 3.3).

Now, let us prove the LLDP for the sequence ξ_n.

Upper bound. We start by establishing the upper bound in LLDP:

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(\rho(\xi_n, f) < \varepsilon) \le -I(f). \tag{3.7}$$

Observe, by the definition of the rate function (2.3), that $I(f)$ takes values different from ∞ only on functions from the set $\widetilde{\mathbb{AC}}_0[0,1]$. Recall that the definition of this set consists of three conditions; see the Introduction section. Indeed, if the first condition is not satisfied, then $I(f) = \infty$ because $\xi_n(0) = 0$ and the sequence w_n is exponentially tight in the space $(\mathbb{C}[0,1], \rho)$ (see Remark 3.2). If the second condition is not satisfied, then $I(f) = \infty$ because, by definition, the random process ξ_n takes the value zero at all points of discontinuity. If the third condition is not satisfied, then $I(f) = \infty$ due to the large deviation principle for w_n (see Proposition 3.1) and the fact that the trajectory of ξ_n between neighboring discontinuities coincides with a shifted trajectory of the process w_n.

Now consider $f \in \widetilde{\mathbb{AC}}_0[0,1]$. If $f \equiv 0$, the inequality (3.7) holds trivially, since $I(f) = 0$. Therefore, we assume that the Lebesgue measure of the set of points where f does not equal zero is non-zero. For a such given f and $\delta > 0$ let $s_0 := 0$ and

$$\begin{aligned} s_{2r-1} &:= 1 \wedge \inf\bigl\{t \in [s_{2r-2}, 1] : |f(t)| = 2\delta\bigr\}, \\ s_{2r} &:= 1 \wedge \inf\bigl\{t \in [s_{2r-1}, 1] : |f(t)| \le \delta\bigr\}, \quad r \in \mathbb{N}. \end{aligned}$$

Let $m_\delta := \min\{l : s_l = 1\}$. Note that $m_\delta < \infty$ since $f \in \widetilde{\mathbb{AC}}_0[0,1]$. For definiteness, let us consider the case where m_δ is odd. It is easy to see that by construction, the function $|f|$ is continuous and its values are no less than δ on each interval $[s_{2r-1}, s_{2r})$, for $1 \le r \le \frac{m_\delta - 1}{2}$. Note also that any zeros of the function f must lie in the intervals $[s_{2r}, s_{2r+1})$.

For $1 \le r \le \frac{m_\delta - 1}{2}$ define the events

$$\begin{aligned} A_r &:= \Bigl\{\omega : \int_{s_{2r-1}}^{s_{2r}} |\xi_n(t) - f(t)|dt < \varepsilon\Bigr\}, \\ B_r &:= \Bigl\{\omega : \nu(ns_{2r}-) - \nu(ns_{2r-1}) = 0\Bigr\}, \\ C_r &:= \Bigl\{\omega : \int_{s_{2r-1}}^{s_{2r}} \bigl|w_n(t) - \frac{1}{n} w(t_{\nu(ns_{2r-1})}) - f(t)\bigr|dt \le \varepsilon\Bigr\}. \end{aligned}$$

It is easy to see that for any $\delta > 0$ on the event A_r, $1 \le r \le \frac{m_\delta - 1}{2}$, for sufficiently small $\varepsilon > 0$ and $t \in [s_{2r-1}, s_{2r})$ the equalities

$$\begin{aligned} &\xi_n(t) = w_n(t) - \frac{1}{n} \sum_{k=1}^{\nu(ns_{2r-1})} \xi(t_k-) = w_n(t) - \frac{1}{n} w(t_{\nu(ns_{2r-1})}), \\ &\nu(nt) - \nu(ns_{2r-1}) = 0, \end{aligned} \tag{3.8}$$

hold, i.e. on the event A_r, for sufficiently small $\varepsilon > 0$, the process $\xi_n(t)$ is continuous on the interval $[s_{2r-1}, s_{2r})$. In other words, the event A_r implies both C_r and B_r, separating the actions of w_n and ν.

Therefore, for any $\delta > 0$, $N > 0$ and sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{P}(\rho_L(\xi_n, f) < \varepsilon) &\le \mathbf{P}\Bigl(w_n \in K_N, \bigcap_{r=1}^{(m_\delta-1)/2} A_r\Bigr) + \mathbf{P}\bigl(w_n \in (K_N)^c\bigr) \\ &\le \mathbf{P}\Bigl(w_n \in K_N, \bigcap_{r=1}^{(m_\delta-1)/2} C_r, \bigcap_{r=1}^{(m_\delta-1)/2} B_r\Bigr) + \mathbf{P}\bigl(w_n \in (K_N)^c\bigr), \end{aligned} \tag{3.9}$$

where K_N is compact in metric space $(\mathbb{C}[0,1],\rho)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\mathbf{P}\big(w_n\in(K_N)^c\big)\le -N. \tag{3.10}$$

(The existence of such a compact set follows from Remark 3.2). Since w_n and ν are independent, we aim to express the event as the intersection of two independent events, each determined by the behavior of w_n (events C_r) and ν (events B_r), respectively.

Let us refine the intersection of events C_r with K_N. Define the following closed set

$$L_{\delta,\varepsilon}:=\bigcap_{r=1}^{(m_\delta-1)/2}\big\{g\in\mathbb{L}[0,1]:\inf_{-c_N\le a_r\le c_N}\int_{s_{2r-1}}^{s_{2r}}|g(t)-a_r-f(t)|dt\le\varepsilon\big\},$$

$c_N:=\sup\limits_{h\in K_N}\sup\limits_{t\in[0,1]}|h(t)|$.

Combining formula (3.9) with the independence of the processes w_n and ν, as well as the independence of increments of ν, we obtain that for any $\delta>0$, $N>0$, and sufficiently small $\varepsilon>0$,

$$\begin{aligned}\mathbf{P}(\rho_L(\xi_n,f)<\varepsilon)&\le\mathbf{P}\Big(w_n\in L_{\delta,\varepsilon},\bigcap_{r=1}^{(m_\delta-1)/2}B_r\Big)+\mathbf{P}\big(w_n\in(K_N)^c\big)\\&=\mathbf{P}\big(w_n\in L_{\delta,\varepsilon}\big)\mathbf{P}\Big(\bigcap_{r=1}^{(m_\delta-1)/2}B_r\Big)+\mathbf{P}\big(w_n\in(K_N)^c\big)\\&=\mathbf{P}\big(w_n\in L_{\delta,\varepsilon}\big)\exp\Big\{-\lambda n\sum_{r=1}^{(m_\delta-1)/2}(s_{2r}-s_{2r-1})\Big\}+\mathbf{P}\big(w_n\in(K_N)^c\big).\end{aligned} \tag{3.11}$$

Using the fact that the rate functional $\hat I$ is lower semicontinuous, we have

$$\lim_{\varepsilon\to0}\hat I(L_{\delta,\varepsilon})\ge\hat I(L_{\delta,0}),$$

where, by definition of $L_{\delta,\varepsilon}$

$$\begin{aligned}L_{\delta,0}&=\bigcap_{r=1}^{(m_\delta-1)/2}\Big\{g\in\mathbb{L}[0,1]:\inf_{-c_N\le a_r\le c_N}\int_{s_{2r-1}}^{s_{2r}}|g(t)-a_r-f(t)|dt=0\Big\}\\&=\bigcap_{r=1}^{(m_\delta-1)/2}\Big\{g\in\mathbb{L}[0,1]:\ \text{exists } a_r\in[-c_N,c_N]\text{ s.t. }g(t)-a_r=f(t)\text{ a.e. in }[s_{2r-1},s_{2r}]\Big\}\\&\subseteq\bigcap_{r=1}^{(m_\delta-1)/2}\Big\{g\in\mathbb{L}[0,1]:\ \dot g(t)=\dot f(t)\text{ a.e. in }[s_{2r-1},s_{2r}]\Big\}.\end{aligned}$$

Therefore, using Proposition 3.1 we obtain,

$$\begin{aligned}\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\ln\mathbf{P}\big(w_n\in L_{\delta,\varepsilon}\big)&\le-\lim_{\varepsilon\to0}\hat I(L_{\delta,\varepsilon})\\&\le-\hat I(L_{\delta,0})=-\frac12\sum_{r=1}^{(m_\delta-1)/2}\int_{s_{2r-1}}^{s_{2r}}\dot f^2(t)dt.\end{aligned} \tag{3.12}$$

For all $\delta > 0$ and $N > 0$, inequalities (3.10), (3.11), and (3.12) yield

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\ln \mathbf{P}(\rho_L(\xi_n, f) < \varepsilon)$$
$$\leq -\min\Bigg(\lambda\sum_{r=1}^{(m_\delta-1)/2}(s_{2r}-s_{2r-1}) + \frac{1}{2}\sum_{r=1}^{(m_\delta-1)/2}\int_{s_{2r-1}}^{s_{2r}}\dot{f}^2(t)dt, N\Bigg).$$

Taking the limits $N\to\infty$ and then $\delta\to 0$, we obtain the inequality

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\ln \mathbf{P}(\rho_L(\xi_n, f) < \varepsilon)$$
$$\leq -\lambda\mathfrak{m}(\{t\in[0,1] : f(t)\neq 0\}) - \frac{1}{2}\int_0^1 \dot{f}^2(t)\mathbf{I}(f(t)\neq 0)dt,$$

where $\mathbf{I}(A)$ denotes the indicator function of the set A. The proof of inequality (3.9) for odd values of m_δ is complete. The even case is analogous and therefore omitted.

Lower bound. Now we prove

$$\liminf_{n\to\infty}\frac{1}{n}\ln \mathbf{P}(\rho_L(\xi_n, f) < \varepsilon) \geq -I(f). \tag{3.13}$$

If $I(f) = \infty$, then inequality (3.13) holds trivially. Therefore, we will henceforth assume that $I(f) < \infty$, and consequently $f \in \widetilde{\mathbb{AC}}_0[0,1]$. Given such a function f, and following the same idea as before to construct the sequence of points, we set $s'_0 := 0$, and for $\varepsilon > 0$ we define

$$s'_{2r-1} := 1\wedge\inf\Big\{t\in[s'_{2r-2},1] : |f(t)| = \frac{\varepsilon}{3}\Big\},$$
$$s'_{2r} := 1\wedge\inf\Big\{t\in[s'_{2r-1},1] : |f(t)| \leq \frac{\varepsilon}{4}\Big\},\quad r\in\mathbb{N}.$$

Let $m_\varepsilon := \min\{l : s'_l = 1\}$. Recall that $m_\varepsilon < \infty$ since $f\in\widetilde{\mathbb{AC}}_0[0,1]$. Consider the case when m_ε is odd. To prove the lower bound, we define "more restrictive" events on the previously specified intervals, in the same spirit: to separate the contributions of the Poisson and Wiener processes, thereby revealing the rate function of the Wiener process.

For $1\leq r\leq \frac{m_\varepsilon+1}{2}$ we define the following events

$$F_r := \Big\{\omega : \sup_{t\in[s'_{2r-2},s'_{2r-1}]}\big|(\nu(nt)-\nu(ns'_{2r-2})) - \lambda n(t-s'_{2r-2})\big| < \frac{n\varepsilon^2}{m_\varepsilon}\Big\},$$
$$G_r := \Big\{\omega : \sup_{t\in[s'_{2r-2},s'_{2r-1}]}\big|w_n(t)-w_n(s'_{2r-2})\big| < \frac{\varepsilon}{24}\Big\},$$

and for $1\leq r\leq \frac{m_\varepsilon-1}{2}$ we define

$$H_r := \Big\{\omega : \sup_{t\in[s'_{2r-1},s'_{2r})}\big|(w_n(t)-w_n(s'_{2r-1})) - (f(t)-f(s'_{2r-1}))\big| < \frac{\varepsilon}{8}\Big\},$$
$$Q_r := \Big\{\omega : \nu(ns'_{2r}-) - \nu(ns'_{2r-1}) = 0\Big\}.$$

Figure 3.3 illustrates the events defined above. Define

$$F := \bigcap_{r=1}^{(m_\varepsilon+1)/2} F_r,\quad G := \bigcap_{k=1}^{(m_\varepsilon+1)/2} G_r,\quad H := \bigcap_{k=1}^{(m_\varepsilon-1)/2} H_r,\quad Q := \bigcap_{k=1}^{(m_\varepsilon-1)/2} Q_r.$$

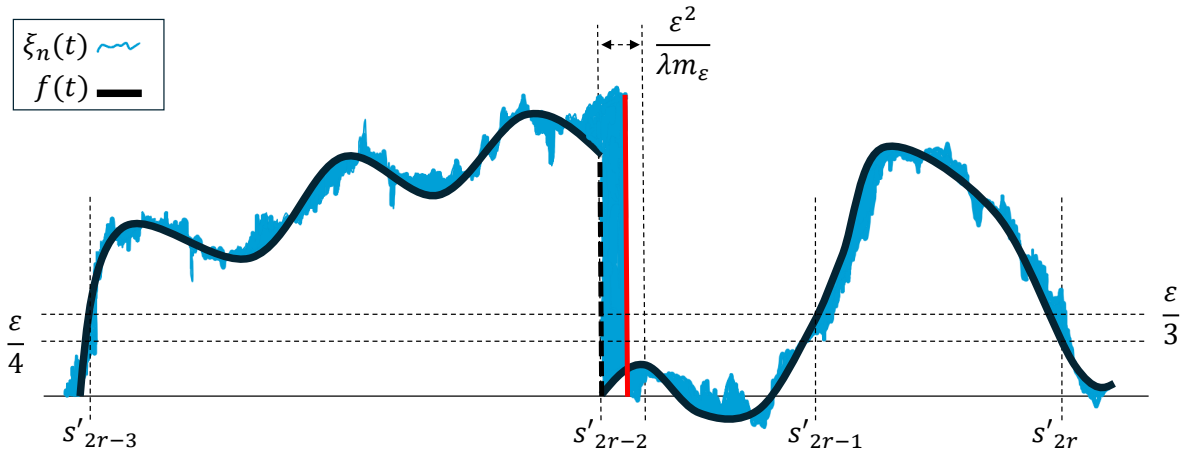

FIGURE 3.3. The figure shows the trajectory of the process ξ_n (in blue) and the function f (thick black line). The blue area represents the integral estimated in formulas (3.17) and (3.18). The red line corresponds to the resetting jump of the process ξ_n triggered by the event F_r.

In particular, the event F ensures the occurrence of at least one jump of the process $\nu(nt)$ in the right neighborhood (of length $\frac{\varepsilon^2}{\lambda m_\varepsilon}$) of the point s'_{2r-2}, for every r such that $1 \le r \le \frac{m_\varepsilon+1}{2}$. If the event G occurs, then for all $t, s \in [s'_{2r-2}, s'_{2r-1}]$, $1 \le r \le \frac{m_\varepsilon+1}{2}$,

$$\left|w_n(t) - w_n(s)\right| < \frac{\varepsilon}{12}. \tag{3.14}$$

Due to the (3.14), it follows that on the event $F \cap G \cap H \cap Q$, for any $\varepsilon > 0$ and $1 \le r \le \frac{m_\varepsilon-1}{2}$, the following inequality holds:

$$\begin{aligned}
\sup_{t\in[s'_{2r-1},s'_{2r})} \left|\xi_n(t) - f(t)\right| &= \sup_{t\in[s'_{2r-1},s'_{2r})} \left|w_n(t) - \frac{1}{n} w\big(t_{\nu(ns'_{2r-1})}\big) - f(t)\right| \\
&\le \left|w_n(s'_{2r-1}) - \frac{1}{n} w\big(t_{\nu(ns'_{2r-1})}\big)\right| \\
&\quad + \sup_{t\in[s'_{2r-1},s'_{2r})} \left|w_n(t) - w_n(s'_{2r-1}) - f(t) + f(s'_{2r-1})\right| + \left|f(s'_{2r-1})\right| \\
&\le \sup_{t,s\in[s'_{2r-2},s'_{2r-1}]} \left|w_n(t) - w_n(s)\right| \\
&\quad + \sup_{t\in[s'_{2r-1},s'_{2r})} \left|w_n(t) - w_n(s'_{2r-1}) - f(t) + f(s'_{2r-1})\right| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{12} + \frac{\varepsilon}{8} + \frac{\varepsilon}{3} = \frac{13\varepsilon}{24}.
\end{aligned} \tag{3.15}$$

From formulas (3.14) and (3.15) it follows that, on the event $F \cap G \cap H \cap Q$, for all $1 \le r \le \frac{m_\varepsilon+1}{2}$, the following inequality

$$\begin{aligned}
\sup_{t,s\in[s'_{2r-2},s^*_{2r-2})} \left|\xi_n(t)\right| &< \left|f(s'_{2r-2}-)\right| + \frac{13\varepsilon}{24} + \sup_{t\in[s'_{2r-2},s^*_{2r-2})} \left|w_n(t) - w_n(s'_{2r-2})\right| \\
&< \sup_{t\in[0,1]} |f(t)| + \frac{13\varepsilon}{24} + \frac{\varepsilon}{12} < \sup_{t\in[0,1]} |f(t)| + 1,
\end{aligned} \tag{3.16}$$

holds for sufficiently small $\varepsilon > 0$, where s^*_{2r-2} denotes the first jump time of the process $\nu(nt)$ within the interval $[s'_{2r-2}, s'_{2r-1})$.

Applying formulas (3.14) and (3.16), we obtain that on the event $F \cap G \cap H \cap Q$, for all sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{r=1}^{(m_\varepsilon+1)/2} \int_{s'_{2r-2}}^{s'_{2r-1}} |\xi_n(t) - f(t)|dt \\
&\qquad = \sum_{r=1}^{(m_\varepsilon+1)/2} \int_{s'_{2r-2}}^{s^*_{2r-2}} |\xi_n(t) - f(t)|dt + \sum_{r=1}^{(m_\varepsilon+1)/2} \int_{s^*_{2r-2}}^{s'_{2r-1}} |\xi_n(t) - f(t)|dt \\
&\qquad < \big(2 \sup_{t\in[0,1]} |f(t)| + 1\big) \cdot \frac{\varepsilon^2}{\lambda m_\varepsilon} \cdot \frac{m_\varepsilon + 1}{2} + \frac{5\varepsilon}{12} \sum_{r=1}^{(m_\varepsilon+1)/2} (s'_{2r-1} - s'_{2r-2}) \\
&\qquad < \frac{2 \sup_{t\in[0,1]} |f(t)| + 1}{\lambda} \varepsilon^2 + \frac{5\varepsilon}{12} \sum_{r=1}^{(m_\varepsilon+1)/2} (s'_{2r-1} - s'_{2r-2}) < \frac{\varepsilon}{24} + \frac{5\varepsilon}{12} = \frac{11\varepsilon}{24},
\end{aligned} \tag{3.17}$$

where the constant $5/12$ is obtained from the following rough estimate of the integrand:

$$\begin{aligned}
\int_{s^*_{2r-2}}^{s'_{2r-1}} |\xi_n(t) - f(t)|dt &\le \Big(\sup_{t\in[s^*_{2r-2}, s'_{2r-1}]} |\xi_n(t)| + \sup_{t\in[s^*_{2r-2}, s'_{2r-1}]} |f(t)| \Big)(s'_{2r-1} - s'_{2r-2}) \\
&\le \Big(\sup_{t,s\in[s'_{2r-2}, s'_{2r-1}]} |w_n(t) - w_n(s)| + \frac{\varepsilon}{3} \Big)(s'_{2r-1} - s'_{2r-2}) \\
&\le \Big(\frac{\varepsilon}{12} + \frac{\varepsilon}{3} \Big)(s'_{2r-1} - s'_{2r-2}) = \frac{5\varepsilon}{12}(s'_{2r-1} - s'_{2r-2}).
\end{aligned}$$

It follows from formula (3.15) that, on the event $F \cap G \cap H \cap Q$,

$$\sum_{r=1}^{(m_\varepsilon-1)/2} \int_{s'_{2r-1}}^{s'_{2r}} |\xi_n(t) - f(t)|dt < \sum_{r=1}^{(m_\varepsilon-1)/2} \int_{s'_{2r-1}}^{s'_{2r}} \frac{13\varepsilon}{24} dt \le \frac{13\varepsilon}{24}. \tag{3.18}$$

From formulas (3.17) and (3.18) it follows that for sufficiently small $\varepsilon > 0$, on the event $F \cap G \cap H \cap Q$, the inequality $\rho_L(\xi_n, f) < \varepsilon$ holds. Therefore, using the independence of the processes ν and w_n, as well as the independence of their increments, we obtain for any sufficiently small $\varepsilon > 0$:

$$\begin{aligned}
\mathbf{P}\big(\rho_L(\xi_n, f) < \varepsilon\big) &\ge \mathbf{P}\big(F, G, H, Q\big) = \mathbf{P}\big(F\big)\mathbf{P}\big(G\big)\mathbf{P}\big(H\big)\mathbf{P}\big(Q\big) \\
&= \mathbf{P}\big(F\big)\mathbf{P}\big(G\big)\mathbf{P}\big(H\big) \exp\Big\{ -\lambda n \sum_{r=1}^{(m_\varepsilon-1)/2} (s'_{2r} - s'_{2r-1}) \Big\}.
\end{aligned} \tag{3.19}$$

Denote $v := \lambda t$, $t \in [0, 1]$. Applying Lemma 4.2, we obtain for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(F\big) \ge \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\sup_{t\in[0,1]} |\nu_n(t) - \lambda t| < \frac{\varepsilon^2}{2m_\varepsilon} \Big) \ge -I_1(v) = 0. \tag{3.20}$$

According to Proposition 3.1, for any $\varepsilon > 0$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(G\big) \ge \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\rho(w_n, 0) < \frac{\varepsilon}{8} \Big) \ge -\hat{I}(0) = 0. \tag{3.21}$$

Consider the following open set of functions

$$C_f := \Big\{ g \in \mathbb{C}[0,1] : \sup_{t\in[s'_{2r-1}, s'_{2r})} \big| (g(t) - g(s'_{2r-1})) - f(t) \big| < \frac{\varepsilon}{8},\ 1 \le r \le \frac{m_\varepsilon - 1}{2} \Big\}.$$

Proposition 3.1 yields that for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(H\big) = \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(w_n \in C_f\big) \geq -\hat{I}(C_f) \geq -\frac{1}{2} \sum_{r=1}^{(m_\varepsilon-1)/2} \int_{s'_{2r-1}}^{s'_{2r}} \dot{f}^2(t)dt. \tag{3.22}$$

Utilizing (3.19)–(3.22), for a sufficiently small $\varepsilon > 0$ we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(\rho_L(\xi_n, f) < \varepsilon) \geq -\lambda \sum_{k=1}^{(m_\varepsilon-1)/2} (s'_{2k} - s'_{2k-1}) - \frac{1}{2} \sum_{k=1}^{(m_\varepsilon-1)/2} \int_{s'_{2k-1}}^{s'_{2k}} \dot{f}^2(t)dt.$$

Taking the limit as $\varepsilon \to 0$ completes the proof of the formula (3.13) when m_ε is odd. The proof for even m_ε is analogous, so we omit it.

Formulas (3.7) and (3.13) establish the LLDP for the sequence ξ_n, and exponential tightness completes the proof of Theorem 2.4. □

3.2. *Proof of Theorem 2.6.* The proof is divided into the upper and lower bounds. We begin with the proof of the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\sup_{t\in[0,1]} |\xi_n(t)| \geq x\Big) \leq -I_{\sup}(x). \tag{3.23}$$

Define the following set of functions

$$A_x := \big\{ f \in \mathbb{L}[0,1] : \operatorname*{ess\,sup}_{t\in[0,1]} |f(t)| \geq x \big\}.$$

Since the process ξ_n has sample paths in the space $\mathbb{D}[0,1]$ almost surely, its supremum coincides almost surely with its essential supremum, i.e.,

$$\big\{\omega : \sup_{t\in[0,1]} |\xi_n(t)| \geq x\big\} = \big\{\omega : \xi_n \in A_x\big\}.$$

Note that the upper bound in the LDP from Theorem 2.4 cannot be applied directly to the set A_x, since it is not closed in the metric space $(\mathbb{L}[0,1], \rho_L)$. However, any set consisting of functions with a maximum of at least x and uniformly continuous at their maximum points is closed. We denote this set by B_x. By the Ascoli–Arzelà theorem and the definition of the process ξ_n, for any compact set K'_N in the space $(\mathbb{C}[0,1], \rho)$ we have

$$\big\{\omega : \xi_n \in B_x\big\} \supseteq \big\{\omega : \xi_n \in A_x, w_n \in K'_N\big\}.$$

We assume that the compact set K'_N is chosen in such a way that inequality (3.10) holds for the probability of its complement. Using the notation introduced above, we obtain

$$\begin{aligned} \mathbf{P}\Big(\sup_{t\in[0,1]} |\xi_n(t)| \geq x\Big) &\leq \mathbf{P}\big(\xi_n \in A_x, w_n \in K'_N\big) + \mathbf{P}\big(w_n \in (K'_N)^c\big) \\ &\leq \mathbf{P}\big(\xi_n \in B_x\big) + \mathbf{P}\big(w_n \in (K'_N)^c\big). \end{aligned} \tag{3.24}$$

Inequality (3.24) together with Theorem 2.4 implies that for any $N > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\sup_{t\in[0,1]} |\xi_n(t)| \geq x\Big) \leq -\min(I(B_x), N) \leq -\min(I(A_x), N).$$

Taking the limit $N \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big(\sup_{t\in[0,1]} |\xi_n(t)| \geq x\Big) \leq -\inf_{f\in A_x} I(f).$$

From the definition of the functional I, it follows that the infimum is attained over functions in the set $A_x \cap \widetilde{\mathbb{AC}}_0[0,1]$.

It is easy to see that for $f \in A_x \cap \widetilde{\mathbb{AC}}_0[0,1]$, the following inequality holds:

$$x \le \sup_{t\in[0,1]} |f(t)| \le \int_{M_f^{\pm}} |\dot f(t)|dt.$$

Therefore, by the Cauchy–Schwarz inequality, we obtain

$$x^2 \le \mathfrak{m}(M_f^{\pm}) \int_{M_f^{\pm}} \dot f^2(t)dt. \tag{3.25}$$

The inequality (3.25) yields

$$\begin{aligned}\inf_{f\in A_x\cap\widetilde{\mathbb{AC}}_0[0,1]} I(f) &\ge \inf_{f\in A_x\cap\widetilde{\mathbb{AC}}_0[0,1]} \left(\lambda\mathfrak{m}(M_f^{\pm}) + \frac{1}{2}\int_{M_f^{\pm}} \dot f^2(t)dt\right)\\ &\ge \inf_{\mathfrak{m}(M_f^{\pm})\in[0,1]} \left(\lambda\mathfrak{m}(M_f^{\pm}) + \frac{x^2}{2\mathfrak{m}(M_f^{\pm})}\right).\end{aligned}$$

Applying standard methods of differential calculus completes the proof of formulas (3.23).

Now we prove the lower bound

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(\sup_{t\in[0,1]} |\xi_n(t)| \ge x\big) \ge -I_{\sup}(x). \tag{3.26}$$

Using Theorem 2.4 and the fact that the set

$$C_x := \big\{f \in \mathbb{L}[0,1] : \operatorname*{ess\,sup}_{t\in[0,1]} |f(t)| > x\big\}$$

is the open set in the metric space $(\mathbb{L}[0,1], \rho_L)$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(\sup_{t\in[0,1]} |\xi_n(t)| \ge x\big) \ge \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\big(\xi_n \in C_x\big) \ge -\inf_{f\in C_x} I(x). \tag{3.27}$$

It is easy to see that

$$\Big\{f \in \mathbb{L}[0,1] : f(t) = 0,\ t \in [0,s];\ f(t) = k(t-s),\ t \in (s,1],\ \text{where } k > \frac{x}{1-s}\Big\} \subset C_x. \tag{3.28}$$

Thus,

$$\inf_{f\in C_x} I(x) \le \inf_{s\in[0,1],\ k>\frac{x}{1-s}} \Big(\lambda\mathfrak{m}([s,1]) + \frac{k^2}{2}(1-s)\Big) = \inf_{s\in[0,1]} \Big(\lambda(1-s) + \frac{x^2}{2(1-s)}\Big). \tag{3.29}$$

Using equality (3.29) and standard methods of differential calculus, we obtain the equality

$$\inf_{f\in C_x} I(x) \le \inf_{s\in[0,1]} \Big(\lambda(1-s) + \frac{x^2}{2(1-s)}\Big) = I_{\sup}(x). \tag{3.30}$$

Formula (3.26) follows from equalities (3.27) and (3.30). It is easy to see that the proof of Theorem 2.6 mainly relies on equalities (3.23) and (3.26). □

4. Auxiliary Results

In this section, we prove some auxiliary lemmas we used to prove the theorems.

Lemma 4.1. *The sequence of random processes η_n satisfies an LDP in the space $(\mathbb{C}[0,1], \rho)$ with speed $\psi(n)$ and the "good" rate function $\tilde I(f)$. Then the sequence η_n also satisfies an LDP in the space $(\mathbb{L}[0,1], \rho_L)$ with the same speed and rate function.*

Proof: Consider the operator $\mathcal{A} : (\mathbb{C}[0,1],\rho) \to (\mathbb{L}[0,1],\rho_L)$, defined by the identity restriction $\mathcal{A}f = f$. Since $\rho(f,g) \geq \rho_L(f,g), \quad f,g \in \mathbb{C}[0,1]$, the operator $\mathcal{A}$ is continuous. Therefore, applying the contraction principle (see, e.g., Dembo and Zeitouni, 1998, Theorem 4.2.1), we obtain

$$I_L(f) = \inf_{g:\mathcal{A}g=f} \tilde{I}(g) = \tilde{I}(f),$$

where I_L is the rate function for the LDP of the sequence η_n in the space $(\mathbb{L}[0,1],\rho_L)$. □

Let $\nu_n(t) := \frac{\nu(nt)}{n}$, $t \in [0,1]$.

Lemma 4.2. *The sequence of random processes ν_n satisfies the LDP in the metric space $(\mathbb{D}[0,1],\rho)$ with normalized function $\psi(n) = n$ and "good" rate function*

$$I_1(f) := \begin{cases} \int_0^1 \left(\dot{f}(t) \ln\left(\frac{\dot{f}(t)}{\lambda}\right) - \dot{f}(t) + \lambda \right) dt, & \text{if } f \in \mathbb{AC}_0^{\uparrow}[0,1], \\ \infty, & \text{if } f \notin \mathbb{AC}_0^{\uparrow}[0,1], \end{cases}$$

Here, $\mathbb{AC}_0^{\uparrow}[0,1]$ denotes the set of monotone non-decreasing absolutely continuous functions defined on the interval $[0,1]$ such that $f(0) = 0$.

Proof: This follows from Mogulskii (1993) and, for example, Lemma 4.4 in Logachov et al. (2024a). □

Note that the LDP for the stochastic process ν_n in the space $(\mathbb{D}[0,1],\rho)$ is formulated for the sequence of probability measures $\mathbf{P}(\nu_n \in A)$, not on the Borel σ-algebra $\mathfrak{B}(\mathbb{D}[0,1])$ generated by all open subsets of the metric space $(\mathbb{D}[0,1],\rho)$, but rather on the σ-algebra generated by cylinder sets defined through finite-dimensional distributions. This stems from the fact that not all sets $A \in \mathfrak{B}(\mathbb{D}[0,1])$ are measurable (for more details, see Billingsley, 1999, Chapter 3, Section 15).

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