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q-invariant functions for some generalizations of the Ornstein-Uhlenbeck semigroup

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Abstract. We show that the multiplication operator associated to a fractional power of a Gamma random variable, with parameter $q > 0$, maps the convex cone of the 1-invariant functions for a self-similar semigroup into the convex cone of the q-invariant functions for the associated Ornstein-Uhlenbeck (for short OU) semigroup. We also describe the harmonic functions for some other generalizations of the OU semigroup. Among the various applications, we characterize, through their Laplace transforms, the laws of first passage times above and overshoot for certain two-sided α-stable OU processes and also for spectrally negative semi-stable OU processes. These Laplace transforms are expressed in terms of a new family of power series which includes the generalized Mittag-Leffler functions and generalizes the family of functions introduced by Patie (2007a).

1. Introduction and main results

Let $\mathrm{E} = \mathbb{R}, \mathbb{R}^+$ or $[0, \infty)$ and let X be the realization of $(P_t)_{t\geq 0}$, a Feller semigroup on E satisfying, for $\alpha > 0$, the α-self-similarity property, i.e. for any $c > 0$ and every $f \in \mathfrak{B}(\mathrm{E})$, the space of bounded Borelian functions on E, we have the following identity

$$P_{c^\alpha t} f(cx) = P_t \left(d_c f\right)(x), \quad x \in \mathrm{E}, \tag{1.1}$$

where d_c is the dilatation operator, i.e. $d_c f(x) = f(cx)$. We denote by $(\mathbb{P}_x)_{x\in \mathrm{E}}$ the family of probability measures of X which act on $D(\mathrm{E})$, the Skorohod space of càdlàg functions from $[0,\infty)$ to E, and by $(F_t^X)_{t\geq 0}$ its natural filtration.

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We also mention that, throughout the paper, $\mathbb{E}$ stands for a reference expectation operator. Moreover, $\mathbf{A}$ (resp. $\mathfrak{D}(\mathbf{A})$) stands for its infinitesimal generator (resp. its domain). We have in mind the following situations

(1) $\mathrm{E} = \mathbb{R}$ and X is an α-stable Lévy process.
(2) $\mathrm{E} = \mathbb{R}^+$ or $[0,\infty)$ and X is a $\frac{1}{\alpha}$-semi-stable processes in the terminology of Lamperti.

More precisely, let ξ be a Lévy process starting from $x \in \mathbb{R}$, Lamperti (1972) showed that the time change process

$$X_t = e^{\xi_{A_t}},\ t \geq 0, \tag{1.2}$$

where

$$A_t = \inf\{u \geq 0;\ V_u := \int_0^u e^{\alpha \xi_s}\, ds > t\},$$

is an $\frac{1}{\alpha}$-semi-stable positive Markov process starting from e^x. Further, we assume that $-\infty < b := \mathbb{E}[\xi_1] < \infty$ and denote the characteristic exponent of ξ by Ψ. We also suppose that ξ is not arithmetic (i.e. does not live on a discrete subgroup $r\mathbb{Z}$ for some $r > 0$). Then, on the one hand, for $b > 0$ (resp. $b \geq 0$ and ξ is spectrally negative), it is plain that X has infinite lifetime and we know from Bertoin and Yor (2002a, Theorem 1) (resp. Bertoin and Yor (2002b, Proposition 1)), that the family of probability measures $(\mathbb{P}_x)_{x>0}$ converges in the sense of finite-dimensional distributions to a probability measure, denoted by $\mathbb{P}_0$, as $x \to 0+$. On the other hand, if $b < 0$, then it is plain that ξ drift towards $-\infty$ and X has a finite lifetime which is $T_0^X = \inf\{u \geq 0;\ X_{u-} = 0, X_u = 0\}$. In this case, we assume that there exists a unique $\theta > 0$ which yields the so-called Cramér condition

$$\mathbb{E}[e^{\theta \xi_1}] = 1. \tag{1.3}$$

Then, under the additional condition $0 < \theta < \alpha$, Rivero (2007) showed that the minimal process (X, T_0^X) admits an unique recurrent extension that hits and leaves 0 continuously a.s. and which is a $\frac{1}{\alpha}$-semi-stable process on $[0,\infty)$. With a slight abuse of notation, we write $(\mathbb{P}_x)_{x>0}$ for the family of laws of such a recurrent extension. We gather the different possibilities in the following.

H0. $\mathrm{E} = [0,\infty)$ and either $b > 0$ or $b < 0$ and (1.3) holds with $\theta < \alpha$. Moreover, if ξ is spectrally negative, the case $b = 0$ is also allowed.

We will also use the following hypothesis

H1. $\mathrm{E} \subseteq [0,\infty)$. ξ has finite exponential moments of arbitrary positive orders, i.e. $\psi(m) < \infty$ for every $m \geq 0$ where $\psi(m) = -\Psi(-im)$.

If ξ is spectrally negative, excluding the degenerate cases, then **H1** holds and for $b \in \mathbb{R}$, we write θ_0 for the largest root of the equation $\psi(u) = 0$. Then, being continuous and increasing on $[\theta_0,\infty)$, ψ has a well-defined inverse function $\phi : [0,\infty) \to [\theta_0,\infty)$ which is also continuous and increasing.

For $r > 0$, we write $P_t^r = e^{-rt}P_t$. We say that a non-negative fu nction, $\mathcal{I}_r$, is r-excessive (resp. r-invariant) for P_t if for any $t > 0$,

$$P_t^r \mathcal{I}_r(x) \leq \mathcal{I}_r(x),$$

(resp. if we have $=$ in place of $\leq$) and $\lim_{t\downarrow 0} P_t^r \mathcal{I}_r(x) = \mathcal{I}_r(x)$ pointwise. Taking $r = 1$ and writing simply $\mathcal{I} = \mathcal{I}_1$, we have $P_t^1 \mathcal{I}(x) = \mathcal{I}(x)$. The self-similarity property (1.1) then yields

$$P_t^r\left(d_{r^{\frac{1}{\alpha}}}\mathcal{I}\right)(r^{-\frac{1}{\alpha}}x) = \mathcal{I}(x), \tag{1.4}$$

which entails the identity $\mathcal{I}_r(x) = \mathcal{I}(r^{\frac{1}{\alpha}}x)$ for all $x \in \mathrm{E}$ and $r > 0$. We denote the convex cone of 1-excessive (resp. 1-invariant) functions for X by $\mathfrak{E}(X)$ (resp. $\mathfrak{I}(X)$).

For any $\lambda > 0$, the Ornstein-Uhlenbeck (for short OU) semigroup, $(Q_t)_{t\geq 0}$, is defined, for $f \in \mathfrak{B}(\mathrm{E})$, by

$$Q_t f(x) = P_{e_\chi(t)}\left(d_{e'_\lambda(-t)}f\right)(x), \quad x \in \mathrm{E},\ t \geq 0, \tag{1.5}$$

where $e_\lambda(t) = \frac{e^{\lambda t}-1}{\lambda}$, $\chi = \alpha\lambda$ and we write $v_\lambda(.)$ for the continuous increasing inverse function of $e_\lambda(.)$. We mention that such a deterministic transformation of self-similar processes traces back to Doob (1942) who studied the generalized OU processes driven by symmetric stable Lévy processes. Moreover, Carmona et al. (1998, Proposition 5.8) showed that $(Q_t)_{t\geq 0}$ is a Feller semigroup with infinitesimal generator, for $f \in \mathfrak{D}(\mathbf{A})$, given by

$$\mathbf{U}f(x) = \mathbf{A}f(x) - \lambda x f'(x).$$

Let U be the realization of the Feller semigroup $(Q_t)_{t\geq 0}$. It follows from (1.5) that

$$U_t = e'_\lambda(-t)X_{e_\chi(t)},\ t \geq 0. \tag{1.6}$$

We deduce, with obvious notation, that $T_0^U = v_\chi(T_0^X)$ a.s.. If $Ee = \mathbb{R}$ (resp. otherwise), we call U a self-similar (resp. semi-stable) OU process. We denote by $(\mathbb{Q}_x)_{x>0}$ the family of probability measures of a *semi-stable* OU process. We deduce, from the Lamperti mapping (1.2) and the discussions above the following.

Proposition 1.1. *For any $x > 0$, there exists a one to one mapping between the law of a Lévy process starting from $\log(x)$ and the law of a semi-stable OU process starting from x. More precisely, we have*

$$U_t = e^{-\lambda t}e^{\xi_{\triangle_t}},\ t < T_0^U, \tag{1.7}$$

where $\triangle_t = \int_0^t U_s^{-\alpha}ds$. Note that for $b > 0$, the previous identity holds for any $t \geq 0$.

Moreover, if (1.3) holds with $0 < \theta < \alpha$, then the minimal process (U, T_0^U) admits a recurrent extension which hits and leaves 0 continuously a.s. which is the OU process associated to $(X, \mathbb{P}_x)$. We write its family of laws by $(\mathbb{Q}_x)_{x>0}$.

Under the condition **H1**, for any $\gamma > 0$, we denote by $\mathbb{Q}_x^{(\gamma)}$ the law of the semi-stable OU process, starting at $x \in \mathbb{R}^+$, associated to the Lévy process having Laplace exponent $\psi_\gamma(u) = \psi(u+\gamma) - \psi(\gamma)$, $u \geq 0$.
We say that a probability measure m is invariant for U if it satisfies, for any $f \in \mathfrak{B}(\mathrm{E})$,

$$\int_{\mathrm{E}} Q_t f(x)m(dx) = \int_{\mathrm{E}} f(x)m(dx).$$

Finally, let G_q be a gamma random variable independent of X, with parameter $q > 0$, whose law is given by $\gamma(dr) = \frac{e^{-r}r^{q-1}}{\Gamma(q)}dr$. We are now ready to state the following.

Theorem 1.2. *If* $E=\mathbb{R}$ *or* **H0** *holds then the Feller process* U *is positively recurrent and its unique invariant measure is* $\chi\,\mathbb{P}_0(X_1\in dx)$.

Next, assume that $\mathcal{I}\in L^1(\gamma(dr))$. *For any* $q>0$, *we introduce the function* $\mathcal{I}(q;x)$ *defined by*

$$\mathcal{I}(q;x)=\chi^{\frac{q}{\chi}}\mathbb{E}\left[\mathcal{I}\left(\left(\chi G_{\frac{q}{\chi}}\right)^{\frac{1}{\alpha}}x\right)\right],\ x\in E. \tag{1.8}$$

Then, if $\mathcal{I}\in\mathfrak{I}(X)$ *(resp.* $\mathfrak{E}(X)$*) then* $\mathcal{I}(q;x)\in\mathfrak{I}^q(U)$ *(resp.* $\mathfrak{E}^q(U)$*).*

Consequently, if $\mathcal{I}\in\mathfrak{I}(X)$, *we have, for any* $q>0$,

$$(1+\chi t)^{-\frac{q}{\chi}}P_t\left(d_{(1+\chi t)^{-\frac{1}{\alpha}}}\,I\right)(q;x)=\mathcal{I}(q;x),\ x\in E. \tag{1.9}$$

Remark 1.3.
(1) We call the multiplication operator (1.8) associated to a fractional power of a Gamma random variable, the Γ-*transform*.
(2) The characterization of time-space invariant functions of the form (1.9), associated to self-similar processes, has been first identified by Shepp (1967) in the case of the Brownian motion and by several authors for some specific processes: Yor (1984) for the Bessel processes, Novikov (1981) and Patie (2007c) for the one sided-stable processes. Whilst in the mentioned papers, the authors made used of specific properties of the studied processes to derive the time-space martingales, we provide a proof which is based simply on the self-similarity property.

We proceed by investigating the process Y, defined, for any $x,\beta\in\mathbb{R}$ and $\xi_0=0$ a.s., by

$$Y_t=e^{\alpha\xi_t}\left(x+\beta\int_0^t e^{-\alpha\xi_s}ds\right),\ t\geq 0. \tag{1.10}$$

We call Y the Lévy OU process. We mention that this generalization of the OU process is a specific instance of the continuous analogue of random recurrence equations, as shown by de Haan and Karandikar (1989). They have been also well-studied by Carmona et al. (1997), Erickson and Maller (2005), Bertoin et al. (2008) and by Kondo et al. (2006). In Carmona et al. (1997), it is proved that Y is a homogeneous Markov process with respect to the filtration generated by ξ. Moreover, they showed, from the stationarity and the independency of the increments of ξ, that, for any fixed $t\geq 0$,

$$Y_t\stackrel{(d)}{=}xe^{\alpha\xi_t}+\beta\int_0^t e^{\alpha\xi_s}ds.$$

Then, if $\mathbb{E}[\xi_1]<0$, they deduced that, as $t\to\infty$, $\xi_t\stackrel{(a.s)}{\to}-\infty$ and $Y_t\stackrel{(d)}{\to}\beta V_\infty=\int_0^\infty e^{\alpha\xi_s}ds$. We refer to Bertoin and Yor (2005) for a thorough survey on the exponential functional of Lévy processes. In the spectrally negative case, it is well know that the law of V_∞ is self-decomposable, hence absolutely continuous and unimodal. Moreover, under the additional assumption that $\theta<\alpha$, its law has been computed in term of the Laplace transform by Patie (2007a). Now, we introduce the process Z defined, for any $x\neq 0,\beta\in\mathbb{R}$ and $\xi_0=0$ a.s., by

$$Z_t=e^{\alpha\xi_t}\left(x+\beta\int_0^t e^{\alpha\xi_s}ds\right)^{-1},\ t\geq 0. \tag{1.11}$$

Before stating the next result, we introduce some notation. Let B be a Borel subset of E and we write T_B^U for the first exit time from B by U. With a slight abuse of terminology, we say that for any $x \in \mathrm{E}$, a non-negative function $\mathcal{H}$ is a $(q\Delta, B)$-harmonic function for $(U, \mathbb{Q}_x)$ if

$$\mathbb{E}_x\left[e^{-q\triangle_{T_B^U}}\mathcal{H}(U_{T_B^U})\mathbb{I}_{\{T_B^U<T_0^U\}}\right] = \mathcal{H}(x). \tag{1.12}$$

When Δ_t is replaced by t in the previous expression, we simply say that $\mathcal{H}$ is a (q, B)-harmonic function for $(U, \mathbb{Q}_x)$. We are ready to state the following.

Theorem 1.4. *Set $\beta = \alpha\lambda x$ in* (1.11)*. Then, to a process Z st arting from $\frac{1}{x}$, with $x \neq 0$, one can associate a semi-stable O U process $(U, \mathbb{Q}_1)$ such that*

$$Z_t = x^{-1}U^{\alpha}_{\triangledown_t},\ t < T_0^U, \tag{1.13}$$

where $\triangledown_t = \int_0^t Z_s ds$ and its inverse is given by $\triangle_t = \int_0^t U_s^{-\alpha} ds$. Note that for $b > 0$, the previous identity holds for any $t \geq 0$. Consequently, with $x > 0$, Z is a Feller process on $(0, \infty)$.

Moreover, let $q > 0$, $0 \leq a < b \leq +\infty$ and $x \in (a, b)$. Then, a $(q\Delta, T^U_{(ax,bx)})$-harmonic function for $(U, \mathbb{Q}_1)$ is a $(q, T^Z_{(a^\alpha,b^\alpha)})$-harmonic function for the process Z starting from $x^{-\alpha}$. Similarly, a $(q\Delta, T^U_{(\frac{x}{b},\frac{x}{a})})$-harmonic for $(U, \mathbb{Q}_1)$ is a $(q, T^{\widehat{Y}}_{(a^\alpha,b^\alpha)})$-harmonic function for the process $\widehat{Y}$ starting from x^α, the Lévy OU process associated to the Lévy process $\widehat{\xi} = -\xi$, the dual of ξ with respect to the Lebesgue measure.

Finally, assume that **H1** *holds and write $p_q(x) = x^q$ for $x, q > 0$. If the function $\mathcal{H}$ is $(\lambda\phi(q), B)$-harmonic function for $(U, \mathbb{Q}_x^{(\phi(q))})$ then the function $p_{\phi(q)}\mathcal{H}$ is $(q\Delta, B)$-harmonic function for $(U, \mathbb{Q}_x)$.*

2. Proofs

2.1. *Proof of Theorem 1.2.* The description of the unique invariant measure is a refinement of Carmona et al. (1998, Proposition 5.7) where therein the proof is provided for $\mathbb{R}$-valued self-similar processes and can be extended readily for the $\mathbb{R}^+$-valued case under the condition **H0**, which ensures that $(X, \mathbb{P}_x)$ admits an entrance law at 0.

Next, let us assume that $\mathcal{I} \in L^1(\gamma(dr)) \cap \mathfrak{I}(X)$. We need to show that for any $q > 0$, $e^{-qt}Q_t\mathcal{I}(q; x) = \mathcal{I}(q; x)$. For $x \in \mathrm{E}$, we deduce from the definition of $(Q_t)_{t\geq 0}$ that

$$\begin{aligned} e^{-qt}Q_t\mathcal{I}(q;x) &= \frac{\chi^{\frac{q}{\chi}}}{\Gamma\left(\frac{q}{\chi}\right)}e^{-qt}\mathbb{E}_x\left[\int_0^\infty \mathcal{I}\left((\chi r)^{\frac{1}{\alpha}}U_t\right)e^{-r}r^{\frac{q}{\chi}-1}dr\right] \\ &= \frac{\chi^{\frac{q}{\chi}}}{\Gamma\left(\frac{q}{\chi}\right)}e^{-qt}\mathbb{E}_x\left[\int_0^\infty \mathcal{I}\left((\chi r)^{\frac{1}{\alpha}}e'_\lambda(-t)X_{e_\chi(t)}\right)e^{-r}r^{\frac{q}{\chi}-1}dr\right]. \end{aligned}$$

Using the change of variable $u = \chi e'_\chi(-t)r$, Fubini theorem and (1.4), we get

$$\begin{aligned} e^{-qt}Q_t\mathcal{I}(q;x) &= \frac{1}{\Gamma\left(\frac{q}{\chi}\right)}\mathbb{E}_x\left[\int_0^\infty e^{-ue_\chi(t)}\mathcal{I}\left(u^{\frac{1}{\alpha}}X_{e_\chi(t)}\right)e^{-\frac{u}{\chi}}u^{\frac{q}{\chi}-1}du\right] \\ &= \frac{1}{\Gamma\left(\frac{q}{\chi}\right)}\int_0^\infty \mathcal{I}\left(u^{\frac{1}{\alpha}}x\right)e^{-\frac{u}{\chi}}u^{\frac{q}{\chi}-1}du \\ &= \mathcal{I}(q;x) \end{aligned}$$

where the last line follows after the change of variable $u = \chi r$. The case $\mathcal{I} \in L^1(\gamma(dr)) \cap \mathfrak{E}(X)$ is obtained by following the same line of reasoning. The last assertion is deduced from (1.5) and (1.8) by performing the change of variable $u = v_\chi(t)$, with $v_\chi(t) = \frac{1}{\chi}\log(1+\chi t)$.

2.2. *Proof of Theorem 1.4.* Setting $\beta = \alpha\lambda x$, the Lamperti mapping (1.2) yields

$$\begin{aligned} Z_t &= x^{-1} e^{\alpha\xi_t}\left(1+\alpha\lambda\int_0^t e^{\alpha\xi_s}ds\right)^{-1} \\ &= x^{-1}\left(\frac{1}{(1+\alpha\lambda.)^{\frac{1}{\alpha}}}X_.\right)^{\alpha}_{V_t} \\ &= x^{-1}\left(U_{v_\chi(.)}\right)^{\alpha}_{V_t} \end{aligned}$$

where the last identity follows from (1.5). The proof of the assertion (1.13) is completed by observing that

$$(v_\chi(V_t))' = e^{\alpha\xi_t}\left(1+\alpha\lambda\int_0^t e^{\alpha\xi_s}ds\right)^{-1}.$$

Moreover, since the mapping $x \mapsto x^\alpha$ is a homeomorphism of $\mathbb{R}^+$, the Feller property follows from its invariance by "nice" time change of Feller processes, see Lamperti (1967, Theorem 1). We also obtain the following identities

$$\begin{aligned} T^Z_{(a,b)} &= \inf\{u \geq 0;\ Z_u \notin (a,b)\} \\ &= \inf\left\{u \geq 0;\ U^\alpha_{\triangledown_u} \notin (ax,bx)\right\} \\ &= \triangle\left(\inf\left\{u \geq 0;\ U_u \notin \left((ax)^{\frac{1}{\alpha}},(bx)^{\frac{1}{\alpha}}\right)\right\}\right). \end{aligned}$$

The characterization of the harmonic functions of Z follows. The characterization of the harmonic functions of $\hat{Y}$ are readily deduced from the ones of Z and the identity

$$\hat{Y}_t = \frac{1}{Z_t},\ t \geq 0.$$

The proof of Theorem is then completed by using the following Lemma together with an application of the optional stopping theorem.

Lemma 2.1. *Assume that* **H1** *holds, then for $\gamma,\delta \geq 0$ and $x > 0$, we have*

$$d\mathbb{Q}^{(\gamma)}_x = \left(\frac{U_t}{x}\right)^{\gamma-\delta} e^{\lambda(\gamma-\delta)t-(\psi(\gamma)-\psi(\delta))\triangle_t}\, d\mathbb{Q}^{(\delta)}_x, \quad \text{on } F^U_t \cap \{t < T^U_0\}. \tag{2.1}$$

Note that for $b > 0$ the condition " on $\{t < T^U_0\}$" can be omitted. For the particular case $\gamma = \theta$ and $\delta = 0$, the absolute continuity relationship (2.1) *reduces to*

$$d\mathbb{Q}^{(\theta)}_x = \left(\frac{U_t}{x}\right)^{\theta} e^{\lambda\theta t}\, d\mathbb{Q}_x, \quad \text{on } F^U_t \cap \{t < T^U_0\}.$$

Proof. We start by recalling that in Patie (2007a), the following power Girsanov transform has been derived, under **H1**, for $\gamma,\delta \geq 0$ and $x > 0$, with obvious notation,

$$d\mathbb{P}^{(\gamma)}_x = \left(\frac{X_t}{x}\right)^{\gamma-\delta} e^{-(\psi(\gamma)-\psi(\delta))A_t}\, d\mathbb{P}^{(\delta)}_x, \quad \text{on } F^X_t \cap \{t < T^X_0\}.$$

The assertion (2.1) follows readily by time change and recalling that

$$A_{e_\chi(t)} = \int_0^t U_u^{-\alpha} du.$$

We complete the proof by recalling that for $b > 0$, U does not reach 0 a.s. □

3. Applications

In this section, we illustrate our results to some new interesting examples.

3.1. *First passage times and overshoot of stable OU processes.* Let X be an α-stable Lévy process whose characteristic exponent satisfy, for $u \in \mathbb{R}$,

$$\Psi(iu) = -c|u|^\alpha \left(1 - i\beta \mathrm{sgn}(u) \tan\left(\frac{\alpha\pi}{2}\right)\right)$$

where $1 < \alpha < 2$ and for convenience we take $c = \left(1 + \beta^2 \tan^2\left(\frac{\alpha\pi}{2}\right)\right)^{-1/2}$. Then, we introduce the constant $\rho = \mathbb{P}(X_1 > 0)$ which was evaluated by Zolotarev (1957) as follows

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1}\left(\beta \tan\left(\frac{\alpha\pi}{2}\right)\right).$$

Following Doney (1987), we introduce, for any integers k, l, the class $C_{k,l}$ of stable processes such that

$$\rho + k = l\tilde{\alpha}$$

where $\tilde{\alpha} = \frac{1}{\alpha}$. For $m \in \mathbb{N}$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$, introduce the function

$$f_m(x, z) = \prod_{i=0}^{m} \left(z + e^{ix(m-2i)\pi}\right).$$

Next, we recall from the Wiener-Hopf factorization of Lévy processes due to Rogozin (1966), that the law of the first passage times τ_0^X and the over(under)shoot of X at the level 0 is described by the following identities, for $\delta, r > 0$ and $p \geq 0$,

$$\begin{aligned}
\int_0^\infty e^{-\delta x} \mathbb{E}_{-x}\left[e^{-r\tau_0^X - pX_{\tau_0^X-}}\right] dx &= \frac{1}{\delta - p}\left(1 - \frac{\Psi^+(-r^{\frac{1}{\alpha}}\delta)}{\Psi^+(-r^{\frac{1}{\alpha}}p)}\right) \\
\int_0^\infty e^{-\delta x} \mathbb{E}_{x}\left[e^{-r\tau_0^X - pX_{\tau_0^X-}}\right] dx &= \frac{1}{\delta - p}\left(1 - \frac{\Psi^-(r^{\frac{1}{\alpha}}\delta)}{\Psi^-(r^{\frac{1}{\alpha}}p)}\right)
\end{aligned}$$

where $(1 - \Psi(\delta))^{-1} = \Psi^-(\delta)\Psi^+(\delta)$. Here $\Psi^+(\delta)$ (resp. $\Psi^-(\delta)$) is analytic in $\Re(\delta) < 0$ (resp. $\Re(\delta) > 0$) continuous and nonvanishing on $\Re(\delta) \leq 0$ (resp. $\Re(\delta) \geq 0$). Doney (1987) computes the Wiener-Hopf factors for stable processes in $C_{k,l}$ as follows

$$\begin{aligned}
\Psi^+(z) &= \frac{f_{k-1}(\alpha, (-1)^l(-z)^\alpha)}{f_{l-1}(\tilde{\alpha}, (-1)^{k+1}z)}, \quad \mathfrak{Arg}(z) \neq 0, \\
\Psi^-(z) &= \frac{f_{l-1}(\tilde{\alpha}, (-1)^{k+1}z)}{f_k(\alpha, (-1)^l z^\alpha)}, \quad \mathfrak{Arg}(z) \neq -\pi,
\end{aligned}$$

where z^β stands for $\sigma^\beta e^{i\beta\phi}$ when $z = \sigma e^{i\phi}$ with $\sigma > 0$ and $-\pi < \phi \leq \pi$. Observe also that $\Psi^+(-x^{\frac{1}{\alpha}}) \sim x^{-\rho}$ for large real x. Moreover, using the fact that the function $\mathbb{E}_x\left[e^{-r\tau_0^X - pX_{\tau_0^X}}\right]$, $x \in \mathbb{R}$, is r-excessive for the semigroup of X, we deduce from the Γ-transform the following.

Corollary 3.1. *For any $q, \delta > 0$, $p \geq 0$, and for any integers k, l such that $X \in C_{k,l}$, we have*

$$\int_{-\infty}^{0} e^{\delta x}\mathbb{E}_x\left[e^{-q\tau_0^U - p\, U_{\tau_0^U -}}\right] dx \;=\; \frac{1}{\delta - p}\left(\chi^{\frac{q}{\chi}} - \frac{1}{\Gamma(\frac{q}{\chi})}\int_0^\infty \frac{\Psi^+(-r^{\frac{1}{\alpha}}\delta)}{\Psi^+(-r^{\frac{1}{\alpha}}p)} e^{-\frac{r}{\chi}} r^{\frac{q}{\chi}-1} dr\right)$$

$$\int_{0}^{\infty} e^{-\delta x}\mathbb{E}_x\left[e^{-q\tau_0^U - p\, U_{\tau_0^U -}}\right] dx \;=\; \frac{1}{\delta - p}\left(\chi^{\frac{q}{\chi}} - \frac{1}{\Gamma(\frac{q}{\chi})}\int_0^\infty \frac{\Psi^-(r^{\frac{1}{\alpha}}\delta)}{\Psi^-(r^{\frac{1}{\alpha}}p)} e^{-\frac{r}{\chi}} r^{\frac{q}{\chi}-1} dr\right).$$

3.2. *First passage times of one-sided semi-stable- and Lévy-OU processes.* We now fix $(P_t)_{t\geq 0}$ to be the semigroup of a spectrally negative $\frac{1}{\alpha}$-semi-stable process X. X is then associated via the L amperti mapping (1.2) to a spectrally negative Lévy process, ξ, which we assume to have a finite mean b. Its characteristic exponent ψ has the well known Lévy-Khintchine representation

$$\psi(u) = bu + \frac{\sigma}{2}u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur)\nu(dr),\ u \geq 0, \tag{3.1}$$

where $\sigma \geq 0$ and the measure ν satisfies the integrability condition $\int_{-\infty}^{0}(r \wedge r^2)\,\nu(dr) < +\infty$. Patie (2007a) computes the Laplace transform of the first passage times above of X as follows. For any $r \geq 0$ and $0 \leq x \leq a$, we have

$$\mathbb{E}_x\left[e^{-rT_a^X}\right] \;=\; \frac{\mathcal{I}_{\alpha,\psi}(rx^\alpha)}{\mathcal{I}_{\alpha,\psi}(ra^\alpha)} \tag{3.2}$$

where the entire function, $\mathcal{I}_{\alpha,\psi}$, is given, for $\gamma \geq 0$ and $\alpha > 0$, by

$$\mathcal{I}_{\alpha,\psi}(z) = \sum_{n=0}^{\infty} a_n(\psi;\alpha) z^n, \quad z \in \mathbb{C}$$

and

$$a_n(\psi;\alpha)^{-1} = \prod_{k=1}^{n} \psi(\alpha k), \quad a_0 = 1.$$

Using the Γ-transform, we introduce the following power series

$$\mathcal{I}_{\alpha,\psi}(q;z) = \sum_{n=0}^{\infty} a_n(\psi;\alpha)(q)_n z^n \tag{3.3}$$

where $(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$ is the Pochhammer symbol and we have used the integral representation of the gamma function $\Gamma(q) = \int_0^\infty e^{-r} r^{q-1} dr$, $\Re(q) > 0$. By means of the following asymptotic formula of ratio of gamma functions, see e.g. Lebedev (1972, p.15), for $\delta > 0$,

$$(z+n)_\delta \;=\; z^\delta\left[1 + \frac{\delta(2n+\delta-1)}{2z} + O(z^{-2})\right],\ |\arg z| < \pi - \epsilon,\ \epsilon > 0, \tag{3.4}$$

we deduce that $\mathcal{I}_{\alpha,\psi}(q;z)$ is an entire function in z and is analytic on the domain $\{q \in \mathbb{C}; \Re(q) > -1\}$. For $b < 0$, we recall that there exists $\theta > 0$ such that $\psi(\theta) = 0$

and thus $\psi_\theta(u) = \psi(\theta + u)$. In this case, by setting $\theta_\alpha = \frac{\theta}{\alpha}$, it is shown in Patie (2007a) that there exists a positive constant C_{θ_α} such that

$$\mathcal{I}_{\alpha,\psi}\left(x^\alpha\right) \sim C_{\theta_\alpha} x^\theta \mathcal{I}_{\alpha,\psi_\theta}\left(x^\alpha\right) \quad \text{as } x \to \infty.$$

We also introduce the function $\mathcal{N}_{\alpha,\psi,\theta}(q; x^\alpha)$ defined by

$$\mathcal{N}_{\alpha,\psi,\theta}(q; x^\alpha) = \mathcal{I}_{\alpha,\psi}(q; x^\alpha) - C_{\theta_\alpha} x^\theta \frac{\Gamma(q+\theta_\alpha)}{\Gamma(q)} \mathcal{I}_{\alpha,\psi_\theta}\left(q+\theta_\alpha; x^\alpha\right), \quad \Re(x) \geq 0. \tag{3.5}$$

Moreover, if we assume that there exists $\beta \in [0,1]$ and a constant $a_\beta > 0$ such that $\lim_{u\to\infty} \psi(u)/u^{1+\beta} = a_\beta$, then C_{θ_α} is characterized by

$$C_{\theta_\alpha} = \begin{cases} \frac{\Gamma(1-\theta_\alpha)}{\alpha} \frac{(\theta_\alpha - 1)!}{\prod_{k=1}^{\theta_\alpha - 1} \psi(\alpha k)}, & \text{if } \theta_\alpha \text{ is a positive integer,} \\ \frac{\Gamma(1-\theta_\alpha)}{\alpha} a_\beta^{-\theta_\alpha} e^{E_\gamma \beta \theta_\alpha} \prod_{k=1}^{\infty} e^{-\frac{\beta\theta_\alpha}{k}} \frac{(k+\theta_\alpha)\psi(\alpha k)}{k\psi(\alpha k + \theta_\alpha)}, & \text{otherwise,} \end{cases}$$

where E_γ stands for the Euler-Mascheroni constant. We recall, also from Patie (2007a), that, for $r, x \geq 0$,

$$\mathbb{E}_x\left[e^{-rT_0^X}\right] = \mathcal{I}_{\alpha,\psi}\left(rx^\alpha\right) - C_{\theta_\alpha}(r^{\frac{1}{\alpha}}x)^\theta \mathcal{I}_{\alpha,\psi_\theta}\left(rx^\alpha\right).$$

We deduce from Theorems 1.2 and 1.4 the following.

Corollary 3.2. *Let $q \geq 0$ and $0 < x \leq a$. Then,*

$$\mathbb{E}_x\left[e^{-qT_a^U}\right] = \frac{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi a^\alpha\right)}$$

and

$$\mathbb{E}_x\left[\left(1+\chi T_{(\alpha)}^X\right)^{-\frac{q}{\chi}}\right] = \frac{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi a^\alpha\right)}$$

where $T_{(\alpha)}^X = \inf\{u \geq 0;\ X_u = a(1+\chi u)^{\frac{1}{\alpha}}\}$. We also deduce that

$$\mathbb{E}_x\left[e^{-q\triangle_{T_a^U}} \mathbb{I}_{\{T_a^U < T_0^U\}}\right] = \left(\frac{x}{b}\right)^\gamma \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi a^\alpha\right)}.$$

Moreover, assume $b > 0$ and set $\beta = \alpha\lambda x$ and $\gamma = \phi(q)$. Then,

$$\mathbb{E}_{\frac{1}{x}}\left[e^{-qT_a^Z}\right] = \left(\frac{1}{bx}\right)^\gamma \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi(ax)^\alpha\right)}, \ 0 < \frac{1}{x} \leq a,$$

$$\mathbb{E}_a\left[e^{-qT_x^{\hat{Y}}}\right] = \left(\frac{x}{a}\right)^{\frac{\gamma}{\alpha}} \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi^{\frac{1}{\alpha}}\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \left(\frac{\chi a}{x}\right)^{\frac{1}{\alpha}}\right)}, \ 0 < x \leq a.$$

Finally, if $b < 0$ and $0 < \theta < \alpha$, we have

$$\mathbb{E}_x\left[e^{-qT_0^U}\right] = \frac{\mathcal{N}_{\alpha,\psi,\theta}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{N}_{\alpha,\psi,\theta}\left(\frac{q}{\chi}; \chi x^\alpha\right)}.$$

Remark 3.3. From the strong Markov property and the absence of positive jumps, we easily get that first passage times above for the processes U and Z are infinitely divisible random variables. Hence, we obtain from Corollary 3.2, that the functions (3.3) and (3.5) are Laplace transforms, with respect to the parameter q, of infinitely divisible distributions concentrated on the positive real line.

We end up by investigating some special cases which allow to make some connections between the power series introduced and some well-known or new special functions.

3.2.1. *The confluent hypergeometric functions.* We first consider a Brownian motion with drift $-\nu$, i.e. $\psi(u) = \frac{1}{2}u^2 - \nu u$. Setting $\alpha = 2$, we have $\theta = 2\nu$ and therefore we assume $\nu < 1$. Its associated semi-stable process is well known to be a Bessel process of index ν and thus the associated Ornstein-Uhlenbeck process is, in the case $n = 2\nu + 1 \in \mathbb{N}$, the radial norm of n-dimensional Ornstein-Uhlenbeck process. We get

$$\mathcal{I}_{2,\psi}(x) = (x/2)^{\nu/2}\Gamma(-\nu+1)\mathrm{I}_{-\nu}\left(\sqrt{2x}\right)$$

where $\mathrm{I}_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$ stands for the modified Bessel function of index ν, see e.g. Lebedev (1972, 5.), and

$$\begin{aligned}\mathcal{I}_{2,\psi}(q;x^2) &= \Phi\left(q, 1-\nu, \frac{x^2}{2}\right)\\ \mathcal{I}_{2,\psi_{2\nu}}(q;x^2) &= \Phi\left(q+\nu, \nu+1, \frac{x^2}{2}\right)\end{aligned}$$

where $\Phi(q,\nu,x) = \sum_{n=0}^{\infty} \frac{(q)_n}{(\nu)_n n!}x^n$ stands for the confluent hypergeometric function of the first kind, see e.g. Lebedev (1972, 9.9). Using the asymptotic behavior of the Bessel function

$$\mathrm{I}_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty,$$

we deduce that $C_{2\nu} = -\frac{\Gamma(-\nu)}{\Gamma(\nu)}$. Hence,

$$\begin{aligned}\mathcal{N}_{\alpha,\psi_{2\nu}}(q;x^2) &= \left(\Phi\left(q,1-\nu,\frac{x^2}{2}\right) + x^{2\nu}\frac{\Gamma(-\nu)\Gamma(q+\nu)}{\Gamma(\nu)\Gamma(q)}\Phi\left(q,1-\nu,\frac{x^2}{2}\right)\right)\\ &= \frac{\Gamma(q)\Gamma(q+\nu)}{\Gamma(\nu)}\Lambda\left(q,\nu+1,\frac{x^2}{2}\right)\end{aligned}$$

where $\Lambda(q,\nu+1,\frac{x^2}{2})$ is the confluent hypergeometric of the second kind. We mention that, in this case, the results of Corollary 3.2 are well-known and can be found in Matsumoto and Yor (2000) and in Borodin and Salminen (2002, II.8.2).

3.2.2. *Some generalization of the Mittag-Leffler function.* Patie (2007b) introduced a new parametric family of one-sided Lévy processes which are characterized by the following Laplace exponents, for any $1 < \alpha < 2$, and $\gamma > 1 - \alpha$,

$$\psi_\gamma(u) = \frac{1}{\alpha}\left((u+\gamma-1)_\alpha - (\gamma-1)_\alpha\right). \tag{3.6}$$

Its characteristic triplet are $\sigma = 0$, $\nu(dy) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{e^{(\alpha+\gamma-1)y}}{(1-e^y)^{\alpha+1}}dy, y < 0$, and $b_\gamma = (\gamma)_\alpha(\Upsilon(\gamma-1+\alpha) - \Upsilon(\gamma-1))$ where $\Upsilon(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$ is the digamma function. In particular, if γ_0 denotes the zero of the function $\gamma \to b_\gamma$, then for $\gamma \geq \gamma_0 \in (1-\alpha, 0)$, $b \geq 0$.

The case $\gamma = 0$. (3.6) reduces to $\psi(u) = \frac{1}{\alpha}(u-1)_\alpha$. Observe that $\theta = 1$, $\psi'(1) = \frac{\Gamma(\alpha)}{\alpha}$ and

$$a_n(\psi;\alpha)^{-1} = \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha)}, \quad a_0 = 1.$$

The series (3.3) can be written as follows

$$\begin{aligned}\mathcal{I}_{1,\psi_1}(q;x) &= \Gamma(\alpha)\mathcal{M}^q_{\alpha,\alpha}(\alpha x)\\ \mathcal{I}_{1,\psi}(q;x) &= \Gamma(\alpha-1)\mathcal{M}^q_{\alpha,\alpha-1}(\alpha x)\end{aligned}$$

where

$$\mathcal{M}^q_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(q)_n z^n}{\Gamma(\alpha n+\beta)}, \quad z \in \mathbb{C},$$

stands for the Mittag-Leffler function of parameter $\alpha, \beta, q > 0$, which was introduced by Prabhakar (1971). Moreover, we have, see e.g. Patie (2007a),

$$\sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\Gamma(\alpha n+\beta)} \sim \frac{1}{\alpha} e^x x^{1-\beta} l(x^\alpha) \quad \text{as } x \to \infty,$$

with l a slowly varying function at infinity. Thus, $C_{\frac{1}{\alpha}} = \frac{\alpha}{\alpha-1}$ and

$$\mathcal{N}_{\alpha,\psi_1}(q;x^\alpha) = \mathcal{M}^q_{\alpha,\alpha-1}(x^\alpha) - \frac{\alpha x}{\alpha-1}\frac{\Gamma(q+\frac{1}{\alpha})}{\Gamma(q)}\mathcal{M}^q_{\alpha,\alpha}(x^\alpha).$$

As concluding remarks, we first mention that in the diffusion case, i.e. when $(U, \mathbb{Q}_x)$ is the Ornstein-Uhlenbeck process associated to a Bessel process, see 3.2.1, the law of the first passage time above can be expressed as an infinite convolution of exponential distributions with parameters given by the sequence of positive zeros of the confluent hypergeometric function, see Kent (1980) for more details. Beside this case, we do not know whether such a representation is available. For instance, the location of the zeros of the generalized Mittag-Leffler functions, considered in the second example treated above, is still an open problem, see e.g. Craven and Csordas (2006).

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