

Large Deviation Principle for Self-Intersection Local Times for Random Walk in \mathbb{Z}^d with $d \geq 5$.

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Abstract. We obtain a large deviation principle for the self-intersection local times for a symmetric random walk in dimension $d \geq 5$. As an application, we obtain moderate deviations for random walk in random sceneries in Region II of [Asselah and Castell \(2007b\)](#).

1. Introduction.

We consider an aperiodic symmetric random walk on the lattice \mathbb{Z}^d , with $d \geq 5$. More precisely, if S_n is the position of the walk at time $n \in \mathbb{N}$, then S_{n+1} chooses uniformly at random a site of $\{z \in \mathbb{Z}^d : |z - S_n| \leq 1\}$, where for $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, the l^1 -norm is $|z| = |z_1| + \dots + |z_d|$. When $S_0 = x$, we denote the law of this walk by P_x , and its expectation by E_x .

We are concerned with estimating the number of trajectories of length n with *many* self-intersections, in the large n -regime. The self-intersection local times process is defined as follows

$$\text{for } n \in \mathbb{N}, \quad B_n = \sum_{0 \leq i < j < n} \mathbb{1}\{S_i = S_j\}. \quad (1.1)$$

The study of self-intersection local times has a long history in probability theory, as well as in statistical physics. Indeed, a caricature of a polymer would be a random walk self-interacting through short-range forces; a simple model arises as we penalize the simple random walk law with $\exp(\beta B_n)$, where $\beta < 0$ corresponds to a weakly self-avoiding walk, and $\beta > 0$ corresponds to a self-attracting walk. The question is whether there is a transition from collapsed paths to diffusive paths, as we change the parameter β . We refer to [Bolthausen \(2002\)](#) Saint-Flour notes for references and a discussion of these models.

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It is useful to represent B_n in terms of local times $\{l_n(x), x \in \mathbb{Z}^d\}$, that is the collection of number of visits of x up to time n , as x spans \mathbb{Z}^d . We set, for $k < n$,

$$l_{[k,n]}(x) = \mathbb{1}\{S_k = x\} + \dots + \mathbb{1}\{S_{n-1} = x\}, \quad l_n = l_{[0,n]}, \quad \text{and} \quad \|l_n\|_2^2 = \sum_{z \in \mathbb{Z}^d} l_n^2(z). \tag{1.2}$$

It is immediate that $\|l_n\|_2^2 = 2B_n + n$. Henceforth, we always consider $\|l_n\|_2^2$ rather than B_n . It turns out useful to think of the self-intersection local times as the square of the l^2 -norm of an additive and positive process (see Section 7.3). Besides, we will deal with other q -norm of l_n (see Proposition 1.4), for which there is no counterpart in terms of multiple self-intersections.

In dimensions $d \geq 3$, a random walk spends, on the average, a time of the order of one on most visited sites, whose number, up to time n , is of order n . More precisely, a result of [Brydges and Slade \(1995\)](#) states

$$\frac{1}{n} \|l_n\|_2^2 \xrightarrow{L^2} \gamma_d = 2G_d(0) - 1, \quad \text{with} \quad \forall z \in \mathbb{Z}^d, \quad G_d(z) = \sum_{n \geq 0} P_0(S_n = z). \tag{1.3}$$

The next question concerns estimating the probabilities of large deviations from the mean: that is $P_0(\|l_n\|_2^2 - E_0[\|l_n\|_2^2] \geq n\xi)$ with $\xi > 0$. In dimension $d \geq 5$, we establish in [Asselah and Castell \(2007b\)](#) the following upper and lower estimates: there are positive constants c_d^\pm such that

$$\exp(-c_d^- \sqrt{n}) \leq P_0(\|l_n\|_2^2 - E_0[\|l_n\|_2^2] \geq n\xi) \leq \exp(-c_d^+ \sqrt{n}). \tag{1.4}$$

Moreover, we show in [Asselah and Castell \(2007b\)](#) that a bounded (uniformly in n) random domain, say \mathcal{D}_n , visited of the order of \sqrt{n} makes a dominant contribution to produce the excess self-intersection.

However, in dimension 3, the correct speed for our large deviations is $n^{1/3}$ (see [Asselah, 2008](#)), and the excess self-intersection is made up by sites visited less than some power of $\log(n)$. It is expected that the walk spends most of its time-period $[0, n]$ on a ball of radius of order $n^{1/3}$. Thus, in this box, sites are visited a time of order unity.

The situation is still different in dimension 2. First, $E_0[B_n]$ is of order $n \log(n)$, and a result of [Le Gall \(1986\)](#) states that $\frac{1}{n}(B_n - E_0[B_n])$ converges in law to a non-Gaussian random variable. The large (and moderate) deviations asymptotics obtained recently in [Bass et al. \(2006\)](#), read as follows. There is some positive constant C_{BCR} , such that for any sequence $\{b_n, n \in \mathbb{N}\}$ going to infinity with $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \log(P_0(B_n - E_0[B_n] \geq b_n n)) = -C_{BCR}. \tag{1.5}$$

For a LDP in the case of $d = 1$, we refer to [Chen and Li \(2004\)](#) (see also [Mansmann, 1991](#) for the case of a Brownian motion instead of a random walk).

Finally, we recall a related result of [Chen and Mörters \(2009\)](#) concerning mutual intersection local times of two independent random walks in infinite time horizon when $d \geq 5$. Let $l_\infty(z) = \lim_{n \rightarrow \infty} l_n(z)$, and denote by \tilde{l}_∞ an independent copy of l_∞ . All symbols related to the second walk differ with a tilde. We denote the average over both walks by \mathbb{E} , and the product law is denoted \mathbb{P} . The intersection

local times of two random walks, in an infinite time horizon, is

$$\langle l_\infty, \tilde{l}_\infty \rangle = \sum_{z \in \mathbb{Z}^d} l_\infty(z) \tilde{l}_\infty(z), \quad \text{and} \quad \mathbb{E} \left[\langle l_\infty, \tilde{l}_\infty \rangle \right] = \sum_{z \in \mathbb{Z}^d} G_d(z)^2 < \infty,$$

where Green’s function, G_d , is square summable in dimension 5 or more. [Chen and Mörters \(2009\)](#) have obtained sharp asymptotics for $\{\langle l_\infty, \tilde{l}_\infty \rangle \geq t\}$ for t large, in dimension 5 or more, by an elegant asymptotic estimation of the moments, improving on the pioneering work of [Khanin et al. \(1994\)](#). Their method provides a variational formula for the rate functional, and their proof produces (and relies on) a finite volume version. Namely, for any finite subset $\Lambda \subset \mathbb{Z}^d$,

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \mathbb{P} \left(\langle \mathbb{1}_\Lambda l_\infty, \tilde{l}_\infty \rangle \geq t \right) = -2\mathcal{I}_{CM}(\Lambda), \quad \text{and} \quad \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathcal{I}_{CM}(\Lambda) = \mathcal{I}_{CM}, \tag{1.6}$$

with

$$\mathcal{I}_{CM} = \inf \{ \|h\|_2 : h \geq 0, \|h\|_2 < \infty, \text{and } \|U_h\| \geq 1 \},$$

where

$$U_h(f)(x) = \sqrt{e^{h(x)} - 1} \sum_{y \in \mathbb{Z}^d} (G_d(x - y) - \delta_x(y)) (f(y) \sqrt{e^{h(y)} - 1}), \tag{1.7}$$

and δ_x is Kronecker’s delta function at x .

In this paper, we consider self-intersection local-times, and we establish a Large Deviations Principle in $d \geq 5$.

Theorem 1.1. *We assume $d \geq 5$. There is a constant $\mathcal{I}(2) > 0$, such that for $\xi > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P_0 \left(\|l_n\|_2^2 - E \left[\|l_n\|_2^2 \right] \geq n\xi \right) = -\mathcal{I}(2)\sqrt{\xi}. \tag{1.8}$$

Moreover,

$$\mathcal{I}(2) = \mathcal{I}_{CM}. \tag{1.9}$$

Remark 1.2. The reason for dividing Theorem 1.1 into two statements (1.8) and (1.9) is that our proof has two steps: (i) The proof of the existence of the limit in (1.8), which relies eventually on a sub-additive argument, in spite of an odd scaling; (ii) An identification with the constant of Chen and Mörters.

Also, we establish later the existence of a limit for other q -norms of the local-times (see Proposition 1.4), for which we have no variational formulas.

The phenomenon behind the identification (1.9) is that a large mutual-intersection local times, as well as an excess self-intersection local times, are essentially realized on finite region independent of n . This is explained heuristically in Remark 1 of [Chen and Mörters \(2009\)](#), and we provide the following mathematical statement of this latter phenomenon.

Proposition 1.3. *Assume dimension is 5 or more.*

$$\limsup_{\epsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \mathbb{P}_0 \left(\sum_{z \in \mathbb{Z}^d} \mathbb{1}_{\{\min(l_\infty(z), \tilde{l}_\infty(z)) < \epsilon\sqrt{t}\}} l_\infty(z) \tilde{l}_\infty(z) > t \right) = -\infty. \tag{1.10}$$

Proposition 1.3 is not needed in proving (1.9), and we include it for the purpose of having a mathematical description of the main phenomenon at play.

Finally, we present applications of our results to Random Walk in Random Sceneries (RWRS). We first describe RWRS. We consider a field $\{\eta(x), x \in \mathbb{Z}^d\}$ independent of the random walk $\{S_k, k \in \mathbb{N}\}$, and made up of symmetric unimodal i.i.d. with law denoted by Q and tail decay characterized by an exponent $\alpha > 1$ and a constant c_α with

$$\lim_{t \rightarrow \infty} \frac{\log Q(\eta(0) > t)}{t^\alpha} = -c_\alpha. \tag{1.11}$$

The RWRS is the process

$$\langle \eta, l_n \rangle = \sum_{z \in \mathbb{Z}^d} \eta(z) l_n(z) = \eta(S_0) + \dots + \eta(S_{n-1}).$$

We obtained in Asselah and Castell (2007b) distinct upper and lower estimates for $P(\langle \eta, l_n \rangle \geq \xi n^\beta)$, for different values of β (corresponding to moderate deviations). We refer to Asselah and Castell (2007b) for references for RWRS, and for a diagram of the speed of moderate deviations $\{\langle \eta, l_n \rangle > \xi n^\beta\}$ with $\xi > 0$, in terms of $\alpha > 1$ and $\beta > \frac{1}{2}$. The moderate deviations correspond to annealed estimates, that is when we average over both the random walk's law P_0 , and the field's law Q . The annealed law is denoted \mathbf{P} . In this paper, we concentrate on what has been called in Asselah and Castell (2007b) Region II:

$$1 < \alpha < \frac{d}{2}, \quad \text{and} \quad 1 - \frac{1}{\alpha + 2} < \beta < 1 + \frac{1}{\alpha}. \tag{1.12}$$

In region II, the random walk is expected to visit often a few sites, and it is therefore natural that our LDP allows for better asymptotics in this regime. We set

$$\zeta = \beta \frac{\alpha}{\alpha + 1} (< 1), \quad \frac{1}{\alpha^*} = 1 - \frac{1}{\alpha}, \quad \text{and for } \xi > 0 \quad \bar{\mathcal{D}}_n(\xi) = \{z : l_n(z) \geq \xi\}. \tag{1.13}$$

In bounding from above the probability of $\{\langle \eta, l_n \rangle \geq \xi n^\beta\}$, we take exponential moments of $\langle \eta, l_n \rangle$, and first integrate with respect to the η -variables. Thus, the behavior of the log-Laplace transform of η , say $\Gamma(x) = \log E_Q[\exp(x\eta(0))]$, either at zero or at infinity, plays a key role. This, in turn, explains why we need a LDP for other powers of the local times. For $q \geq 1$, the q -norm of function $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}$ is

$$\|\varphi\|_q^q = \sum_{z \in \mathbb{Z}^d} |\varphi(z)|^q.$$

Before dealing with $\{\langle \eta, l_n \rangle > \xi n^\beta\}$, we give estimates for the α^* -norm of the local-times, for $\alpha^* > \frac{d}{d-2}$.

Proposition 1.4. *Choose ζ as in (1.13) with α, β in Region II. Choose χ such that $\zeta > \chi \geq \frac{\zeta}{d/2}$, and any $\xi > 0$. There is a positive constant $\mathcal{I}(\alpha^*)$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n^\zeta} \log (P_0 (\|\mathbb{1}_{\bar{\mathcal{D}}_n(n\chi)} l_n\|_{\alpha^*} \geq \xi n^\zeta)) = -\xi \mathcal{I}(\alpha^*). \tag{1.14}$$

Our moderate deviations estimates for RWRS is as follows.

Theorem 1.5. *Assume α, β are in Region II given in (1.12). With ζ given in (1.13), and any $\xi > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n^\zeta} \log (\mathbf{P} (\langle \eta, l_n \rangle \geq \xi n^\beta)) = -(\alpha + 1) c_\alpha^{\frac{1}{\alpha+1}} \left(\frac{\mathcal{I}(\alpha^*)}{\alpha} \right)^{\frac{\alpha}{\alpha+1}} \xi^{\frac{\alpha}{\alpha+1}}. \quad (1.15)$$

We now wish to outline schematically the main ideas and limitations in our approach. This serves also to describe the organization of the paper. First, we use a shorthand notation for the centered self-intersection local times process,

$$\overline{\|l_n\|_2^2} = \|l_n\|_2^2 - E_0 [\|l_n\|_2^2]. \quad (1.16)$$

Theorem 1.1 relies on the following intermediary result interesting on its own.

Proposition 1.6. *Assume $d \geq 5$. There is $\beta > 0$, such that for any $\epsilon > 0$, there is $\alpha_\epsilon > 0$, and Λ_ϵ a finite subset of \mathbb{Z}^d , such that for any $\alpha > \alpha_\epsilon$, for any $\Lambda \supset \Lambda_\epsilon$ finite, and n large enough*

$$\begin{aligned} & \frac{1}{2} P_0 (\|\mathbb{1}_\Lambda l_{\lfloor \alpha \sqrt{n} \rfloor}\|_2^2 \geq n\xi(1 + \epsilon), S_{\lfloor \alpha \sqrt{n} \rfloor} = 0) \\ & \leq P_0 (\overline{\|l_n\|_2^2} \geq n\xi) \leq e^{\beta \epsilon \sqrt{n}} P_0 (\|\mathbb{1}_\Lambda l_{\lfloor \alpha \sqrt{n} \rfloor}\|_2^2 \geq n\xi(1 - \epsilon), S_{\lfloor \alpha \sqrt{n} \rfloor} = 0). \end{aligned} \quad (1.17)$$

We use $\lfloor x \rfloor$ for the integer part of x .

The upper bound for $P_0(\overline{\|l_n\|_2^2} \geq n\xi)$ in (1.17) is the main technical result of the paper.

From our previous work in Asselah and Castell (2007b), we know that the main contribution to the excess self-intersection comes from level set $\mathcal{D}_n = \{x : l_n(x) \sim \sqrt{n}\}$. This is the place where $d \geq 5$ is crucial. Indeed, this latter fact is false in dimension 3 as shown in Asselah (2008), and unknown in $d = 4$. In Section 2, we recall and refine the results of Asselah and Castell (2007b). We establish that \mathcal{D}_n is a finite set. More precisely, for any $\epsilon > 0$ and L large enough, there is a constant C_ϵ such that for n large enough

$$P_0 (\overline{\|l_n\|_2^2} \geq n\xi) \leq C_\epsilon P_0 (\|\mathbb{1}_{\mathcal{D}_n} l_n\|_2^2 \geq n \xi(1 - \epsilon), |\mathcal{D}_n| < L). \quad (1.18)$$

Then, our main objective is to show that the time spent on \mathcal{D}_n is of order \sqrt{n} . However, this is only possible if some control on the diameter of \mathcal{D}_n is first established. This is the main difficulty. Note that \mathcal{D}_n is visited by the random walk within the time-period $[0, n[$, and from (1.18), a crude uniform estimate yields

$$P_0 (\overline{\|l_n\|_2^2} \geq n\xi) \leq C_\epsilon (2n)^{dL} \sup_{\Lambda \in]-n, n[^d, |\Lambda| \leq L} P_0 (\|\mathbb{1}_\Lambda l_n\|_2^2 \geq n \xi(1 - \epsilon)). \quad (1.19)$$

Now, we can replace the time period $[0, n[$, in the right hand side of (1.19), by an infinite interval $[0, \infty)$ since the local time increases with time. Consider $\Lambda_n \subset]-n, n[^d$ which realizes the supremum in (1.19). Next, we construct two maps: a local map \mathcal{L} in Section 3.2, and a global map f in Section 5. A finite number of iterates of \mathcal{L} (at most L), say \mathcal{L}^L , transforms Λ_n into a subset of finite diameter. On the other hand, f maps $\{\mathcal{D}_n = \Lambda_n\}$ into $\{\mathcal{D}_n = \mathcal{L}(\Lambda_n)\}$, allowing us to compare the probabilities of these two events. Thus, the heart of our argument has two ingredients.

- A marriage theorem (classical in combinatorics) which is recalled in Section 5.1. It is then used to perform surgery on the circuits.

- Classical potential estimates of Sections 4.2 and 4.3. This is the place where the random walk’s features enter the play. Our estimates relies on basic estimates (Green’s function asymptotics, Harnack’s inequalities and heat kernel asymptotics), which are known to hold for general symmetric random walks (see Lawler and V. (2008)). Though we have considered the simplest aperiodic symmetric random walk, all our results hold when the basic potential estimates hold.

We then iterate f a finite number of time to reach $\{\mathcal{D}_n = \mathcal{L}^L(\Lambda_n)\}$. To control the cost of this transformation, it is crucial that only a finite number of iterations of f is needed. The construction of \mathcal{L} and f requires as well many preliminary steps.

- (1) Section 3 deals with *clusters*. In Section 3, we introduce a partition of Λ_n into a collection of nearby points, called *clusters*. In Section 3.2, we define a map \mathcal{L} acting on *clusters*, by translating one *cluster* at a time.
- (2) Section 4 deals with *circuits*. In Section 4.1, we decompose a trajectory in $\{\mathcal{D}_n = \Lambda_n\}$ into all possible *circuits*. We introduce the notions of *trip* and *loop*.

We show in Proposition 6.1, that for trajectories in $\{\mathcal{D}_n = \mathcal{L}^L(\Lambda_n)\}$, no time is wasted on lengthy excursions, and the total time needed to visit \mathcal{D}_n is less than $\alpha\sqrt{n}$, for some large α . This steps also relies on assuming $d \geq 5$. Indeed, we have been using that conditioned on returning to the origin, the expected return time is finite in dimension 5 or more. This concludes the outline of the proof of the upper bound in Proposition 1.6. The lower bound is easy, and is done in Section 7.2.

Assuming Proposition 1.6, we are in a situation where a certain l^2 -norm of an additive process is larger than $\sqrt{n\xi}$ over a time-period of $\alpha\sqrt{n}$. Section 7.1 presents a sub-additive argument yielding the existence of a limit (1.8). We identify the limit in Section 8.1. We prove Proposition 1.3 in Section 8. Finally, the proof of Theorem 1.5 is given in Section 9.

2. Preliminaries on Level Sets.

In this section, we recall and refine the analysis of Asselah and Castell (2007b). Their approach focuses on the contribution of each level set of the local times to the event $\{||l_n||_2^2 - E[||l_n||_2^2] > n\xi\}$.

We first recall Proposition 1.6 of Asselah and Castell (2007b). For $\epsilon_0 > 0$, set

$$\mathcal{R}_n = \left\{ x \in \mathbb{Z}^d : n^{1/2-\epsilon_0} \leq l_n(x) \leq n^{1/2+\epsilon_0} \right\}.$$

Then, for any $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P_0 \left(||\mathbb{1}_{\mathcal{R}_n^c} l_n||_2^2 - E_0 [||l_n||_2^2] \geq n\epsilon\xi \right) = -\infty. \tag{2.1}$$

Thus, we have for any $0 < \epsilon < 1$, and $\xi > 0$

$$\begin{aligned} P_0 \left(\overline{||l_n||_2^2} \geq n\xi \right) &\leq P_0 \left(||\mathbb{1}_{\mathcal{R}_n^c} l_n||_2^2 - E_0 [||l_n||_2^2] \geq n\epsilon\xi \right) \\ &\quad + P_0 \left(||\mathbb{1}_{\mathcal{R}_n} l_n||_2^2 \geq n\xi(1 - \epsilon) \right). \end{aligned} \tag{2.2}$$

We only need to focus on the second term of the right hand side of (2.2), and for simplicity here, we use $\xi > 0$ instead of $\xi(1 - \epsilon)$. First, we show in Lemma 2.1 that when asking $\{||l_n||_2^2 \geq E[||l_n||_2^2] + n\xi\}$ with $\xi > 0$, we can assume $\{||l_n||_2^2 \leq An\}$ for

some large A . Then, in Lemma 2.2, we show that the only sites which matter are those whose local times is within $[\frac{\sqrt{n}}{A}, A\sqrt{n}]$ for some large constant A .

Lemma 2.1. *For A positive, there are constants $C, \kappa > 0$ such that*

$$P_0 \left(\|\overline{l_n}\|_2^2 \geq nA \right) \leq C \exp \left(-\kappa\sqrt{An} \right). \tag{2.3}$$

Proof: We rely on Proposition 1.6 and the proof of Lemma 3.1 of Asselah and Castell (2007b) (with $p = 2$ and $\gamma = 1$), for the same subdivision $\{b_i, i = 1, \dots, M\}$ of $[1/2 - \epsilon, 1/2]$, and the same $\{y_i\}$ such that $\sum y_i \leq 1$, but the level sets are here of the form

$$\mathcal{D}_i = \left\{ x \in \mathbb{Z}^d : A^{\frac{1}{2}}n^{b_i} \leq l_n(x) < A^{\frac{1}{2}}n^{b_{i+1}} \right\}. \tag{2.4}$$

Using Lemma 2.2 of Asselah and Castell (2007b), we obtain the second line of (2.5),

$$\begin{aligned} P_0 \left(\sum_{\cup \mathcal{D}_i} l_n^2(x) \geq nA \right) &\leq \sum_{i=1}^{M-1} P_0 \left(|\mathcal{D}_i| (A^{\frac{1}{2}}n^{b_{i+1}})^2 \geq ny_i A \right) \\ &\leq \sum_{i=1}^{M-1} P_0 \left(|\mathcal{D}_i| \geq y_i n^{1-2b_{i+1}} \right) \\ &\leq \sum_{i=1}^{M-1} (n^d)^{n^{1-2b_{i+1}} y_i} \exp \left(-\kappa_d A^{\frac{1}{2}} n^{b_i + (1-\frac{2}{d})(1-2b_{i+1})} y_i^{1-\frac{2}{d}} \right) \\ &\leq \sup_{i \leq M} \left\{ \mathcal{C}_i(n) \exp \left(-\kappa_d A^{\frac{1}{2}} n^{b_i + (1-\frac{2}{d})(1-2b_{i+1})} y_i^{1-\frac{2}{d}} \right) \right\}, \end{aligned} \tag{2.5}$$

where $\mathcal{C}_i(n) = M(n^d)^{n^{1-2b_{i+1}} y_i}$. The constant κ_d is linked with estimating the probability of spending a given time in a given domain Λ of prescribed volume; this latter inequality is derived in Lemma 1.2 of Asselah and Castell (2007a). We first need $\mathcal{C}_i(n)$ to be negligible, which imposes

$$n^{1-2b_{i+1}} y_i \log(n^d) \ll A^{\frac{1}{2}} n^{b_i + (1-\frac{2}{d})(1-2b_{i+1})} y_i^{1-\frac{2}{d}} \tag{2.6}$$

Inequality (2.6) is easily seen to hold when b_i is larger than $1/2 - \epsilon$, for ϵ small. Now, we need that for some $\kappa > 0$

$$\kappa_d A^{\frac{1}{2}} n^{b_i + (1-\frac{2}{d})(1-2b_{i+1})} (y_i)^{1-\frac{2}{d}} \geq 2\kappa A^{\frac{1}{2}} \sqrt{n}. \tag{2.7}$$

This holds with the choice of y_i as in Lemma 3.1 of Asselah and Castell (2007b). We use one κ of (2.7) to match $\mathcal{C}_i(n)$ in (2.5), and we are left with a constant C such that

$$P_0 \left(\sum_{\cup \mathcal{D}_i} l_n^2(x) \geq nA \right) \leq C \exp \left(-\kappa\sqrt{An} \right) \tag{2.8}$$

■

For any positive reals A and ζ , and $k \in \mathbb{N} \cup \{\infty\}$, we define

$$\mathcal{D}_k(A, \xi) := \left\{ x \in \mathbb{Z}^d : \frac{\xi}{A} \leq l_k(x) < A\xi \right\}. \tag{2.9}$$

Lemma 2.2. Fix $\xi > 0$. For any $M > 0$, there is $A > 0$ so that

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \left(P_0 \left(\sum_{\mathcal{R}_n \setminus \mathcal{D}_n(A, \sqrt{n})} l_n^2(x) > n\xi \right) \right) \leq -M. \tag{2.10}$$

Also,

$$P_0 (|\mathcal{D}_n(A, \sqrt{n})| \geq A^3) \leq C \exp \left(-\kappa \sqrt{An} \right). \tag{2.11}$$

Proof: We consider an increasing sequence $\{a_i, i = 1, \dots, N\}$ to be chosen later, and form

$$\mathcal{B}_i = \left\{ x : \frac{\sqrt{n}}{a_i} \leq l_n(x) < \frac{\sqrt{n}}{a_{i-1}} \right\}, \tag{2.12}$$

where a_0 will be chosen as a large constant, and $a_N \sim n^\epsilon$. In view of Lemma 2.1, it is enough to show that the probability of the event $\{\sum_{\mathcal{B}_i} l_n^2(x) \geq n\xi\}$ is negligible. First, from Lemma 2.1, we can restrict attention to $\{An \geq \sum_{\mathcal{B}_i} l_n^2(x) \geq n\xi_i\}$ for some large constant A and with $\xi = \sum \xi_i$ a decomposition to be chosen later. When considering the sum over $x \in \mathcal{B}_i$, we obtain

$$\sum_{x \in \mathcal{B}_i} l_n^2(x) \leq nA \implies |\mathcal{B}_i| \left(\frac{\sqrt{n}}{a_i} \right)^2 \leq An \implies |\mathcal{B}_i| \leq a_i^2 A. \tag{2.13}$$

Similarly, we obtain the lower bound $|\mathcal{B}_i| \geq \xi_i a_{i-1}^2$. If we call

$$H_i = \{a_{i-1}^2 \xi_i < |\mathcal{B}_i| \leq a_i^2 A\}, \tag{2.14}$$

then by Lemma 2.1, if we set $l_n(\mathcal{B}_i) = \sum_{x \in \mathcal{B}_i} l_n(x)$

$$\begin{aligned} P_0 \left(\sum_{x \in \mathcal{B}_i} l_n^2(x) > n\xi_i \right) &\leq P_0 \left(\sum_{x \in \mathcal{B}_i} l_n^2(x) > nA \right) \\ &\quad + P_0 \left(H_i \cap \{l_n(\mathcal{B}_i) \geq a_{i-1} \xi_i \sqrt{n}\} \right) \\ &\leq C e^{-\kappa \sqrt{An}} \\ &\quad + (n^d)^{a_i^2 A} \exp \left(-\kappa_d \frac{a_{i-1} \xi_i \sqrt{n}}{(a_i^2 A)^{2/d}} \right). \end{aligned} \tag{2.15}$$

Since we assume $a_i \leq n^\epsilon$, the term $(n^d)^{a_i^2 A}$ is innocuous. It remains to find, for any large constant M , two sequences $\{a_i, \xi_i, i = 1, \dots, N\}$ such that

$$\kappa_d \frac{a_{i-1} \xi_i}{(a_i^2 A)^{2/d}} = M, \quad \text{and} \quad \sum \xi_i = \xi. \tag{2.16}$$

Fix an arbitrary $\delta > 0$ and set

$$a_i := (1 + \delta)^i a_0, \quad \xi_i := \frac{z(\delta)}{(1 + \delta)^{\gamma i}} \xi, \quad \text{and} \quad \gamma = 1 - \frac{4}{d}, \tag{2.17}$$

where $z(\delta)$ is a normalizing constant ensuring that $\sum \xi_i = \xi$. Using the values (2.17) in (2.16), we obtain

$$\frac{\kappa_d z(\delta) \xi}{(1 + \delta) A^{2/d}} a_0^{1-4/d} = M. \tag{2.18}$$

Now, for any constant M , we can choose an a_0 large enough so that none of the level \mathcal{B}_i contributes. Note also that $N = \min \{n : a_n \geq n^\epsilon\}$.

Finally, (2.11) follows from Lemma 2.1, once we note that

$$P_0(|\mathcal{D}_n(A, \sqrt{n})| \geq A^3) \leq P_0(\|\mathbb{1}_{\mathcal{D}_n(A, \sqrt{n})} l_n\|_2^2 \geq An).$$

■

We will need estimates for other powers of the local times. We choose two parameters (α, β) satisfying (1.12), and we further define

$$\zeta = \beta \frac{\alpha}{\alpha + 1}, \quad b = \frac{\beta}{\alpha + 1}, \quad \frac{1}{\alpha^*} = 1 - \frac{1}{\alpha}, \quad \text{and} \quad \bar{\mathcal{D}}_n(n^b) := \{z : l_n(z) \geq n^b\}. \quad (2.19)$$

When dealing with the α^* -norm of l_n , we only focus on sites with large local times. Among those sites, we show that finitely many contribute to making the α^* -norm of l_n large. To appreciate the first estimate, similar in spirit and proof to Lemma 2.2, recall that $\zeta < 1$, $\alpha^* > 1$, and $\|l_n\|_{\alpha^*} \geq \|l_n\|_1 = n$.

Lemma 2.3. *Choose ζ, b as in (2.19) with α, β in Region II. For any $\xi > 0$, there are constants $C, \kappa > 0$ such that*

$$P_0(\|\mathbb{1}_{\bar{\mathcal{D}}_n(n^b)} l_n\|_{\alpha^*} \geq \xi n^\zeta) \leq C \exp(-\kappa \xi n^\zeta). \quad (2.20)$$

Moreover, for any $M > 0$, there is $A > 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^\zeta} \log P_0(\|\mathbb{1}_{\mathcal{D}_n(A, n^\zeta)^c \cap \bar{\mathcal{D}}_n(n^b)} l_n\|_{\alpha^*} > \xi n^\zeta) \leq -M. \quad (2.21)$$

Finally, from (2.20), we have

$$P_0(|\mathcal{D}_n(A, n^\zeta)| \geq A^2) \leq P_0(\|\mathbb{1}_{\mathcal{D}_n(A, n^\zeta)} l_n\|_{\alpha^*} \geq An^\zeta) \leq C \exp(-\kappa An^\zeta). \quad (2.22)$$

The proof is similar to that of Lemmas 2.1 and 2.2, and we omit the details. We point out that Lemma 3.1 of Asselah and Castell (2007b) has to be used with $p = \alpha^*$ and $\gamma = \alpha^* \zeta$. Also, Proposition 3.3 of Asselah and Castell (2007b) holds on $\bar{\mathcal{D}}_n(n^b)$ since the condition $b \frac{d}{2} \geq \zeta$ is fulfilled in Region II.

3. Clusters' Decomposition.

From Lemma 2.2, for any $\epsilon > 0$, and A large enough, we have $C_\epsilon > 0$ such that for any $\xi > 0$ and n large enough

$$P_0\left(\overline{\|l_n\|_2^2} \geq n\xi(1 + \epsilon)\right) \leq C_\epsilon P_0\left(\|\mathbb{1}_{\mathcal{D}_n(A, \sqrt{n})} l_n\|_2^2 \geq n\xi, |\mathcal{D}_n(A, \sqrt{n})| \leq A^3\right). \quad (3.1)$$

Since $\mathcal{D}_n(A, \sqrt{n}) \subset]-n, n[^d$, we bound the right hand side of (3.1) by a uniform bound. For any $A' > 0$, we have

$$\begin{aligned} P_0\left(\overline{\|l_n\|_2^2} \geq n\xi(1 + \epsilon)\right) &\leq C_\epsilon (2n)^{dA^3} \sup_{\Lambda} P_0\left(\|\mathbb{1}_{\Lambda} l_n\|_2^2 \geq n\xi, \mathcal{D}_n(A, \sqrt{n}) = \Lambda\right) \\ &\leq C_\epsilon (2n)^{dA^3} \sup_{\Lambda} P_0\left(\|\mathbb{1}_{\Lambda} l_\infty\|_2^2 \geq n\xi, \Lambda \subset \mathcal{D}_\infty(A', \sqrt{n})\right) \\ &\quad + C_\epsilon (2n)^{dA^3} P_0(\exists z \in]-n, n[^d, l_\infty(z) > A' \sqrt{n}), \end{aligned} \quad (3.2)$$

where in the supremum over Λ we assumed that $\Lambda \subset]-n, n[^d$, $|\Lambda| \leq A^3$. In the last term in (3.2), we can use that $l_\infty(z)$ is bounded by a geometric random variable (since the walk is transient) to neglect this very last term for A' large enough in view of the lower bound in (1.4). For simplicity, we still denote A instead of A' .

If we denote by Λ_n the finite subset of \mathbb{Z}^d which realizes the last supremum in (3.2), then our starting point, in this section, is the collection $\{\Lambda_n, n \in \mathbb{N}\}$ of finite subsets of \mathbb{Z}^d .

3.1. *Defining Clusters.* In this section, we partition an arbitrary finite subset of \mathbb{Z}^d , say Λ into subsets of nearby sites, with the feature that these subsets are far apart. More precisely, this partitioning goes as follows.

Lemma 3.1. *Fix Λ finite subset of \mathbb{Z}^d , and L an integer. There is a partition of Λ whose elements are called L -clusters with the property that two distinct L -clusters \mathcal{C} and $\tilde{\mathcal{C}}$ satisfy*

$$\text{dist}(\mathcal{C}, \tilde{\mathcal{C}}) := \inf \left\{ |x - y|, x \in \mathcal{C}, y \in \tilde{\mathcal{C}} \right\} \geq 4 \max \left(\text{diam}(\mathcal{C}), \text{diam}(\tilde{\mathcal{C}}), L \right). \quad (3.3)$$

Also, there is a positive constant $C(\Lambda)$ which depends on $|\Lambda|$, such that for any L -cluster \mathcal{C}

$$\text{diam}(\mathcal{C}) \leq C(\Lambda) L. \quad (3.4)$$

L is called the *cluster-length*.

Remark 3.2. If we define an L -shell $\mathcal{S}_L(\mathcal{C})$ around \mathcal{C} by

$$\mathcal{S}_L(\mathcal{C}) = \left\{ z \in \mathbb{Z}^d : \text{dist}(z, \mathcal{C}) \leq \max(L, \text{diam}(\mathcal{C})) \right\}, \quad \text{then} \quad \mathcal{S}_L(\mathcal{C}) \cap \Lambda = \mathcal{C}. \quad (3.5)$$

We deduce from (3.4), and (3.5), that for any \mathcal{C} and any $x, y \in \mathcal{C}$, there is a finite sequence of points $x_0 = x, \dots, x_k = y$ (not necessarily in Λ), such that for $i = 1, \dots, k$

$$|x_i - x_{i-1}| \leq L, \quad \text{and} \quad B(x_i, L) \subset \mathcal{S}(\mathcal{C}), \quad (3.6)$$

where $B(x_i, L) = \{z \in \mathbb{Z}^d : |x_i - z| \leq L\}$.

Proof: We build clusters by a bootstrap algorithm. At level 0, we define a *linking* relation for $x, y \in \Lambda$: $x \overset{0}{\leftrightarrow} y$ if $|x - y| \leq 4L$, and an equivalent relation $x \overset{0}{\sim} y$ if there is a (finite) path $x = x_1, x_2, \dots, x_k = y \in \Lambda$ such that for $i = 1, \dots, k - 1$, $x_i \overset{0}{\leftrightarrow} x_{i+1}$. The cluster at level 0 are the equivalent classes of Λ . We denote by $\mathcal{C}^{(0)}(x)$ the class which contains x , and by $|\mathcal{C}^{(0)}|$ the number of clusters at level 0 which is bounded by $|\Lambda|$. It is important to note that the diameter of a cluster is bounded independently of the diameter of Λ . Indeed, it is easy to see that for any $x \in \Lambda$, we have $\text{diam}(\mathcal{C}^{(0)}(x)) \leq 4L(|\mathcal{C}^{(0)}(x)| - 1)$, so that

$$\text{diam}(\mathcal{C}^{(0)}(x)) \leq 4L|\Lambda|. \quad (3.7)$$

Then, we set

$$x \overset{1}{\leftrightarrow} y \quad \text{if} \quad |x - y| \leq 4 \max \left(\text{diam}(\mathcal{C}^{(0)}(x)), \text{diam}(\mathcal{C}^{(0)}(y)), L \right). \quad (3.8)$$

As before, relation $\overset{1}{\leftrightarrow}$ is associated with an equivalence relation $\overset{1}{\sim}$ which defines clusters $\mathcal{C}^{(1)}$. Note also that $x \overset{0}{\sim} y$ implies that $x \overset{1}{\sim} y$, and that for any $x \in \Lambda$,

$$\text{diam}(\mathcal{C}^{(1)}(x)) \leq 5|\mathcal{C}^{(0)}| \max \left\{ \text{diam}(\mathcal{C}) : \mathcal{C} \in \mathcal{C}^{(0)} \right\} \leq 5|\Lambda|(4L|\Lambda|), \quad (3.9)$$

since we produce $\mathcal{C}^{(1)}$'s by multiple concatenations of pairs of $\mathcal{C}^{(0)}$ -clusters at a distance of at most four times the maximum diameters of the clusters making up level 0, those latter clusters being less in number than $|\Lambda|$. In the worst scenario,

there is one cluster at level 1 made up of all clusters of $\mathcal{C}^{(0)}$ at a distance of at most $4 \max \{ \text{diam}(\mathcal{C}) : \mathcal{C} \in \mathcal{C}^{(0)} \}$.

If the number of clusters at level 0 is the same as those of level 1, then the algorithm stops and we have two distinct clusters $\mathcal{C}, \tilde{\mathcal{C}} \in \mathcal{C}^{(0)}$

$$\text{dist}(\mathcal{C}, \tilde{\mathcal{C}}) := \inf \{ |x - y|, x \in \mathcal{C}, y \in \tilde{\mathcal{C}} \} \geq 4 \max \left(\text{diam}(\mathcal{C}), \text{diam}(\tilde{\mathcal{C}}), L \right).$$

Otherwise, the number of cluster at level 1 has decreased by at least one. Now, assume by way of induction, that we have reached level $k - 1$. We define $\overset{k}{\leftrightarrow}$ as follows

$$x \overset{k}{\leftrightarrow} y \quad \text{if} \quad |x - y| \leq 4 \max \left(\text{diam}(\mathcal{C}^{(k-1)}(x)), \text{diam}(\mathcal{C}^{(k-1)}(y)), L \right). \quad (3.10)$$

Now, since $|\Lambda|$ is finite, the algorithm stops in a finite number of steps. The clusters we obtain eventually are called L -clusters. Note that two distinct L -clusters satisfy (3.3). Property (3.4) with $C(\Lambda) = (5|\Lambda|)^{|\Lambda|}$, follows by induction with the same argument used to prove (3.9). ■

3.2. Transforming Clusters. For a subset Λ and a *cluster-length* L , assume that we have a partition in terms of L -clusters as in Lemma 3.1. We define the following map on the partition of Λ .

Lemma 3.3. *There is a map \mathcal{L} on the L -clusters of Λ such that $\mathcal{L}(\mathcal{C}) = \mathcal{C}$, but for one cluster, say \mathcal{C}_1 where $\mathcal{L}(\mathcal{C}_1)$ is a translate of \mathcal{C}_1 such that, when the following minimum is taken over all L -clusters*

$$0 = \min \{ \text{dist}(\mathcal{C}, \mathcal{L}(\mathcal{C}_1)) - (\text{diam}(\mathcal{C}) + \text{diam}(\mathcal{L}(\mathcal{C}_1))) \}. \quad (3.11)$$

Also, for any L -cluster $\mathcal{C} \neq \mathcal{C}_1$, we have

$$\text{dist}(\mathcal{C}, \mathcal{L}(\mathcal{C}_1)) \leq 2 \text{dist}(\mathcal{C}, \mathcal{C}_1). \quad (3.12)$$

We denote by $\mathcal{L}(\Lambda) = \cup \mathcal{L}(\mathcal{C})$. Also, we can define \mathcal{L} as a map on \mathbb{Z}^d : for a site $z \in \mathcal{C}_1$ $\mathcal{L}(z)$ denotes the translation of z , otherwise $\mathcal{L}(z) = z$. Finally, we can define the inverse of \mathcal{L} , which we denote \mathcal{L}^{-1} .

Remark 3.4. We can \mathcal{C}_0 the cluster which realizes the minimum in (3.11). Note that $\mathcal{L}(\Lambda)$ has at least one L -cluster less than Λ since (3.3) does not hold for $(\mathcal{C}_0, \mathcal{L}(\mathcal{C}_1))$. Thus, if we apply to L -cluster partition of Lemma 3.1 to $\mathcal{L}(\Lambda)$, \mathcal{C}_0 and $\mathcal{L}(\mathcal{C}_1)$ would merge into one L -cluster, possibly triggering other merging.

Proof: We start with two clusters which minimize the distance among clusters. Let \mathcal{C}_0 and \mathcal{C}_1 be such that

$$\text{dist}(\mathcal{C}_0, \mathcal{C}_1) = \min \{ \text{dist}(\mathcal{C}, \mathcal{C}') : \mathcal{C}, \mathcal{C}' \text{ distinct clusters} \}. \quad (3.13)$$

Now, let $(x_0, x_1) \in \mathcal{C}_0 \times \mathcal{C}_1$ such that $|x_0 - x_1| = \text{dist}(\mathcal{C}_0, \mathcal{C}_1)$, and note that by (3.3), $|x_0 - x_1| \geq 2(\text{diam}(\mathcal{C}_0) + \text{diam}(\mathcal{C}_1))$. Assume that $\text{diam}(\mathcal{C}_0) \geq \text{diam}(\mathcal{C}_1)$. We translate sites of \mathcal{C}_1 by a vector whose coordinates are the integer parts of the following vector

$$u = (x_0 - x_1) \left(1 - \frac{\text{diam}(\mathcal{C}_0) + \text{diam}(\mathcal{C}_1)}{|x_0 - x_1|} \right), \quad (3.14)$$

in such a way that the translated cluster, say $\mathcal{L}(\mathcal{C}_1)$, is at a distance $\text{diam}(\mathcal{C}_0) + \text{diam}(\mathcal{C}_1)$ of \mathcal{C}_0 . We now see that $\mathcal{L}(\mathcal{C}_1)$ is far enough from other clusters. Let \mathcal{C} be a cluster different from \mathcal{C}_0 and \mathcal{C}_1 . Let $z \in \mathcal{C}$, and $y \in \mathcal{C}_1$.

$$\begin{aligned} |z - \mathcal{L}y| &\geq |z - x_0| - |x_0 - \mathcal{L}y| \geq |z - x_0| - (|x_0 - \mathcal{L}x_1| + |\mathcal{L}x_1 - \mathcal{L}y|) \\ &\geq 4 \max(\text{diam}(\mathcal{C}), \text{diam}(\mathcal{C}_0)) - (\text{diam}(\mathcal{C}_0) + \text{diam}(\mathcal{C}_1) + \text{diam}(\mathcal{C}_1)) \\ &\geq \max\{\text{diam}(\mathcal{C}), \text{diam}(\mathcal{C}_0)\}. \end{aligned} \tag{3.15}$$

Thus, for any cluster \mathcal{C} , we have

$$\text{dist}(\mathcal{C}, \mathcal{L}(\mathcal{C}_1)) \geq \max\{\text{diam}(\mathcal{C}), \text{diam}(\mathcal{L}(\mathcal{C}_0))\}. \tag{3.16}$$

Finally, we prove (3.12). Let $z \in \mathcal{C}$, and $y \in \mathcal{C}_1$. Then, using that $\text{dist}(\mathcal{C}_0, \mathcal{C}_1)$ minimizes the distance among distinct clusters

$$\begin{aligned} |z - \mathcal{L}y| &\leq |z - y| + |y - \mathcal{L}y| \leq |z - y| + \text{dist}(\mathcal{C}_0, \mathcal{C}_1) \\ &\leq |z - y| + \text{dist}(\mathcal{C}, \mathcal{C}_1) \leq 2|z - y|. \end{aligned} \tag{3.17}$$

■

4. On Circuits.

4.1. *Definitions and Notations.* Let $\Lambda_n \subset \mathbb{Z}^d$ maximizes the supremum in the last term of (3.2). Assume we have partitioned Λ_n into L -clusters, as done in Section 3.

We decompose the paths realizing $\{\|1_{\Lambda_n} l_\infty\|_2^2 \geq n\xi\}$ with $\{\Lambda_n \subset \mathcal{D}_\infty(A, \sqrt{n})\}$ into the successive visits to $\Lambda'_n = \Lambda_n \cup \mathcal{L}(\Lambda_n)$. For ease of notations, we drop the subscript n in Λ though it is important to keep in mind that Λ varies as we increase n .

We consider the collection of integer-valued vectors over Λ' which we think of as candidates for the local times over Λ' . Thus, after we have fixed a *cluster-length* L , we set

$$V_L(\Lambda', n) := \left\{ \mathbf{k} \in \mathbb{N}^{\Lambda'} : \inf_{x \in \Lambda} k(x) \geq \frac{\sqrt{n}}{A}, \sup_{x \in \Lambda'} k(x) \leq A\sqrt{n}, \sum_{x \in \Lambda} k^2(x) \geq n\xi \right\}. \tag{4.1}$$

Also, for $\mathbf{k} \in V_L(\Lambda', n)$, we set

$$|\mathbf{k}| = \sum_{x \in \Lambda'} k(x), \quad \text{and note that } |\mathbf{k}| \leq |\Lambda'|A\sqrt{n} \leq 2A^4\sqrt{n}. \tag{4.2}$$

We need now more notations. For $U \subset \mathbb{Z}^d$, we call $T(U)$ the first hitting time of U , and we denote by $T = T(\Lambda') = \inf\{n \geq 0 : S_n \in \Lambda'\}$. We also use the notation $\hat{T}(U) = \inf\{n \geq 1 : S_n \in U\}$. For a trajectory in the event $\{l_\infty(x) = k(x), \forall x \in \Lambda'\}$, we call $\{T^{(i)}, i \in \mathbb{N}\}$ the successive times of visits of Λ' : $T^{(1)} = \inf\{n \geq 0 : S_n \in \Lambda'\}$, and by induction for $i \leq |\mathbf{k}|$ when $\{T^{(i-1)} < \infty\}$

$$T^{(i)} = \inf\{n > T^{(i-1)} : S_n \in \Lambda'\}. \tag{4.3}$$

The first observation is that the number of *long trips* cannot be too large.

Lemma 4.1. *For any $\epsilon > 0$, and $M > 0$, there is $L > 0$ such that for each $\mathbf{k} \in V_L(\Lambda', n)$,*

$$P_0\left(l_\infty|_{\Lambda'} = \mathbf{k}, \left| \left\{ i \leq |\mathbf{k}| : |S_{T^{(i)}} - S_{T^{(i-1)}}| > \sqrt{L} \right\} \right| \geq \epsilon\sqrt{n} \right) \leq e^{-M\sqrt{n}}. \tag{4.4}$$

We know from Asselah and Castell (2007b) that the probability that $\{\|\overline{l_n}\|_2^2 \geq n\xi\}$ is bounded from below by $\exp(-\bar{c}\sqrt{n})$ for some positive constant \bar{c} . We assume $M > 2\bar{c}$ (and $L > L(M)$ given in Lemma 4.1), and the left hand side of (4.4) is negligible. The proof of this Lemma is postponed to the Appendix.

We consider now the collections of possible sequence of visited sites of Λ' , and in view of Lemma 4.1, we consider at most $\epsilon\sqrt{n}$ consecutive sites at a distance larger than \sqrt{L} . First, for $\mathbf{k} \in V_L(\Lambda', n)$, and each $\mathbf{z} \in (\Lambda')^{|\mathbf{k}|}$, and $x \in \mathbb{Z}^d$, we denote by $l_{\mathbf{z}}(x)$ the *local times* of \mathbf{z} at x , that is the number of occurrences of x in the string \mathbf{z} . Then,

$$\mathcal{E}_L(\mathbf{k}) = \left\{ \mathbf{z} \in (\Lambda')^{|\mathbf{k}|} : l_{\mathbf{z}}(x) = k(x), \forall x \in \Lambda', \quad \sum_{i < |\mathbf{k}|} \mathbb{1}_{\{|z(i+1) - z(i)| > \sqrt{L}\}} < \epsilon\sqrt{n} \right\}. \quad (4.5)$$

Definition 4.2. For $\mathbf{k} \in V_L(\Lambda', n)$, a *circuit* is an element of $\mathcal{E}_L(\mathbf{k})$. The random walk follows circuit $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$, if it belongs to the event

$$\{S_{T^{(i)}} = z(i), i = 1, \dots, |\mathbf{k}|\} \cap \{T^{(|\mathbf{k}|+1)} = \infty\}. \quad (4.6)$$

When we remove the second constraint in (4.6), we obtain from Lemma 4.1 that for any $\epsilon > 0$ and $M > 0$, there is a *cluster-length* L large enough such that (with $z(0) = 0$)

$$\begin{aligned} P_0(\|\mathbb{1}_\Lambda l_\infty\|_2^2 \geq n\xi, \Lambda \subset \mathcal{D}_\infty(A, \sqrt{n})) &\leq e^{-M\sqrt{n}} \\ &+ \sum_{\mathbf{k} \in V_L(\Lambda', n)} \sum_{\mathbf{z} \in \mathcal{E}_L(\mathbf{k})} \prod_{i=1}^{|\mathbf{k}|} P_{z(i-1)}(S_T = z(i)). \end{aligned} \quad (4.7)$$

We come now to the definitions of *trips* and *loops*.

Definition 4.3. After a *cluster-length* L is fixed, let $\mathbf{k} \in V_L(\Lambda', n)$ and $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$. A *trip* is a pair $(z(i), z(i+1))$, where $z(i)$ and $z(i+1)$ do not belong to the same cluster. A *loop* is a maximal substring of \mathbf{z} belonging to the same cluster.

Remark 4.4. We think of a circuit as a succession of loops connected by trips. Recall that (3.3) tells us that two points of a trip are at a distance larger than L . Thus, trips are necessarily long journeys, whereas loops may contain many short journeys, typically of the order of \sqrt{n} . For $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$, the number of trips is less than $\epsilon\sqrt{n}$, so is the number of loops, since a loop is followed by a trip.

We recall the notations of Section 3.2: $\Lambda = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_k\}$ with $\text{dist}(\mathcal{C}_0, \mathcal{C}_1)$ minimizing distance among the clusters. The map \mathcal{L} translates only cluster \mathcal{C}_1 .

We now fix $\mathbf{k} \in V_L(\Lambda', n)$ and $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$. We number the different points of entering and exiting from \mathcal{C}_1 .

$$\tau_1 = \inf \{n > 0 : z(n) \in \mathcal{C}_1\}, \quad \text{and} \quad \sigma_1 = \inf \{n > \tau_1 : z(n) \notin \mathcal{C}_1\}, \quad (4.8)$$

and by induction, if we assume $\{\tau_2, \sigma_2, \dots, \tau_i, \sigma_i\}$ defined with $\sigma_i < \infty$, then

$$\tau_{i+1} = \inf \{n > \sigma_i : z(n) \in \mathcal{C}_1\}, \quad \text{and} \quad \sigma_{i+1} = \inf \{n > \tau_{i+1} : z(n) \notin \mathcal{C}_1\}. \quad (4.9)$$

Definition 4.5. For a configuration $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$, its i -th \mathcal{C}_1 -loop is

$$\mathbf{L}(i) = \{z(\tau_i), z(\tau_i + 1), \dots, z(\sigma_i - 1)\}. \quad (4.10)$$

We associate with $L(i)$ the entering and exiting site from C_1 , $p(i) = \{z(\tau_i), z(\sigma_i - 1)\}$, which we think of as the *type* of the C_1 -loop.

The construction is identical for $\mathcal{L}(C_1)$ (usually with a tilde put on all symbols).

4.2. *Encaging Loops.* We wish eventually to transform a piece of random walk associated with a C_1 -loop, into a piece of random walk associated with a $\mathcal{L}(C_1)$ -loop. We explain one obvious problem we face when acting with \mathcal{L} on circuits. Consider a C_1 -loop in a circuit \mathbf{z} . Assume for simplicity, that it corresponds to the i -th C_1 -loop. In general,

$$\prod_{k=\tau_i}^{\sigma_i-2} P_{z(k)}(S_T = z(k+1)) \neq \prod_{k=\tau_i}^{\sigma_i-2} P_{\mathcal{L}(z(k))}(S_T = \mathcal{L}(z(k+1))). \tag{4.11}$$

However, if while travelling from $z(k)$ to $z(k+1)$, the walk were forced to stay inside an L -shell of C_1 during $[\tau_i, \sigma_i[$, then under \mathcal{L} , we would have a walk travelling from $\mathcal{L}(z(k))$ to $\mathcal{L}(z(k+1))$, inside an L -shell of $\mathcal{L}(C_1)$.

To give a precise meaning to our use of the expression *encage*, we recall that for any cluster \mathcal{C} , the L -shell around \mathcal{C} is denoted

$$\mathcal{S}(\mathcal{C}) = \{z : \text{dist}(z, \mathcal{C}) = \max(L, \text{diam}(\mathcal{C}))\}.$$

Now, for $x, y \in \mathcal{C}$, the random walk is encaged inside \mathcal{S} while flying from x to y if it does not exit \mathcal{S} before touching y . The main result in this section is the following proposition.

Proposition 4.6. *For any $\epsilon > 0$, there is a cluster-length L , and a constant $\beta > 0$ (independent of ϵ and L), such that for a circuit $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$ with $\mathbf{k} \in V_L(\Lambda, n)$, and the notations*

$$P_{z(i)}^L(S_T = z(i+1)) = \mathbb{I}_{\{z(i+1) \in \mathcal{C}_i\}} P_{z(i)}(S_T = z(i+1), T < T(\mathcal{S}(\mathcal{C}_i))) + \mathbb{I}_{\{z(i+1) \notin \mathcal{C}_i\}} P_{z(i)}(S_T = z(i+1)), \quad \text{with } \mathcal{C}_i = \mathcal{C}(z(i)), \tag{4.12}$$

we have

$$\prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}(S_T = z(i+1)) \leq e^{\beta \epsilon \sqrt{n}} \prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}^L(S_T = z(i+1)). \tag{4.13}$$

Remark 4.7. Consider a \mathcal{C} -loop, say L , and assume that for some integer i , L corresponds to the i -th \mathcal{C} -loop in circuit \mathbf{z} . We use the shorthand notation $\text{Weight}(L)$ to denote the probability associated with L

$$\text{Weight}(L) = \prod_{k=\tau_i}^{\sigma_i} P_{z(k-1)}^L(S_T = z(k)). \tag{4.14}$$

Note that $\text{Weight}(L)$ includes the probabilities of the entering and exiting *trip*. The point of encaging loop is the following identity

$$\prod_{k=\tau_i}^{\sigma_i-2} P_{z(k)}^L(S_T = z(k+1)) = \prod_{k=\tau_i}^{\sigma_i-2} P_{\mathcal{L}(z(k))}^L(S_T = \mathcal{L}(z(k+1))).$$

Thus, if we set $z = z(\tau_i - 1)$ and $z' = z(\sigma_i)$

$$\text{Weight}(L) = \frac{P_z(S_T = z(\tau_i))}{P_z(S_T = \mathcal{L}(z(\tau_i)))} \frac{P_{z(\sigma_i-1)}(S_T = z')}{P_{\mathcal{L}(z(\sigma_i-1))}(S_T = z')} \text{Weight}(\mathcal{L}(L)). \tag{4.15}$$

The proof of Proposition 4.6 is divided in two lemmas. The first lemma deals with excursions between *close* sites. Such excursions are abundant. The larger L is, the better the estimate (4.16) of Lemma 4.8. The second result, Lemma 4.9, deals with excursions between *distant* sites of the same cluster. Such excursions are rare, and even a large constant in the bound (4.17) is innocuous.

Lemma 4.8. *For any $\epsilon > 0$, there is L , such that for any L -cluster \mathcal{C} , and $x, y \in \mathcal{C}$, with $|x - y| \leq \sqrt{L}$, we have*

$$P_x(S_T = y) \leq e^\epsilon P_x(S_T = y, T < T(\mathcal{S})). \tag{4.16}$$

Lemma 4.9. *There is C_B independent of L , such that for any L -cluster \mathcal{C} , and $x, y \in \mathcal{C}$, with $|x - y| > \sqrt{L}$, we have*

$$P_x(S_T = y) \leq C_B P_x(S_T = y, T < T(\mathcal{S})). \tag{4.17}$$

Lemmas 4.8 and 4.9 are proved in the Appendix. We explain how they yield (4.13), that is how to bound the cost of engaging a *loop*. Consider a circuit associated with $\mathbf{k} \in V_L(\Lambda', n)$ and $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$.

- (i) Each journey between sites at a distance less than \sqrt{L} brings a cost e^ϵ from (4.16), and even if \mathbf{z} consisted only of such journeys, the cost would be negligible, since the total number of visits of Λ is $|\mathbf{k}| \leq 2A^4\sqrt{n}$ as seen in (4.2).
- (ii) Each journey between sites at a distance larger than \sqrt{L} brings a constant C_B , but their total number is less than $\epsilon\sqrt{n}$ by the second constraint in (4.5).

Combining (i) and (ii), we obtain (4.13).

4.3. *Local Circuits Surgery.* In this section, we first estimate the cost of wiring differently trips. More precisely, we have the following two lemmas.

Lemma 4.10. *There is a constant $C_T > 0$, such that for any $y \in \Lambda \setminus \mathcal{C}$ and $x \in \mathcal{C}$, we have*

$$P_y(S_T = x) \leq C_T P_y(S_T = \mathcal{L}(x)). \tag{4.18}$$

Remark 4.11. By noting that for any $x, y \in \Lambda$, $P_x(S_T = y) = P_y(S_T = x)$, we have also (4.18) with the role of x and y interchanged. However, it is important to see that the following inequality with C independent of n

$$P_y(S_T = \mathcal{L}(x)) \leq C P_y(S_T = x) \quad \text{is wrong !} \tag{4.19}$$

Indeed, the distance between y and $\mathcal{L}(x)$ might be considerably shorter than the distance between y and x , and the constant C in (4.19) should depend on this ratio of distances, and thus on n .

Secondly, we need to wire different points of the same cluster to an outside point.

Lemma 4.12. *There is a constant $C_I > 0$, such that for all $x, x' \in \mathcal{C}$, and for $y \in \Lambda \setminus \mathcal{C}$*

$$\begin{aligned} P_y(S_T = x) &\leq C_I P_y(S_T = x'), \text{ and for } y \in \Lambda \setminus \mathcal{L}(\mathcal{C}), P_y(S_T = \mathcal{L}(x)) \\ &\leq C_I P_y(S_T = \mathcal{L}(x')). \end{aligned} \tag{4.20}$$

Moreover, (4.20) holds when we interchange initial and final conditions.

Finally, we compare the cost of different trips joining \mathcal{C} and $\mathcal{L}(\mathcal{C})$. This is a corollary of Lemma 4.12.

Corollary 4.13. *For all $x, x' \in \mathcal{C}$ and $y, y' \in \mathcal{C}$,*

$$P_x(S_T = \mathcal{L}(y)) \leq C_I^2 P_{x'}(S_T = \mathcal{L}(y')), \text{ and } P_{\mathcal{L}(x)}(S_T = y) \leq C_I^2 P_{\mathcal{L}(x')}(S_T = y'). \tag{4.21}$$

5. Global Circuits Surgery.

In this section, we discuss the following key result. We use the handy notation $\mathcal{A}_n(\Lambda) = \{\Lambda \subset \mathcal{D}_\infty(A, \sqrt{n})\}$, as well as the notations of Section 4.1.

Proposition 5.1. *There is $\beta > 0$, such that for any $\epsilon > 0$, $M > 0$, and L large enough*

$$P(\|\mathbb{I}_\Lambda l_\infty\|_2^2 \geq n\xi, \mathcal{A}_n(\Lambda)) \leq e^{\beta\epsilon\sqrt{n}} P(\|\mathbb{I}_{\mathcal{L}(\Lambda)} l_\infty\|_2^2 \geq n\xi, \mathcal{A}_n(\mathcal{L}(\Lambda))) + e^{-M\sqrt{n}}. \tag{5.1}$$

We iterate a finite number of times Proposition 5.1, with starting set $\mathcal{L}(\Lambda)$, then $\mathcal{L}^2(\Lambda)$ and so forth (at most $|\Lambda|$ -iterations are enough), and end up with a set $\tilde{\Lambda}$ made up of just one L -cluster.

If $\text{dist}(0, \tilde{\Lambda})$ is larger than $2\text{diam}(\tilde{\Lambda})$, then we can choose an arbitrary point z^* at a distance $\text{diam}(\tilde{\Lambda})$ from $\tilde{\Lambda}$, and replace in the circuit decomposition of (4.7) $P_0(S_T = z(1))$, for any $z(1) \in \tilde{\Lambda}$, by $P_{z^*}(S_T = z(1))$ at the cost of a constant, by arguments similar to those of Section 4.3, and then use translation invariance to translate $\tilde{\Lambda}$ by z^* back to the origin. Thus, from Proposition 5.1, we obtain easily the following result.

Proposition 5.2. *There is $\tilde{\Lambda} \ni 0$ a subset of \mathbb{Z}^d whose diameter depends on ϵ but not on n , such that for n large enough*

$$P(\|\mathbb{I}_\Lambda l_\infty\|_2^2 \geq n\xi, \mathcal{A}_n(\Lambda)) \leq e^{\beta\epsilon\sqrt{n}} P_0\left(\|\mathbb{I}_{\tilde{\Lambda}} l_\infty\|_2^2 \geq n\xi, \mathcal{A}_n(\tilde{\Lambda})\right) + e^{-M\sqrt{n}}. \tag{5.2}$$

First steps of proof of Proposition 5.1 Fix $\epsilon > 0$. Proposition 4.6 produces a cluster-length L , which in turn allows us to define *circuits*. Also, the constant β in (4.13) is independent of ϵ . Recalling (4.7) together with (4.13), we obtain

$$P(\|\mathbb{I}_\Lambda l_\infty\|_2^2 \geq n\xi, \mathcal{A}_n(\Lambda)) \leq e^{\beta\epsilon\sqrt{n}} \sum_{\mathbf{k} \in V_L(\Lambda', n)} \sum_{\mathbf{z} \in \mathcal{E}_L(\mathbf{k})} \prod_{i=1}^{|\mathbf{k}|} P_{z(i-1)}^L(S_T = z(i)) + e^{-M\sqrt{n}}. \tag{5.3}$$

Recall that for $\mathbf{k} \in V_L(\Lambda', n)$, $\mathcal{E}_L(\mathbf{k})$ is the collection of possible circuits producing local times \mathbf{k} with $\{\sum_\Lambda k(x)^2 \geq n\xi\}$. The aim of this section is to modify the circuits so as to interchange the role of \mathcal{C}_1 and $\mathcal{L}(\mathcal{C}_1)$.

We aim at building a map f on circuits with the following three properties: (i) when $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$

$$\forall x \in \Lambda \setminus \mathcal{C}, l_{f(\mathbf{z})}(x) = k(x), \forall x \in \mathcal{C}_1, l_{f(\mathbf{z})}(\mathcal{L}(x)) \geq k(x), \text{ and } l_{f(\mathbf{z})}(x) \leq k(\mathcal{L}(x)). \tag{5.4}$$

(ii) for $\beta > 0$ and a constant $C(\Lambda) > 0$ depending only on $|\Lambda|$,

$$\forall z \in f(\mathcal{E}_L(\mathbf{k})), |f^{-1}(z)| \leq C(\Lambda)e^{\beta\epsilon\sqrt{n}}, \tag{5.5}$$

(iii) for $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$

$$\prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}^L(S_T = z(i+1)) \leq e^{\beta\epsilon\sqrt{n}} \prod_{i=0}^{|\mathbf{k}|-1} P_{f(z(i))}^L(S_T = f(z(i+1))). \quad (5.6)$$

Assume, for a moment, that we have f with (i),(ii) and (iii). Then, summing over $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$,

$$\begin{aligned} \sum_{\mathbf{z} \in \mathcal{E}_L(\mathbf{k})} \prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}^L(S_T = z(i+1)) &\leq e^{\beta\epsilon\sqrt{n}} \sum_{\mathbf{z} \in \mathcal{E}_L(\mathbf{k})} \prod_{i=0}^{|\mathbf{k}|-1} P_{f(z(i))}^L(S_T = f(z(i+1))) \\ &\leq e^{\beta\epsilon\sqrt{n}} \sum_{\mathbf{z} \in f(\mathcal{E}_L(\mathbf{k}))} |f^{-1}(z)| \prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}^L(S_T = z(i+1)) \leq C(\Lambda) e^{2\beta\epsilon\sqrt{n}} \\ &\times P_0(l_\infty|_{\Lambda \setminus \mathcal{C}} = k|_{\Lambda \setminus \mathcal{C}}, \forall x \in \mathcal{C}_1, l_\infty(\mathcal{L}(x)) \geq k(x), \text{ and } l_\infty(x) \leq k(\mathcal{L}(x))) \end{aligned} \quad (5.7)$$

We further sum over $\mathbf{k} \in V_L(\Lambda', n)$, replace the sum over the $\{k(y) \leq A\sqrt{n}\}$ for $y \in \mathcal{L}(\mathcal{C}_1)$ by a factor $(A\sqrt{n})^{|\Lambda|}$, and rearrange the sum over $\{k(y), y \in \mathcal{C}_1\}$, to obtain

$$\begin{aligned} \sum_{\mathbf{k} \in V_L(\Lambda', n)} \sum_{\mathbf{z} \in \mathcal{E}_L(\mathbf{k})} \prod_{i=0}^{|\mathbf{k}|-1} P_{z(i)}^L(S_T = z(i+1)) &\leq e^{2\beta\epsilon\sqrt{n}} (A\sqrt{n})^{|\Lambda|} \\ &\times E \left[\prod_{y \in \mathcal{C}_1} l_\infty(\mathcal{L}(y)), \mathcal{L}(\Lambda) \subset \mathcal{D}_\infty(A, \sqrt{n}), \|\mathbb{1}_{\mathcal{L}(\Lambda)} l_\infty\|_2^2 \geq n\xi \right]. \end{aligned} \quad (5.8)$$

Note that in (5.8), we can assume $l_\infty(\mathcal{L}(y)) \leq A\sqrt{n}$ for all $y \in \mathcal{C}_1$, since for a transient walk, the number of visits to a given site is bounded by a geometric random variable. Thus, in the expectation of (5.8), we bound $l_\infty(\tilde{y})$ by $A\sqrt{n}$, and $|\mathcal{C}_1|$ by $|\Lambda|$.

Providing we can show the existence of a map f with properties (5.4), (5.5) and (5.6), we would have proved Proposition 5.1. Sections 5.1, 5.2 and 5.3 are devoted to constructing the map f .

5.1. A Marriage Theorem. This section deals with *global* modifications of circuits. For this purpose, we rely on an old *Marriage Theorem* (see e.g. [Jukna, 2001](#)), which seems to have been first proved by [Frobenius \(1997\)](#) in our setting. Since we rely heavily on this classical result, we quote it for the ease of reading. First, we recall that a k -regular bipartite graph, with bipartition (G_1, G_2) is a graph whose vertices have degree k , and can be divided into two disjoint sets G_1 and G_2 such that every edge connects a vertex in G_1 to one in G_2 .

Theorem 5.3. Marriage Theorem. *Let $\mathcal{G} = (G, E)$ be a k -regular bipartite graph with bipartition (G_1, G_2) . Then, there is a bijection $\varphi : G_1 \rightarrow G_2$ such that $\{(x, \varphi(x)), x \in G_1\} \subset E$.*

Now, to see how we use the Marriage Theorem, we need more notations. First, for two integers n and m , we call

$$\Omega_{n,m} = \left\{ \eta \in \{0,1\}^{n+m} : \sum_{i=1}^{n+m} \eta(i) = n \right\}. \tag{5.9}$$

Now, when $n > m$, we define the graph $\mathcal{G}_{n,m} = (G_{n,m}, E_{n,m})$ with $G_{n,m} = \Omega_{n,m} \cup \Omega_{m,n}$, and

$$E_{n,m} = \{(\eta, \zeta) \in \Omega_{n,m} \times \Omega_{m,n} : \zeta(x) \leq \eta(x), \forall x \leq n+m\}. \tag{5.10}$$

With $k = n - m$, $\mathcal{G}_{n,m}$ is a k -regular graph with bipartition $(\Omega_{n,m}, \Omega_{m,n})$, and the Marriage Theorem gives us a bijection $\varphi_{n,m} : \Omega_{n,m} \rightarrow \Omega_{m,n}$. Thus, under the action of $\varphi_{n,m}$ a 1 can become a 0, but a 0 stays 0. The importance of this feature is explained below in Remark 5.5. When $n = m$, we call $\varphi_{n,n}$ the identity on $\Omega_{n,n}$.

We use the Marriage Theorem to select pairs of *trips* with the same *type*, one *trip* to \mathcal{C} and one *trip* to $\tilde{\mathcal{C}}$ which are interchanged. Then, we describe how the associated loops are interchanged. However, some *patterns* of loops cannot be handled using the Marriage Theorem, and we call these loops *improper*. For the ease of notations, we call $\mathcal{C} = \mathcal{C}_1$ and $\tilde{\mathcal{C}} = \mathcal{L}(\mathcal{C}_1)$.

Definition 5.4. A \mathcal{C} -loop is called *proper* if it is preceded by a trip from Λ to \mathcal{C} , and the other \mathcal{C} -loops are called *improper*. Similarly, a $\tilde{\mathcal{C}}$ -loop is called *proper* if it is preceded by a trip from $\mathcal{L}(\Lambda)$ to $\tilde{\mathcal{C}}$.

We describe in the two next sections, how to define a map f satisfying (5.4), (5.5) and (5.6). This map only transforms \mathcal{C} and $\tilde{\mathcal{C}}$ -loops. It acts on each *proper* loop of a certain *type*, say p and \tilde{p} , by a global action that we denote f_p . Also, there will be an action f_i on *improper* loops which we describe in Section 5.3. Thus, f is a composition of $\{f_p, p \in \mathcal{C}^2\}$ and f_i , taken in the the order we wish. Note that for any $\mathbf{z} \in f(\mathcal{E}_L(\mathbf{k}))$, we have

$$|f^{-1}(\mathbf{z})| = \prod_{p \in \mathcal{C}^2} |f_p^{-1}(\mathbf{z})| \times |f_i^{-1}(\mathbf{z})|. \tag{5.11}$$

Thus, property (5.5) holds for f , if it holds for f_i , and for each f_p as $p \in \mathcal{C}^2$. We describe the $\{f_p, p \in \mathcal{C}^2\}$ in Section 5.2, and f_i in Section 5.3.

5.2. Proper Loops. We fix $\mathbf{k} \in V_L(\Lambda', n)$ and $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$. We fix a *type* $p = (z, z') \in \mathcal{C}^2$, and we call $\nu(p)$ the number of proper \mathcal{C} -loops of type p in \mathbf{z} . Similarly, $\nu(\tilde{p})$ is the number of proper $\tilde{\mathcal{C}}$ -loops of type $\tilde{p} = (\mathcal{L}(z), \mathcal{L}(z'))$. To each *type* p corresponds a configuration $\eta_p \in \Omega_{\nu(p), \nu(\tilde{p})}$ which encodes the successive occurrences of proper \mathcal{C} and $\tilde{\mathcal{C}}$ -loops of type p : a mark 1 for a \mathcal{C} -loop and a mark 0 for a $\tilde{\mathcal{C}}$ -loop.

Assume that $n = \nu(p) \geq m = \nu(\tilde{p})$, and $\eta_p \in \Omega_{n,m}$. All \mathcal{C} -loop (*proper* and of *type* p) are translated by \mathcal{L} , and all $\tilde{\mathcal{C}}$ -loop (*proper* and of *type* \tilde{p}) are translated by \mathcal{L}^{-1} . The bijection $\varphi_{n,m}$ encodes the positions of the translated loops, as follows.

- The \mathcal{C} -loop associated with the i -th occurrence of a 1 in η_p , is transformed into a $\tilde{\mathcal{C}}$ -loop associated with the i -th occurrence of a 0 in $\varphi_{n,m}(\eta_p)$.
- The $\tilde{\mathcal{C}}$ -loop associated with the i -th occurrence of a 0 in η_p , is transformed into a \mathcal{C} -loop associated with the i -th occurrence of a 1 in $\varphi_{n,m}(\eta_p)$.

After acting with f_p , the number of $\tilde{\mathcal{C}}$ -loops of type \tilde{p} increases by $\nu(p) - \nu(\tilde{p}) \geq 0$.

For definiteness, we illustrate this algorithm on a simple example (see Figure 5.1). Assume that circuit $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$ has 3 proper \mathcal{C} -loops of type p , say $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 , and 1 proper $\tilde{\mathcal{C}}$ -loop of type p , say $\tilde{\mathcal{L}}_1$. Let us make visible in \mathbf{z} only these very loops and the trips joining them:

$$\mathbf{z} : \quad \dots y_1 \mathcal{L}_1 y'_1 \dots y_2 \mathcal{L}_2 y'_2 \dots y_3 \tilde{\mathcal{L}}_1 y'_3 \dots y_4 \mathcal{L}_3 y'_4 \dots, \quad (5.12)$$

for $\{y_i, y'_i, i = 1, \dots, 4\}$ in $\Lambda \setminus \mathcal{C}$. For such a circuit, we would have $\nu(p) = 3$ and $\nu(\tilde{p}) = 1$ and $\eta_p = (1101)$. Furthermore, assume that $\varphi_3(1101) = 0100$. Then, the p, \tilde{p} proper loops are transformed into

$$\mathbf{f}_p(\mathbf{z}) : \quad \dots y_1 \mathcal{L}(\mathcal{L}_1) y'_1 \dots y_2 \mathcal{L}^{-1}(\tilde{\mathcal{L}}_1) y'_2 \dots y_3 \mathcal{L}(\mathcal{L}_2) y'_3 \dots y_4 \mathcal{L}(\mathcal{L}_3) y'_4 \dots \quad (5.13)$$

We end up with 3 $\tilde{\mathcal{C}}$ -loops of type \tilde{p} , $\mathcal{L}(\mathcal{L}_1), \mathcal{L}(\mathcal{L}_2)$ and $\mathcal{L}(\mathcal{L}_3)$, and one \mathcal{C} -loop $\mathcal{L}^{-1}(\tilde{\mathcal{L}}_1)$. Note that in both \mathbf{z} and $\mathbf{f}_p(\mathbf{z})$, the second loop (of type p or \tilde{p}) is a \mathcal{C} -loop, as required by the map φ_3 . The configuration z in (5.12) is represented on the left hand side of Figure 5.1, whereas $f_p(z)$ is shown on its right hand side. Note that we put most of the sites $\{y_i, y'_i, i = 1, \dots, 4\}$ close to $\mathcal{L}(\mathcal{C})$. This is the desired feature of \mathcal{L} as established in Lemma 3.3.

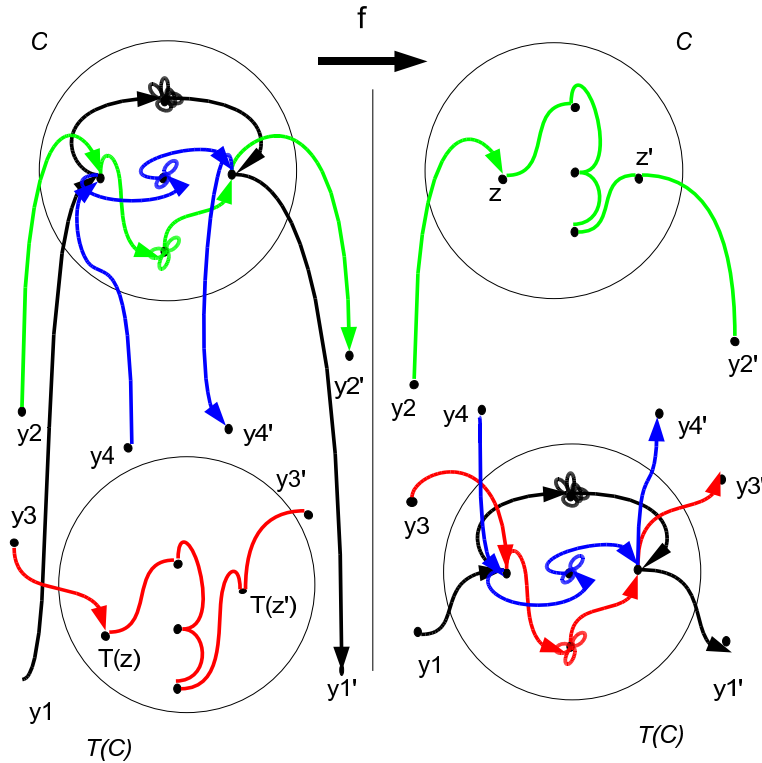


FIGURE 5.1. Action of f on proper loops.

Remark 5.5. One implication of the key feature of $\varphi_{n,m}$, namely that $(\eta_p, \varphi_{n,m}(\eta_p)) \in E_{n,m}$, is that a trip $(y, \mathcal{L}(z))$ or $(\mathcal{L}(z'), y')$ is invariant under f_p .

Note that in Figure 5.1, $(y_3, \mathcal{L}(z))$ and $(\mathcal{L}(z'), y'_3)$ are invariant, whereas (y_1, z) becomes $(y_1, \mathcal{L}(z))$ and fortunately $|y_1 - \mathcal{L}(z)| \leq |y_1 - z|$ on the drawing.

Note that f_p satisfies (5.4). Indeed, if we call z_p the substring of z made up of only sites represented in (5.12), and $f_p(z_p)$ the substring of $f_p(z)$ made up of only sites represented in (5.13), we have $l_{f_p(z_p)}(x) = l_{z_p}(x)$ for $x \in \Lambda \setminus \mathcal{C}$,

$$\forall x \in \mathcal{C}, \quad l_{f_p(z_p)}(\mathcal{L}(x)) = l_{z_p}(x), \quad \text{and} \quad l_{f_p(z_p)}(x) = l_{z_p}(\mathcal{L}(x)). \quad (5.14)$$

Now, we estimate the cost of going from z_p to $f_p(z_p)$. We consider encaged loops as described in Section 4.2. The purpose of having defined *types*, and of having *encaged* loops, is the following two simple observations, which we deduce from (4.15) in Remark 4.7.

$$(i) \quad \text{Weight}(\tilde{\mathbf{L}}_1) \text{Weight}(\mathbf{L}_2) = \text{Weight}(\mathcal{L}^{-1}(\tilde{\mathbf{L}}_1)) \text{Weight}(\mathcal{L}(\mathbf{L}_2)), \quad (5.15)$$

and, if $p = (z, z') \in \mathcal{C}^2$

$$(ii) \quad \text{Weight}(\mathbf{L}_1) = \frac{P_{y_1}(S_T = z)}{P_{y_1}(S_T = \mathcal{L}(z))} \frac{P_{z'}(S_T = y'_1)}{P_{\mathcal{L}(z')}(S_T = y'_1)} \text{Weight}(\mathcal{L}(\mathbf{L}_1)), \quad (5.16)$$

and a similar equality linking $\text{Weight}(\mathbf{L}_3)$ and $\text{Weight}(\mathcal{L}(\mathbf{L}_3))$. Thus, the cost of transformation (5.13) is C_T^4 , where C_T appears in Lemma 4.10, since only 2 entering trips and 2 exiting trips have been wired differently.

Now, for any $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$, the number of loops which undergo a transformation is less than the total number of loops, which is bounded by $\epsilon\sqrt{n}$. The maximum cost (maximum over $\mathbf{z} \in \mathcal{E}_L(\mathbf{k})$) of such an operation is $2C_T$ to the power $\epsilon\sqrt{n}$.

The case (rare but possible) where $\nu(p) < \nu(\tilde{p})$ has to be dealt with differently. Indeed, for an arbitrary cluster \mathcal{C}' , we cannot transform a trip between \mathcal{C}' and $\tilde{\mathcal{C}}$ into a trip between \mathcal{C}' and \mathcal{C} at a constant cost, since $\text{dist}(\mathcal{C}', \tilde{\mathcal{C}})$ might be much smaller than $\text{dist}(\mathcal{C}', \mathcal{C})$.

We propose that f_p performs the following changes:

- Act with \mathcal{L} on all \mathcal{C} -loops of *type* p .
- Act with \mathcal{L}^{-1} only on the first $\nu(p)$ $\tilde{\mathcal{C}}$ -loops of *type* \tilde{p} .
- Interchange the position of the $\nu(p)$ first \mathcal{C} -loops with $\nu(p)$ first $\tilde{\mathcal{C}}$ -loops.

For instance, in the following example, \mathbf{z} has three $\tilde{\mathcal{C}}$ -loops $\tilde{\mathbf{L}}_1, \tilde{\mathbf{L}}_2$ and $\tilde{\mathbf{L}}_3$ and one \mathcal{C} -loop \mathbf{L}_1 ,

$$\mathbf{z} : \dots y_1 \tilde{\mathbf{L}}_1 y'_1 \dots y_2 \mathbf{L}_1 y'_2 \dots y_3 \tilde{\mathbf{L}}_2 y'_3 \dots y_4 \tilde{\mathbf{L}}_3 y'_4.$$

$\nu(p) = 1 < \nu(\tilde{p}) = 3$, and we have

$$\mathbf{z} : \longrightarrow f_p(\mathbf{z}) : \dots y_1 \mathcal{L}(\mathbf{L}_1) y'_1 \dots y_2 \mathcal{L}^{-1}(\tilde{\mathbf{L}}_1) y'_2 \dots y_3 \tilde{\mathbf{L}}_2 y'_3 \dots y_4 \tilde{\mathbf{L}}_3 y'_4. \quad (5.17)$$

In so doing, note that the cost is 1, but instead of (5.14), we have

$$\forall x \in \mathcal{C}, \quad l_{f_p(z_p)}(\mathcal{L}(x)) \geq l_{z_p}(x), \quad \text{and} \quad \forall x \in \mathcal{C}, \quad l_{f_p(z_p)}(x) \leq l_{z_p}(\mathcal{L}(x)). \quad (5.18)$$

Also, we have brought a multiplicity of pre-images. Indeed, note that the final circuit of (5.17) could have been obtained, following the rule of (5.13), by a circuit \mathbf{z}' where $\nu(p) \geq \nu(\tilde{p})$:

$$\mathbf{z}' : \dots y_1 \mathbf{L}_1 y'_1 \dots y_2 \mathcal{L}^{-1}(\mathbf{L}'_2) y'_2 \dots y_3 \tilde{\mathbf{L}}_1 y'_3 \dots y_4 \mathcal{L}^{-1}(\mathbf{L}'_3) y'_4 \dots \longrightarrow f_p(\mathbf{z}). \quad (5.19)$$

Also, f_p maps a *proper* loop into a *proper* loop, and a pre-image under f_p has either $\nu(p) \geq \nu(\tilde{p})$ or $\nu(p) < \nu(\tilde{p})$, and so only two possible pre-images. Since this is true for any *type*, an upper bound on the number of pre-images of the composition of all

f_p , is bounded by 2 to the power $|\mathcal{C}|^2$ (which is the number of *types*). Since $\mathcal{C} \subset \Lambda$ whose volume is independent of n , the multiplicity is innocuous in this case.

5.3. *Improper Loops.* In this section, we deal with trips in $\mathcal{C} \times \tilde{\mathcal{C}} \cup \tilde{\mathcal{C}} \times \mathcal{C}$. The notion of *type* is not useful here. We call f_i the action of f on *improper* loops.

To grasp the need to distinguish *proper* loops from *improper* loops, assume that we have a trip from a $\mathcal{L}(\mathcal{C})$ -loop to a \mathcal{C} -loop. If we could allow the \mathcal{C} -loop to become a $\mathcal{L}(\mathcal{C})$ -loop, we could reach a situation with two successive $\mathcal{L}(\mathcal{C})$ -loops linked with no trip. They would merge into one $\mathcal{L}(\mathcal{C})$ -loop by our definition 4.3. This may increase dramatically the number of pre-images of a given $f(\mathbf{z})$, violating (5.5). We illustrate this with a concrete example drawn in Figure 5.2, below. We have

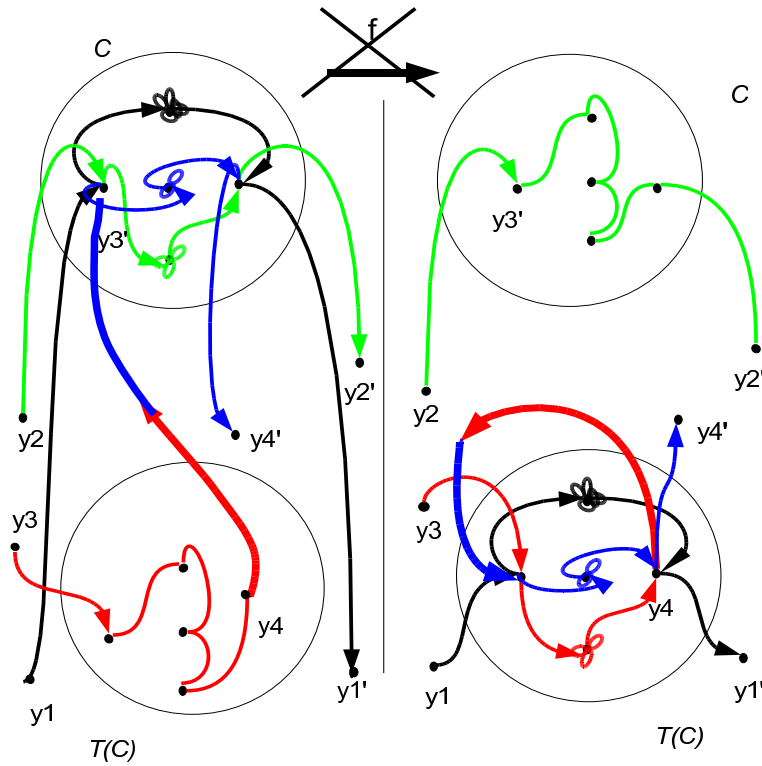


FIGURE 5.2. Red and blue loops merging.

considered the same example as in (5.12), but now there is a trip between \tilde{L}_1 to L_3 , so that y'_3 is in loop L_3 whereas $y_4 \in \tilde{L}_1$, as shown in Figure 5.2. If we were to apply the algorithm of Section 5.2, we would obtain the image shown on the right hand side of in Figure 5.2. There, the loops $\mathcal{L}(L_2)$ and $\mathcal{L}(L_3)$ (that we obtain in (5.13)) would have to merge.

Consider first a circuit with a string of successive *improper loops* of type p , such that the number of \mathcal{C} -loops matches the number of $\tilde{\mathcal{C}}$ -loops. For instance, assume that the i -th $\tilde{\mathcal{C}}$ -loop is *improper* and followed by the j -th \mathcal{C} -loop, and so forth. For definiteness, assume that \mathbf{z} contains z_i (i for *improper*) with

$$z_i = y_1 \tilde{L}(i) L(j) \dots \tilde{L}(i+k) L(j+k) y'_1, \quad \text{with } k \geq 0, \quad \text{and } y_1, y'_1 \notin \mathcal{C} \cup \tilde{\mathcal{C}}. \quad (5.20)$$

Our purpose is to transform such a sequence of alternating \mathcal{C} - $\tilde{\mathcal{C}}$ loops into a similar alternating sequence, such that $f_i(z_i)$ satisfies (5.4), (5.5) and (5.6).

One constraint is that we cannot replace the entering trip, and exiting trip in general, which in turn fixes the order of visits to \mathcal{C} and $\tilde{\mathcal{C}}$. Indeed, as in the previous section, if $p = (z, z')$ and $|y_1 - \mathcal{L}(z)| \ll |y_1 - z|$, then we cannot map the trip $(y_1, \mathcal{L}(z))$ to (y_1, z) at a small cost. We propose to following map

$$f_i(z_i) = y_1 \mathcal{L}(\mathbf{L}(j)) \mathcal{L}^{-1}(\tilde{\mathbf{L}}(i)) \dots \mathcal{L}(\mathbf{L}(j+k)) \mathcal{L}^{-1}(\mathbf{L}(i+k)) y'_1 \tag{5.21}$$

Note that (5.14) holds. With an abuse of notations we represent the probability associated with $f_i(z_i)$, as

$$\text{Weight}(f_i(z_i)) = \prod_{l=0}^{k-1} \text{Weight}(\tilde{\mathbf{L}}(i+l)) \text{Weight}(\mathbf{L}(j+l)), \tag{5.22}$$

even though we mean now that the trips joining successive journeys between \mathcal{C} - $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}$ - \mathcal{C} are counted only once. Thus, the estimates we need concern *trips* joining *improper loops* together, in addition to the first entering and the last exiting trip from γ . These estimates are the content of Lemma 4.12. The cost $\text{Weight}(z_i)/\text{Weight}(f_i(z_i))$ is bounded by $C_I^{2(k+1)+1}$, where $k+1$ is the number of successive blocks of $\tilde{\mathcal{C}}$ - \mathcal{C} loops. Since the total number of improper loops of all *types* is bounded by $\epsilon\sqrt{n}$, the total cost is negligible in our order of asymptotics.

The case where the number of \mathcal{C} and $\tilde{\mathcal{C}}$ -loops does not match is trickier. First, assume that we deal with

$$z_i = y_1 \mathbf{L}(i) \tilde{\mathbf{L}}(j) \dots \mathbf{L}(i+k) y'_1. \tag{5.23}$$

Here, we have no choice but to replace z_i with

$$f_i(z_i) = y_1 \mathcal{L}(\mathbf{L}(i)) \mathcal{L}^{-1}(\mathbf{L}(j)) \dots \mathcal{L}(\mathbf{L}(i+k)) y'_1. \tag{5.24}$$

Note that (5.18) holds.

Lastly, consider the case with more $\tilde{\mathcal{C}}$ -loops. For instance,

$$z_i = y_1 \tilde{\mathbf{L}}(i) \mathbf{L}(j) \tilde{\mathbf{L}}(i+1) \dots \mathbf{L}(j+k-1) \tilde{\mathbf{L}}(i+k) y'_1. \tag{5.25}$$

For reasons already mentioned, we cannot map the first $\tilde{\mathcal{C}}$ -loop into a \mathcal{C} -loop. We propose to keep the first loop unchanged, and act on the remaining loops, in the following way

$$f_i(z_i) = y_1 \tilde{\mathbf{L}}(i) \mathcal{L}^{-1}(\tilde{\mathbf{L}}_{i+1}) \mathcal{L}(\mathbf{L}(j)) \dots \mathcal{L}^{-1}(\tilde{\mathbf{L}}_{i+k}) \mathcal{L}(\mathbf{L}(j+k-1)) y'_1. \tag{5.26}$$

Here, as in (5.17), (5.18) holds, and this choice brings a multiplicity of pre-images. Indeed, $f_i(z_i)$ could have come from

$$z'_i = y_1 \mathcal{L}^{-1}(\tilde{\mathbf{L}}(i)) \tilde{\mathbf{L}}(i+1) \mathbf{L}(j) \dots \tilde{\mathbf{L}}_{i+k} \mathcal{L}(\mathbf{L}(j+k-1)) y'_1 \longrightarrow f_i(z_i).$$

So, in estimating the number of pre-images of a circuit, we find that it is at most 2 to the power of the number of *improper* loops. Now, the maximum number of *improper* loops is $\epsilon\sqrt{n}$. Also, the cost of transforming all *improper* loops is uniformly bounded by C_I to the power $\epsilon\sqrt{n}$.

6. Time-Scale.

In this section, we pass from an infinite time horizon to the *correct* time-scale \sqrt{n} . The simple idea is that we have to visit of the order of \sqrt{n} -times, a region of diameter uniformly bounded (in n). A renewal type argument shows that, with a large enough probability, this is realized in a time of order \sqrt{n} . Our main result is the following.

Proposition 6.1. *For any finite domain $\tilde{\Lambda} \subset \mathbb{Z}^d$, there are positive constants α_0, γ , such that for any large integer n , there is a sequence $\mathbf{k}_n^* = \{k_n^*(z), z \in \tilde{\Lambda}\}$ with*

$$\sum_{z \in \tilde{\Lambda}} k_n^*(z) \leq n, \quad \left\{ k_n^*(z) \in \left[\frac{\sqrt{n}}{A}, A\sqrt{n} \right], z \in \tilde{\Lambda} \right\}, \quad \text{and} \quad \sum_{z \in \tilde{\Lambda}} k_n^*(z)^2 \geq n\xi, \quad (6.1)$$

such that for any $\alpha > \alpha_0$

$$P_0 \left(\|\mathbb{1}_{\tilde{\Lambda}} l_\infty\|_2^2 \geq n\xi, \quad \tilde{\Lambda} \subset \mathcal{D}_\infty(A, \sqrt{n}) \right) \leq n^\gamma P_0 \left(l_{\lfloor \alpha\sqrt{n} \rfloor} |_{\tilde{\Lambda}} = \mathbf{k}_n^*, \quad S_{\lfloor \alpha\sqrt{n} \rfloor} = 0 \right). \quad (6.2)$$

Proof: We first use a rough upper bound

$$P_0 \left(\|\mathbb{1}_{\tilde{\Lambda}} l_\infty\|_2^2 \geq n\xi, \tilde{\Lambda} \subset \mathcal{D}_\infty(A, \sqrt{n}) \right) \leq \left| \left\{ \mathbf{k}_n \in \left[\frac{\sqrt{n}}{A}, A\sqrt{n} \right]^{\tilde{\Lambda}} \right\} \right| \times \max_{\mathbf{k}_n \text{ in (6.1)}} P(l_\infty |_{\tilde{\Lambda}} = \mathbf{k}_n). \quad (6.3)$$

We choose a sequence \mathbf{k}_n^* which maximizes the last term in (6.3). Then, we decompose $\{l_\infty |_{\tilde{\Lambda}} = \mathbf{k}_n^*\}$ into all possible circuits in a manner similar to the circuit decomposition of Section 4: We set $\nu = \sum_{\tilde{\Lambda}} k_n^*(x)$ (and $\nu \leq |\tilde{\Lambda}|A\sqrt{n}$), and

$$\mathcal{E}^* = \left\{ \mathbf{z} = (z(1), \dots, z(\nu)) \in \tilde{\Lambda}^\nu : l_{\mathbf{z}}(x) = k_n^*(x), \forall x \in \tilde{\Lambda} \right\}. \quad (6.4)$$

Then, we recall the notations before (4.3), and set $\tilde{T} = \inf \{n \geq 1 : S_n \in \tilde{\Lambda}\}$, (and $z(0) = 0$)

$$P_0(l_\infty |_{\tilde{\Lambda}} = \mathbf{k}_n^*) = \sum_{\mathbf{z} \in \mathcal{E}^*} \prod_{i=0}^{\nu-1} P_{z(i)} \left(\tilde{T}(z(i+1)) = \tilde{T} < \infty \right) P_{z(\nu)}(\tilde{T} = \infty). \quad (6.5)$$

For a fixed $\mathbf{z} \in \mathcal{E}^*$, we call $\tau^{(i)}$ the duration of the flight from $z(i-1)$ and $z(i)$ which avoids other sites of $\tilde{\Lambda}$, and $\sigma^{(i)} = \tau^{(1)} + \dots + \tau^{(i)}$. Thus, $\tau^{(1)} \stackrel{\text{law}}{=} \tilde{T}(z(1)) \mathbb{1}\{\tilde{T}(z(1)) = T\}$, when restricting on the values $\{1, 2, \dots\}$, and by induction

$$\tau^{(i)} \stackrel{\text{law}}{=} \tilde{T}(z(i)) \circ \theta_{\sigma^{(i-1)}} \mathbb{1}\left\{ \tilde{T}(z(i)) \circ \theta_{\sigma^{(i-1)}} = T \circ \theta_{\sigma^{(i-1)}} \right\} \quad (6.6)$$

If $\mathbb{T}(\mathbf{z}) = \{0 < \tau^{(i)} < \infty, \forall i = 1, \dots, \nu\}$, we have

$$P_0(\mathbb{T}(\mathbf{z})) = \prod_{i=0}^{\nu-1} P_{z(i)} \left(\tilde{T}(z(i+1)) = T < \infty \right). \quad (6.7)$$

Now, we fix $\mathbf{z} \in \mathcal{E}^*$ such that $P_0(\mathbb{T}(\mathbf{z})) > 0$, and we fix $i < \nu$. For ease of notations, we rename $x = z(i-1)$ and $y = z(i)$. Now, note that $\{0 < \tau^{(i)} < \infty\}$ contributes to (6.7) if $P_x(S_T = y) > 0$, or in other words, if there is at least one path going

from x to y avoiding other sites of $\tilde{\Lambda}$. Since $\tilde{\Lambda}$ has finite diameter, we can choose a finite length self-avoiding paths, and have

$$P_x \left(T(y) < \tilde{T}(x) \right) \geq c_\Lambda(x, y) = P_x(S_T = y, T < \infty) > c_\Lambda > 0, \quad (6.8)$$

where c_Λ is the minimum of $c_\Lambda(z, z')$ over all $z, z' \in \tilde{\Lambda}$ with $c_\Lambda(z, z') > 0$. Now, note that, when $S_0 = y$

$$\tilde{T}(y) \mathbb{1}_{T(x) < \tilde{T}(y) < \infty} \leq \tilde{T}(y) \mathbb{1}_{\tilde{T}(y) < \infty}. \quad (6.9)$$

Thus,

$$\begin{aligned} E_y \left[\tilde{T}(y) \mathbb{1}_{\tilde{T}(y) < \infty} \right] &\geq E_y \left[\mathbb{1}_{T(x) < \tilde{T}(y) < \infty} \left(\tilde{T}(y) \circ \theta_{T(x)} + T(x) \right) \right] \\ &= E_y \left[\mathbb{1}_{T(x) < \tilde{T}(y) < \infty} T(x) \right] + E_y \left[\mathbb{1}_{T(x) < \tilde{T}(y) < \infty} \tilde{T}(y) \circ \theta_{T(x)} \right]. \end{aligned} \quad (6.10)$$

Now, by the strong Markov's property

$$E_y \left[\tilde{T}(y) \mathbb{1}_{\tilde{T}(y) < \infty} \right] \geq P_y \left(T(x) < \tilde{T}(y) \right) E_x \left[T(y) \mathbb{1}_{T(y) < \infty} \right]. \quad (6.11)$$

By using translation invariance of the walk and (6.11), we obtain

$$E_x \left[T(y) \mathbb{1}_{T(y) = T < \infty} \right] \leq E_x \left[T(y) \mathbb{1}_{T(y) < \infty} \right] \leq \frac{E_0 \left[\tilde{T}(0) \mathbb{1}_{\tilde{T}(0) < \infty} \right]}{P_y \left(T(x) < \tilde{T}(y) \right)}. \quad (6.12)$$

Now, it is well known that there is a constant $c_d > 0$ such that for any integer k , $P_0(\tilde{T}(0) = k) \leq c_d/k^{\frac{d}{2}}$, which implies that $E_0 \left[\tilde{T}(0) \mathbb{1}_{\tilde{T}(0) < \infty} \right] < \infty$ in $d \geq 5$, and

$$\begin{aligned} E_x \left[T(y) | T(y) = T < \infty \right] &= \frac{E_x \left[T(y) \mathbb{1}_{T(y) = T < \infty} \right]}{P_x(T(y) = T < \infty)} \\ &\leq \frac{E_0 \left[\tilde{T}(0) \mathbb{1}_{\tilde{T}(0) < \infty} \right]}{P_x(T(y) = T < \infty) P_y(T(x) < \tilde{T}(y))} \\ &\leq \frac{E_0 \left[\tilde{T}(0) \mathbb{1}_{\tilde{T}(0) < \infty} \right]}{c_\Lambda^2}. \end{aligned} \quad (6.13)$$

When translating (6.13) in terms of the $\{\tau^{(i)}\}$, we obtain for any $\beta > 0$

$$P \left(\sum_{i=1}^{\nu} \tau^{(i)} > \beta \nu | \mathbb{T}(\mathbf{z}) \right) \leq \frac{E_0 \left[\sum_{i=1}^{\nu} \tau^{(i)} | \mathbb{T}(\mathbf{z}) \right]}{\beta \nu} \leq \frac{E_0 \left[\tilde{T}(0) \mathbb{1}_{\tilde{T}(0) < \infty} \right]}{c_\Lambda^2} \times \frac{1}{\beta}. \quad (6.14)$$

Thus, we can choose β_0 large enough (independent of \mathbf{z}) so that

$$P_0 \left(\sum_{i=1}^{\nu} \tau^{(i)} > \beta_0 \nu | \mathbb{T}(\mathbf{z}) \right) \leq \frac{1}{2}. \quad (6.15)$$

We use now

$$\begin{aligned} P_0(\mathbb{T}(\mathbf{z})) &= P_0 \left(\sum_{i=1}^{\nu} \tau^{(i)} > \beta_0 \nu | \mathbb{T}(\mathbf{z}) \right) P_0(\mathbb{T}(\mathbf{z})) + \\ &\quad P_0 \left(\sum_{i=1}^{\nu} \tau^{(i)} \leq \beta_0 \nu | \mathbb{T}(\mathbf{z}) \right) P_0(\mathbb{T}(\mathbf{z})), \end{aligned}$$

to conclude that

$$P_0(\mathbb{T}(\mathbf{z})) \leq 2P_0 \left(\left\{ \sum_{i=1}^{\nu} \tau^{(i)} \leq \beta_0 \nu \right\} \cap \mathbb{T}(\mathbf{z}) \right). \tag{6.16}$$

Now, there is α_0 such that $\beta_0 \nu \leq \alpha_0 \sqrt{n}$. Also, note that there is n_0 such that for any $z(\nu) \in \tilde{\Lambda}$, there is a path of length n_0 joining $z(\nu)$ to 0. Now, fix $\alpha > 2\alpha_0$, take n large enough so that $\lfloor \alpha \sqrt{n} \rfloor \geq \lfloor \alpha_0 \sqrt{n} \rfloor + n_0$, and use classical estimates on return probabilities, to obtain that for a constant C_d

$$P_0(\mathbb{T}(\mathbf{z})) \leq C_d(\alpha n)^{d/2} \sum_{\nu \leq k \leq \beta_0 \nu} P_0 \left(\left\{ \sum_{i=1}^{\nu} \tau^{(i)} = k \right\} \cap \mathbb{T}(\mathbf{z}) \right) \times P_{z(\nu)}(S_{n_0} = 0) P_0(S_{\lfloor \alpha \sqrt{n} \rfloor - (k+n_0)} = 0). \tag{6.17}$$

After summing over $z \in \mathcal{E}^*$, we obtain for any $\alpha > 2\alpha_0$

$$\sum_{z \in \mathcal{E}^*} P_0(\mathbb{T}(\mathbf{z})) \leq C_d(\alpha n)^{d/2} P_0 \left(\|\mathbb{I}_{\tilde{\Lambda}} l_{\lfloor \alpha \sqrt{n} \rfloor}\|_2^2 \geq n\xi, S_{\lfloor \alpha \sqrt{n} \rfloor} = 0 \right). \tag{6.18}$$

Note that another power of n arises from the first factor in the right hand site of (6.3) yielding the desired result. ■

7. Existence of a Limit.

We keep notations of Section 6. We reformulate Proposition 6.1 as follows. For any finite domain $\tilde{\Lambda} \subset \mathbb{Z}^d$, there are positive constants α_0, γ , such that for any $\alpha > \alpha_0$, and n large

$$P_0 \left(\|\mathbb{I}_{\tilde{\Lambda}} l_{\infty}\|_2^2 \geq n\xi, \tilde{\Lambda} \subset \mathcal{D}_{\infty}(A, \sqrt{n}) \right) \leq n^{\gamma} P_0 \left(\|\mathbb{I}_{\tilde{\Lambda}} l_{\lfloor \alpha \sqrt{n} \rfloor}\|_2 \geq \sqrt{n\xi}, S_{\lfloor \alpha \sqrt{n} \rfloor} = 0 \right). \tag{7.1}$$

Thus, (7.1) is the starting point in this section.

7.1. *A Sub-additive Argument.* We consider a fixed region $\Lambda \ni 0$, and first show the following lemma.

Lemma 7.1. *Let $q > 1$. For any $\xi > 0$ and Λ finite subset of \mathbb{Z}^d , the following limit exists*

$$\lim_{n \rightarrow \infty} \frac{\log(P_0(\|\mathbb{I}_{\Lambda} l_n\|_q \geq n\xi, S_n = 0))}{n} = -I(\xi, \Lambda). \tag{7.2}$$

Proof: We fix two integers K and n , with K to be taken first to infinity. Let m, r be integers such that $K = mn + r$, and $0 \leq r < n$. The phenomenon behind the sub-additive argument is that

$$\mathcal{A}_K(\xi, \Lambda) = \{ \|\mathbb{I}_{\Lambda} l_K\|_q \geq K\xi, S_K = 0 \} \tag{7.3}$$

is built by concatenating the *same* optimal scenario realizing $\mathcal{A}_n(\xi, \Lambda)$ on m consecutive time-periods of length n , and one last time-period of length r where the scenario is necessarily special and its cost innocuous. The crucial independence between the different periods is obtained as we force the walk to return to the origin at the end of each time period.

Our first step is to exhibit an optimal strategy realizing $\mathcal{A}_n(\xi, \Lambda)$. By optimizing over a finite number of variables $\{k_n(x), x \in \Lambda\}$ satisfying

$$\sum_{x \in \Lambda} k_n(x)^q \geq (n\xi)^q, \quad \text{and} \quad \sum_{x \in \Lambda} k_n(x) \leq n, \quad (7.4)$$

there is a sequence $\mathbf{k}_n^* = \{k_n^*(x), x \in \Lambda\}$ and $\gamma > 0$ (both depend on Λ) such that

$$P_0(\mathcal{A}_n(\xi, \Lambda)) \leq n^\gamma P_0(\mathcal{A}_n^*(\xi, \Lambda)), \quad \text{with} \quad \mathcal{A}_n^*(\xi, \Lambda) = \{l_n|_\Lambda = \mathbf{k}_n^*, S_n = 0\}. \quad (7.5)$$

Let $z^* \in \Lambda$, be the site where \mathbf{k}_n^* reaches its maximum. We start with the case $z^* = 0$, and postpone the case $z^* \neq 0$ to Remark 7.2. When $z^* = 0$, for any integer r , we call

$$\mathcal{R}_r = \{l_r(0) = r\}, \quad \text{and note that} \quad P_0(\mathcal{R}_r) = P_0(S_1 = 0)^{r-1} > 0. \quad (7.6)$$

Now, denote by $\mathcal{A}_n^{(1)}, \dots, \mathcal{A}_n^{(m)}$ m independent copies of $\mathcal{A}_n^*(\xi, \Lambda)$ which we realize on the successive increments of the random walk

$$\forall i = 1, \dots, m, \quad \mathcal{A}_n^{(i)} = \{l_{[(i-1)n, in]}|_\Lambda = \mathbf{k}_n^*, S_{in} = 0\}.$$

Make a copy of \mathcal{R}_r independent of $\mathcal{A}_n^{(1)}, \dots, \mathcal{A}_n^{(m)}$, by using increments after time nm : that is $\mathcal{R}_r = \{S_j = 0, \forall j \in [nm, K]\}$. Note that by independence

$$\begin{aligned} P_0(\mathcal{A}_n(\xi, \Lambda))^m P_0(\mathcal{R}_r) &\leq (n^\gamma)^m P_0(\mathcal{A}_n^{(1)}) \dots P_0(\mathcal{A}_n^{(m)}) P_0(\mathcal{R}_r) \\ &\leq (n^\gamma)^m P_0\left(\bigcap_{j=1}^m \mathcal{A}_n^{(j)}\right) \cap \mathcal{R}_r. \end{aligned} \quad (7.7)$$

Now, the local times is positive, so that

$$\begin{aligned} &\bigcap_{i=1}^m \{l_{[(i-1)n, in]}|_\Lambda = \mathbf{k}_n^*, S_{in} = 0\} \cap \{l_{[mn, K]}(0) = r\} \\ &\subset \left\{ \sum_{x \in \Lambda} \left[\sum_{i=1}^m l_{[(i-1)n, in]}(x) + l_{[mn, K]}(x) \right]^q \geq \sum_{x \in \Lambda} (mk_n^*(x) + r\delta_0(x))^q, S_K = 0 \right\}. \end{aligned}$$

At this point, observe the following fact whose simple inductive proof we omit: for $q > 1$, and for φ and ψ are positive functions on Λ , and for $z^* \in \Lambda$, $\varphi(z^*) = \max \varphi$, then

$$(\varphi(z^*) + \sum_{z \in \Lambda} \psi(z))^q + \sum_{z \neq z^*} \varphi(z)^q \geq \sum_{z \in \Lambda} (\varphi(z) + \psi(z))^q. \quad (7.8)$$

(7.8) implies that for any integer m

$$\begin{aligned} \sum_{x \in \Lambda} (mk_n^*(x) + r\delta_{z^*}(x))^q &\geq \sum_{x \in \Lambda} \left(mk_n^*(x) + \frac{r}{n} k_n^*(x) \right)^q \\ &= \left(m + \frac{r}{n} \right)^q \sum_{x \in \Lambda} k_n^*(x)^q \geq (mn + r)^q \xi^q = (K\xi)^q. \end{aligned} \quad (7.9)$$

Using (7.9), (7.7) yields

$$\begin{aligned} P_0(\mathcal{A}_n(\xi, \Lambda))^m P_0(\mathcal{R}_r) &\leq (n^\gamma)^m P_0(\|\mathbf{1}_\Lambda l_K\|_q \geq K\xi, S_K = 0) \\ &\leq (n^\gamma)^m P_0(\mathcal{A}_K(\xi, \Lambda)). \end{aligned} \quad (7.10)$$

We now take the logarithm on each side of (7.7)

$$\frac{nm}{nm+r} \frac{\log(P_0(\mathcal{A}_n(\xi, \Lambda)))}{n} + \frac{\log(P_0(\mathcal{R}_r))}{K} \leq \frac{m(\log(n^\gamma))}{nm+r} + \frac{\log(P_0(\mathcal{A}_K(\xi, \Lambda)))}{K}. \tag{7.11}$$

We take now the limit $K \rightarrow \infty$ while n is kept fixed (e.g. $m \rightarrow \infty$) so that

$$\frac{\log(P_0(\mathcal{A}_n(\xi, \Lambda)))}{n} \leq \frac{\log(n^\gamma)}{n} + \liminf_{K \rightarrow \infty} \frac{\log(P_0(\mathcal{A}_K(\xi, \Lambda)))}{K}. \tag{7.12}$$

By taking the limit sup in (7.12) as $n \rightarrow \infty$, we conclude that the limit in (7.2) exists.

Remark 7.2. We treat here the case $z^* \neq 0$. In this case, we cannot consider \mathcal{R}_r since to use (7.8), we would need the walk to start on site z^* , whereas each period of length n sees the walk returning to the origin. Note that this problem is related to the strategy on a single time-period of length r . The remedy is simple: we insert a period of length r into the first time-period of length n at the first time the walk hits z^* . During this period of length r , the walk stays at z^* during $r - 1$ steps. In other words, let $\tau^* = \inf\{n \geq 0 : S_n = z^*\}$, $\mathcal{R}_r^* = \{l_r(z^*) = r\}$ and note that

$$\begin{aligned} P_0(\mathcal{A}_n^{(1)})P_{z^*}(\mathcal{R}_r^*) &= \sum_{i=1}^n P_0(\mathcal{A}_n^{(1)}, \tau^* = i)P_{z^*}(l_r(z^*) = r) \\ &\leq P_0(l_{[0, n+r[} | \Lambda = \mathbf{k}_n^* + r\delta_{z^*}). \end{aligned} \tag{7.13}$$

Note that $P_{z^*}(\mathcal{R}_r^*) = P_0(\mathcal{R}_r)$, and that $\bigcap_{j=1}^m \mathcal{A}_n^{(j)}$ is a subset of

$$\left\{ \sum_{x \in \Lambda} \left[l_{[0, n+r[}(x) + \sum_{i=2}^m l_{[(i-1)n, in[}(x) \right]^q \geq \sum_{x \in \Lambda} (mk_n^*(x) + r\delta_{z^*}(x))^q, S_K = 0 \right\}.$$

We can now resume the proof of the case $z^* = 0$ at step (7.9). ■

7.2. Lower Bound in Proposition 1.6. We prove here the lower bound of (1.17). Call t_n be the integer part of $\alpha\sqrt{n}$, and consider the following scenario

$$\begin{aligned} \mathcal{S}_n(\Lambda, \alpha, \epsilon) &= \{ \|\mathbb{1}_\Lambda l_{[0, t_n[}\|_2^2 \geq n\xi(1 + \epsilon), S_{t_n} = 0 \} \\ &\quad \cap \{ \|\mathbb{1}_{[t_n, n[}\|_2^2 - E_0[\|l_n\|_2^2] \geq n\xi(1 - \epsilon) \}, \end{aligned} \tag{7.14}$$

Note that $\mathcal{S}_n(\Lambda, \alpha, \epsilon) \subset \{ \|\mathbb{1}_\Lambda l_n\|_2^2 \geq n\xi \}$. Indeed, note that for any $\beta \geq 1$, and $a, b > 0$ we have $a^\beta + b^\beta \leq (a + b)^\beta$. Thus, for any $x \in \mathbb{Z}^d$

$$l_{[0, t_n[}^2(x) + l_{[t_n, n[}^2(x) \leq l_n^2(x), \tag{7.15}$$

and we obtain on $\mathcal{S}_n(\Lambda, \alpha, \epsilon)$

$$E_0[\|l_n\|_2^2] + n\xi \leq \sum_{x \in \Lambda} l_{[0, t_n[}^2(x) + \sum_{x \in \mathbb{Z}^d} l_{[t_n, n[}^2(x) \leq \|l_n\|_2^2. \tag{7.16}$$

Note that $\|\mathbb{1}_\Lambda l_{[0, t_n[}\|_2$ and $S_{t_n} = 0$ only depend on the increments of the random walk in the time period $[0, t_n[$, whereas $\|\mathbb{1}_{[t_n, n[}\|_2$ depends on the increments in $[t_n, n[$. Thus,

$$\begin{aligned} P(\mathcal{S}_n(\Lambda, \alpha, \epsilon)) &= P_0(\|\mathbb{1}_\Lambda l_{[0, t_n[}\|_2^2 \geq n\xi(1 + \epsilon), S_{t_n} = 0) \\ &\quad \times P_0(\|\mathbb{1}_{[t_n, n[}\|_2^2 - E_0[\|l_n\|_2^2] \geq n\xi(1 - \epsilon)). \end{aligned} \tag{7.17}$$

Now, since $\frac{1}{n} \|l_n\|_2^2$ converges in L^1 towards γ_d , we have $E_0[\|l_n\|_2^2] \leq n\gamma_d(1 + \epsilon/2)$ for n large enough, and we have

$$P_0 \left(\|l_{[t_n, n]}\|_2^2 - E_0[\|l_n\|_2^2] \geq n\xi(1 - \epsilon) \right) \leq P_0 \left(\frac{\|l_{[0, n-t_n]}\|_2^2}{n - t_n} \geq \frac{\gamma_d - \frac{\epsilon}{2}\xi}{1 - \frac{t_n}{n}} \right) \rightarrow 1. \tag{7.18}$$

Remark 7.3. Note that for any Λ finite subset of \mathbb{Z}^d , any $\beta > 0$ and $\epsilon > 0$ small, we have for $\chi < \zeta < 1$, and n large enough

$$\{ \|\mathbb{1}_\Lambda l_{[\beta n^\zeta, \cdot]}\|_{\alpha^*} \geq \xi n^\zeta(1 + \epsilon), S_{[\beta n^\zeta, \cdot]} = 0 \} \subset \{ \|\mathbb{1}_{\bar{D}_n(\chi)} l_n\|_{\alpha^*} \geq \xi n^\zeta \}. \tag{7.19}$$

7.3. Proof of Theorem 1.1. First, the upper bound of Proposition 1.6 follows after combining inequalities (3.2), (4.7), (5.1) and (7.1). The lower bound of Proposition 1.6 is shown in the previous section. Then, we invoke Lemma 7.1 with $q = 2$, we take the logarithm on each sides of (1.17), we normalize by \sqrt{n} , and take the limit n to infinity. We obtain that for any $\epsilon > 0$, there are α_ϵ and Λ_ϵ such that for $\Lambda, \Lambda' \supset \Lambda_\epsilon$, and $\alpha, \alpha' > \alpha_\epsilon$

$$\begin{aligned} -\alpha' I\left(\frac{\sqrt{\xi(1 + \epsilon)}}{\alpha'}, \Lambda'\right) &\leq \liminf_{n \rightarrow \infty} \frac{\log \left(P_0(\|l_n\|_2^2 \geq n\xi) \right)}{\sqrt{n}} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \left(P_0(\|l_n\|_2^2 \geq n\xi) \right)}{\sqrt{n}} \leq -\alpha I\left(\frac{\sqrt{\xi(1 - \epsilon)}}{\alpha}, \Lambda\right) + C\epsilon. \end{aligned} \tag{7.20}$$

By using (7.20), we obtain for any $\Lambda, \Lambda' \supset \Lambda_\epsilon$, and $\alpha, \alpha' > \alpha_\epsilon$

$$\frac{\alpha'}{\sqrt{\xi(1 + \epsilon)}} I\left(\frac{\sqrt{\xi(1 + \epsilon)}}{\alpha'}, \Lambda'\right) \geq \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \frac{\alpha}{\sqrt{\xi(1 - \epsilon)}} I\left(\frac{\sqrt{\xi(1 - \epsilon)}}{\alpha}, \Lambda\right) - \frac{C\epsilon}{\sqrt{\xi(1 + \epsilon)}}. \tag{7.21}$$

Thus, if we call $\varphi(x, \Lambda) = I(x, \Lambda)/x$, we have: $\forall \epsilon > 0$, there is $x_\epsilon, \Lambda_\epsilon$ such that for $x, x' < x_\epsilon$ and $\Lambda, \Lambda' \supset \Lambda_\epsilon$

$$\varphi(x', \Lambda') \geq \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \varphi(x, \Lambda) - \frac{C\epsilon}{\sqrt{\xi(1 + \epsilon)}}. \tag{7.22}$$

By taking the limit $\Lambda' \nearrow \mathbb{Z}^d, x' \rightarrow 0$, and then $\Lambda \nearrow \mathbb{Z}^d$ and $x \rightarrow 0$, we reach for any $\epsilon > 0$

$$\liminf_{\Lambda \nearrow \mathbb{Z}^d, x \rightarrow 0} \varphi(x, \Lambda) \geq \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \limsup_{\Lambda \nearrow \mathbb{Z}^d, x \rightarrow 0} \varphi(x, \Lambda) - \frac{C\epsilon}{\sqrt{\xi(1 + \epsilon)}}. \tag{7.23}$$

Since (7.23) is true for $\epsilon > 0$ arbitrarily small, this implies that the limit of $\varphi(x, \Lambda)$ exists as x goes to 0 and Λ increases toward \mathbb{Z}^d . We call this latter limit $\mathcal{I}(2)$, where the label 2 stresses that we are dealing with the l^2 -norm of the local times.

Now, recall that the result of Asselah and Castell (2007b), (see Lemma 2.1) says that there are two positive constants \underline{c}, \bar{c} such that for x small enough $\underline{c} \leq I(x, \Lambda)/x \leq \bar{c}$, which together with (7.23) imply $0 < \underline{c} \leq \mathcal{I}(2) \leq \bar{c} < \infty$. Now, using (7.22) again, we obtain

$$\alpha I\left(\frac{\sqrt{\xi(1 + \epsilon)}}{\alpha}, \Lambda\right) \leq \frac{1 + \epsilon}{\sqrt{1 - \epsilon}} \mathcal{I}(2) \sqrt{\xi} + C\epsilon \sqrt{\frac{1 - \epsilon}{1 + \epsilon}}, \tag{7.24}$$

and,

$$\alpha I \left(\frac{\sqrt{\xi(1-\epsilon)}}{\alpha}, \Lambda \right) \geq \frac{1-\epsilon}{\sqrt{1+\epsilon}} \mathcal{I}(2) \sqrt{\xi} - C\epsilon \sqrt{\frac{1-\epsilon}{1+\epsilon}}. \quad (7.25)$$

This establishes the Large Deviations Principle of (1.8) as ϵ is sent to zero. \blacksquare

Proof of Proposition 1.4 Looking at the proof of Theorem 1.1, we notice that the only special feature of $\{\|l_n\|_2^2 \geq n\xi\}$ which we used, was that the excess self-intersection was realized on a **finite** set $\mathcal{D}_n(A, \sqrt{n})$. Similarly, when considering $\{\|\mathbb{1}_{\mathcal{D}_n(n^b)} l_n\|_{\alpha^*} \geq \xi n^\zeta\}$, inequality (2.21) of Lemma 2.3, ensures that our large deviation is realized on $\mathcal{D}_n(A, n^\zeta)$, and by (2.22), we make a negligible error assuming it is not finite. Thus, our key steps work in this case as well: *circuit surgery*, *renormalizing time*, and the *sub-additive argument*. Besides, by Remark 7.3, the lower bound follows trivially as well. Instead of (1.17), we would have that there is a constant β such that for any $\epsilon > 0$, there is $\tilde{\Lambda}$ set of finite diameter, and $a_0 > 0$, such that for Λ finite with $\Lambda \supset \tilde{\Lambda}$ and $a \geq a_0$,

$$\begin{aligned} P_0 \left(\|\mathbb{1}_{\Lambda} l_{[an^\zeta]}\|_{\alpha^*} \geq \xi(1+\epsilon)n^\zeta, S_{[an^\zeta]} = 0 \right) &\leq P_0 \left(\|\mathbb{1}_{\mathcal{D}_n(n^b)} l_n\|_{\alpha^*} \geq \xi n^\zeta \right) \\ &\leq e^{\beta \epsilon n^\zeta} P_0 \left(\|\mathbb{1}_{\tilde{\Lambda}} l_{[an^\zeta]}\|_{\alpha^*} \geq \xi(1-\epsilon)n^\zeta, S_{[an^\zeta]} = 0 \right). \end{aligned} \quad (7.26)$$

Following the last step of the proof of Theorem 1.1, we prove Proposition 1.4. \blacksquare

8. On Mutual Intersections.

8.1. *Identification of the rate function (1.9).* The main observation is that the proof of Theorem 1.1 yields also

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P \left(\|\mathbb{1}_{\Lambda} l_\infty\|_2^2 > n \right) = -\mathcal{I}(2). \quad (8.1)$$

Indeed, in order to use our sub-additive argument, Lemma 7.1, we need first to show that for some $\gamma > 0$, for any α large enough, and for n large enough

$$\begin{aligned} P_0 \left(\|\mathbb{1}_{\Lambda} l_{[\alpha\sqrt{n}]}\|_2^2 \geq n, S_{[\alpha\sqrt{n}]} = 0 \right) \\ \leq P \left(\|\mathbb{1}_{\Lambda} l_\infty\|_2^2 > n \right) \leq n^\gamma P_0 \left(\|\mathbb{1}_{\Lambda} l_{[\alpha\sqrt{n}]}\|_2^2 \geq n, S_{[\alpha\sqrt{n}]} = 0 \right). \end{aligned} \quad (8.2)$$

The upper bound in (8.2) is obtained from Proposition 6.1, whereas the lower bound is immediate.

Now, we proceed with the link with intersection local times. First, as mentioned in (1.6), Chen and Mörters prove also that for any finite $\Lambda \subset \mathbb{Z}^d$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \log P \left(\left\langle \mathbb{1}_{\Lambda} l_\infty, \tilde{l}_\infty \right\rangle > n \right) = -2I_{CM}(\Lambda),$$

with $I_{CM}(\Lambda)$ converging to I_{CM} as Λ increases to cover \mathbb{Z}^d . The important feature is that for any fixed $\epsilon > 0$, we can fix a finite Λ subset of \mathbb{Z}^d such that $|I_{CM}(\Lambda) - I_{CM}| \leq \epsilon$. Note now that by Cauchy-Schwarz' inequality, and for finite set Λ

$$\left\langle \mathbb{1}_{\Lambda} l_\infty, \tilde{l}_\infty \right\rangle \leq \|\mathbb{1}_{\Lambda} l_\infty\|_2 \|\mathbb{1}_{\Lambda} \tilde{l}_\infty\|_2. \quad (8.3)$$

Inequalities (8.1) and (8.3) imply by routine consideration that

$$\limsup_{\Lambda \nearrow \mathbb{Z}^d} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log P \left(\left\langle \mathbb{1}_{\Lambda} l_\infty, \tilde{l}_\infty \right\rangle > n \right) \leq -\mathcal{I}(2) \inf_{\alpha > 0} \left\{ \alpha + \frac{1}{\alpha} \right\} = -2\mathcal{I}(2). \quad (8.4)$$

When \mathbf{k}_n^* is the sequence which enters into defining $\mathcal{A}_n^*(1, \Lambda)$ in (7.5) (see also (7.4)), we have the lower bound

$$P\left(\left\langle \mathbb{1}_\Lambda l_\infty, \tilde{l}_\infty \right\rangle > n\right) \geq P\left(l_{\lfloor \alpha\sqrt{n} \rfloor} |_\Lambda = \mathbf{k}_n^* |_\Lambda, S_{\lfloor \alpha\sqrt{n} \rfloor} = 0\right)^2. \tag{8.5}$$

Following the same argument as in the proof of Section 7.3, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}\left(\left\langle l_\infty, \tilde{l}_\infty \right\rangle > n\right) \geq -2\mathcal{I}(2). \tag{8.6}$$

(8.4) and (8.6) conclude the proof (1.9).

8.2. *Proofs of Proposition 1.3.* Proposition 1.3 is based on the idea that $\langle l_\infty, \tilde{l}_\infty \rangle$ is not *critical* in the sense that even when *weighting less* intersection local times, the strategy remains the same. In other words, define for $1 < q \leq 2$

$$\zeta(q) = \sum_{z \in \mathbb{Z}^d} l_\infty(z) \tilde{l}_\infty^{q-1}(z). \tag{8.7}$$

Then, we have the following lemma, interesting on its own.

Lemma 8.1. *Assume that $d \geq 5$. For any $2 \geq q > \frac{d}{d-2}$, there is $\kappa_q > 0$ such that*

$$\mathbb{P}(\zeta(q) > t) \leq \exp(-\kappa_q t^{\frac{1}{q}}). \tag{8.8}$$

We prove Lemma 8.1 in the next section. Proposition 1.3 follows easily from Lemma 8.1. Indeed, if $\mathcal{D}(\xi) = \{z : \min(l_\infty(z), \tilde{l}_\infty(z)) \leq \xi\}$ and $q < 2$

$$\begin{aligned} \left\{ \left\langle \mathbb{1}_{\mathcal{D}(\epsilon\sqrt{t})} l_\infty, \tilde{l}_\infty \right\rangle > t \right\} &\subset \left\{ \sum_{l_\infty(z) \leq \epsilon\sqrt{t}} l_\infty(z)^{q-1} \tilde{l}_\infty(z) > \frac{t}{2} \left(\frac{1}{\epsilon\sqrt{t}} \right)^{2-q} \right\} \\ &\cup \left\{ \sum_{\tilde{l}_\infty(z) \leq \epsilon\sqrt{t}} l_\infty(z) \tilde{l}_\infty(z)^{q-1} > \frac{t}{2} \left(\frac{1}{\epsilon\sqrt{t}} \right)^{2-q} \right\}. \end{aligned} \tag{8.9}$$

Then, since $1 > \frac{2-q}{2}$, Lemma 8.1 applied to (8.15) implies that for large t

$$\mathbb{P}\left(\left\langle \mathbb{1}_{\mathcal{D}(\epsilon\sqrt{t})} l_\infty, \tilde{l}_\infty \right\rangle > t\right) \leq 2 \exp\left(-\kappa_d \left(\frac{1}{\epsilon}\right)^{\frac{2-q}{q}} t^{1/2}\right), \quad \text{since} \quad \frac{1}{q} \left(1 - \frac{2-q}{2}\right) = \frac{1}{2}. \tag{8.10}$$

8.3. *Proof of Lemma 8.1.* We assume $d \geq 5$. Lemma 8.1 can be thought of as an interpolation inequality between Lemma 1 and Lemma 2 of Khanin et al. (1994), whose proofs follow a classical pattern (in statistical physics) of estimating all moments of $\zeta(q)$. This control is possible since all quantities are expressed in terms of iterates of the Green’s function, whose asymptotics are well known (see for instance Theorem 1.5.4 of Lawler, 1991).

From Khanin et al. (1994), it is enough that for a positive constant C_q , we establish the following control on the moments

$$\forall n \in \mathbb{N}, \quad \mathbb{E}[\zeta(q)^n] \leq C_q^n (n!)^q. \tag{8.11}$$

First, noting that $q - 1 \leq 1$, we use Jensen’s inequality in the last inequality

$$\mathbb{E}[\zeta(q)^n] \leq \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} E_0 \left[\prod_{i=1}^n l_\infty(z_i) \right] E_0 \left[\prod_{i=1}^n l_\infty(z_i)^{q-1} \right]$$

$$\leq \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} \left(E_0 \left[\prod_{i=1}^n l_\infty(z_i) \right] \right)^q \quad (8.12)$$

If \mathcal{S}_n is the set of permutation of $\{1, \dots, n\}$ (with the convention that for $\pi \in \mathcal{S}_n$, $\pi(0) = 0$) we have,

$$\begin{aligned} E \left[\prod_{i=1}^n l_\infty(z_i) \right] &= \sum_{s_1, \dots, s_n \in \mathbb{N}} P_0(S_{s_i} = z_i, \forall i = 1, \dots, n) \\ &\leq \sum_{\pi \in \mathcal{S}_n} \sum_{s_1 \leq s_2 \leq \dots \leq s_n \in \mathbb{N}} P_0(S_{s_i} = z_{\pi(i)}, \forall i = 1, \dots, n) \\ &\leq \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)}). \end{aligned} \quad (8.13)$$

Now, by Hölder's inequality

$$\begin{aligned} \sum_{z_1, \dots, z_n} \left(\sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)}) \right)^q &\leq \sum_{z_1, \dots, z_n} (n!)^{q-1} \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n G_d(z_{\pi(i-1)}, z_{\pi(i)})^q \\ &= (n!)^q \sum_{z_1, \dots, z_n} \prod_{i=1}^n G_d(z_{i-1}, z_i)^q. \end{aligned} \quad (8.14)$$

Classical estimates for the Green's function, (8.14) implies that

$$\begin{aligned} \sum_{z_1, \dots, z_n \in \mathbb{Z}^d} \left(E_0 \left[\prod_{i=1}^n l_\infty(z_i) \right] \right)^q &\leq (n!)^q C^n \sum_{z_1, \dots, z_n} \prod_{i=1}^n (1 + \|z_i - z_{i-1}\|)^{q(2-d)} \\ &\leq (n!)^q C^n \left(\sum_{z \in \mathbb{Z}^d} (1 + \|z\|)^{q(2-d)} \right)^n. \end{aligned} \quad (8.15)$$

Thus, when $d \geq 5$ and $q > \frac{d}{d-2}$, we have a constant $C_q > 0$ such that

$$\mathbb{E}[\zeta(q)^n] \leq C_q^n (n!)^q \quad (8.16)$$

The proof concludes now by routine consideration (see e.g. [Khanin et al., 1994](#) or [Chen and Mörters, 2009](#)).

9. Applications to RWRS.

We consider a the range of parameters (α, β) given in (1.12). Also, if $\Gamma(x) = \log(E_Q[\exp(x\eta(0))])$, then there are positive constants Γ_0 and Γ_∞ (see [Asselah and Castell, 2007b](#)), such that

$$\lim_{x \rightarrow 0} \frac{\Gamma(x)}{x^2} = \Gamma_0, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{x^{\alpha^*}} = \Gamma_\infty, \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\alpha^*} = 1. \quad (9.1)$$

A classical way of obtaining large deviations is through exponential bounds for $\mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta)$. For instance, if we expect the latter quantity to be of order

$\exp(-cn^\zeta)$, then a first tentative would be to optimize over $\lambda > 0$ with $b = \beta - \zeta$ in the following

$$\begin{aligned} \mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta) &\leq e^{-\lambda n^{\beta-b}} E[\exp\left(\lambda \frac{\langle \eta, l_n \rangle}{n^b}\right)] \\ &\leq e^{-\lambda n^\zeta} E_0[\exp\left(\sum_{z \in \mathbb{Z}^d} \Gamma\left(\frac{\lambda l_n(z)}{n^b}\right)\right)]. \end{aligned} \quad (9.2)$$

We need to distinguish asymptotic regimes at zero or at infinity for $\Gamma(\frac{\lambda l_n(z)}{n^b})$ according to whether $l_n(x) < n^{b-\epsilon}$ or $l_n(x) > n^{b+\epsilon}$ respectively. For $\epsilon > 0$, we introduce

$$\bar{\mathcal{D}}_{b+\epsilon} = \{x \in \mathbb{Z}^d : l_n(x) \geq n^{b+\epsilon}\}, \quad \underline{\mathcal{D}}_{b-\epsilon} = \{x \in \mathbb{Z}^d : 0 < l_n(x) \leq n^{b-\epsilon}\},$$

and,

$$\mathcal{R}_\epsilon = \{x \in \mathbb{Z}^d : n^{b-\epsilon} \leq l_n(x) \leq n^{b+\epsilon}\}.$$

Then, for any $\epsilon_0 > 0$ small

$$\mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta) \leq \mathbf{P}\left(\langle \eta, \mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n \rangle \geq (1 - \epsilon_0)yn^\beta\right) + I_1 + I_2, \quad (9.3)$$

where

$$I_1 = \mathbf{P}\left(\langle \eta, \mathbb{1}_{\underline{\mathcal{D}}_{b-\epsilon}} l_n \rangle \geq \frac{\epsilon_0}{2}yn^\beta\right), \quad \text{and} \quad I_2 = \mathbf{P}\left(\langle \eta, \mathbb{1}_{\mathcal{R}_\epsilon} l_n \rangle \geq \frac{\epsilon_0}{2}yn^\beta\right). \quad (9.4)$$

We have now to show that the contribution of $\underline{\mathcal{D}}_{b-\epsilon}$ and \mathcal{R}_ϵ which concerns the *low* level sets, is negligible. We gather the two estimates in the next subsection. We treat afterwards $\bar{\mathcal{D}}_{b+\epsilon}$.

9.1. Contribution of small local times. We first show that I_1 is negligible. Set $\mathcal{B} = \left\{\|\mathbb{1}_{\underline{\mathcal{D}}_{b-\epsilon}} l_n\|_2^2 \geq \delta n^{\beta+b}\right\}$, for a $\delta > 0$ to be chosen later. For any $\lambda > 0$

$$\begin{aligned} \mathbf{P}\left(\langle \eta, \mathbb{1}_{\underline{\mathcal{D}}_{b-\epsilon}} l_n \rangle \geq \frac{\epsilon_0}{2}yn^\beta\right) &\leq \\ &P_0(\mathcal{B}) + e^{-\lambda n^{\beta-b} \frac{\epsilon_0}{2}y} E_0 \left[\mathbb{1}_{\mathcal{B}^c} \exp\left(\sum_{\underline{\mathcal{D}}_{b-\epsilon}} \Gamma\left(\frac{\lambda l_n(x)}{n^b}\right)\right) \right]. \end{aligned} \quad (9.5)$$

Now, for any $\lambda > 0$ and n large enough, we have for $x \in \underline{\mathcal{D}}_{b-\epsilon}$ that

$$\Gamma\left(\frac{\lambda l_n(x)}{n^b}\right) \leq \Gamma_0(1 + \epsilon_0)\left(\frac{\lambda l_n(x)}{n^b}\right)^2,$$

so that

$$\mathbf{P}\left(\langle \eta, \mathbb{1}_{\underline{\mathcal{D}}_{b-\epsilon}} l_n \rangle \geq \frac{\epsilon_0}{2}yn^\beta\right) \leq P_0(\mathcal{B}) + \exp\left(-n^\zeta \left(\lambda \frac{\epsilon_0}{2}y - \lambda^2 \Gamma_0(1 + \epsilon_0)\delta\right)\right). \quad (9.6)$$

Since $\beta+b > 1$, Lemma 1.8 of [Asselah and Castell \(2007b\)](#) gives that $-\log(P_0(\mathcal{B})) \geq Mn^\zeta$, for any $\delta > 0$, and any large constant M . Finally, for any ϵ_0 fixed, and a large constant M , we first choose λ so that $\lambda \frac{\epsilon_0}{2}y \geq 2M$. Then, we choose δ small enough so that $\lambda \Gamma_0 \delta \leq \frac{\epsilon_0}{4}y$.

We consider the contribution of \mathcal{R}_ϵ . We use here our hypothesis that the η are bell-shaped random variables, since it leads to clearer derivations. Thus, according

to Lemma 2.1 of Asselah and Castell (2007a), we have when integrating only over the field

$$Q(\langle \eta, 1\{\mathcal{R}_\epsilon\}l_n \rangle \geq yn^\beta) \leq Q\left(\sum_{\mathcal{R}_\epsilon} \eta(x) \geq n^{\beta-b-\epsilon}\right). \tag{9.7}$$

By Proposition 1.9 of Asselah and Castell (2007b), we can assume that $|\mathcal{R}_\epsilon| < n^\gamma$, with

$$\gamma < \gamma_0 = \frac{1}{1-\frac{2}{d}} \frac{\alpha-1}{\alpha+1} \beta = \frac{1-\frac{1}{\alpha}}{1-\frac{2}{d}} \zeta < \zeta \quad \text{if } \alpha < \frac{d}{2}. \tag{9.8}$$

Note that γ_0 given in (9.8) is lower than ζ when $\alpha < d/2$. Using Lemma A.4 of Asselah and Castell (2007a), we obtain

$$Q\left(\sum_{\mathcal{R}_\epsilon} \eta(x) \geq n^{\beta-b-\epsilon}, |\mathcal{R}_\epsilon| \leq n^\gamma\right) \leq \exp\left(-Cn^{\gamma+\alpha(\beta-b-\epsilon-\gamma)}\right). \tag{9.9}$$

For the left hand side of (9.9) to be negligible, we would need (recall that $\alpha > 1$)

$$\gamma + \alpha(\beta - b - \epsilon - \gamma) > \beta - b \iff (\beta - b)(\alpha - 1) > (\alpha - 1)\gamma \iff \zeta > \gamma. \tag{9.10}$$

This last inequality has already been noticed to hold in (9.8).

9.2. Contribution of large local times.

9.2.1. Upper Bound. We deal now with the contributions of $\bar{\mathcal{D}}_{b+\epsilon}$. For any $\lambda > 0$ (recalling that $\beta - b = \zeta = \frac{\alpha\beta}{\alpha+1}$), we integrate first the field

$$Q(\langle \eta, 1\{\bar{\mathcal{D}}_{b+\epsilon}\}l_n \rangle \geq (1 - \epsilon_0)yn^\beta) \leq e^{-\lambda n^{\beta-b}(1-\epsilon_0)y} \exp\left(\sum_{x \in \bar{\mathcal{D}}_{b+\epsilon}} \Gamma\left(\frac{\lambda l_n(x)}{n^b}\right)\right). \tag{9.11}$$

Now, for λ not too small, when n is large enough we have

$$\sum_{x \in \bar{\mathcal{D}}_{b+\epsilon}} \Gamma\left(\frac{\lambda l_n(x)}{n^b}\right) \leq (\Gamma_\infty + \epsilon_0)\lambda^{\alpha^*} \left(\frac{\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*}}{n^b}\right)^{\alpha^*}. \tag{9.12}$$

Thus, (9.11) yields that $Q(\langle \eta, 1\{\bar{\mathcal{D}}_{b+\epsilon}\}l_n \rangle \geq (1 - \epsilon_0)yn^\beta)$ is less than

$$\exp\left(-n^\zeta \left(\lambda(1 - \epsilon_0)y - \lambda^{\alpha^*}(\Gamma_\infty + \epsilon_0) \frac{\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*}}{n^{b\alpha^*+\zeta}}\right)\right). \tag{9.13}$$

Now, optimizing in λ in (9.13), we obtain

$$(1 - \epsilon_0)y = \alpha^*(\Gamma_\infty + \epsilon_0) \frac{\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*}}{n^{b\alpha^*+\zeta}} \lambda^{\alpha^*-1}. \tag{9.14}$$

Now, recall that in order to fall in the asymptotic regime of Γ at infinity, we assumed that λ were not too small. In other words, in view of (9.14), we would need to restrict to an event of the type $\{\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*} \leq An^\zeta\}$ for a large constant A . Now, using Proposition 1.4, there is a constant $\mathcal{I}(\alpha^*)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\zeta} \log P_0\left(\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*} \geq An^\zeta\right) \leq -\mathcal{I}(\alpha^*)A. \tag{9.15}$$

When restricting to $\{\|\mathbb{1}_{\mathcal{D}_{b+\epsilon}} l_n\|_{\alpha^*} \leq An^\zeta\}$, we can assume that λ satisfying (9.14) is bounded from below. Also, replacing the value of λ obtained in (9.14) in inequality (9.13), and using that $\Gamma_\infty^{-1} = \alpha^*(\alpha c_\alpha)^{\alpha^*-1}$, we obtain that

$$\begin{aligned} \mathbf{P}\left(\langle \eta, \mathbb{1}_{\{\bar{\mathcal{D}}_{b+\epsilon}\}} l_n \rangle \geq (1 - \epsilon_0)yn^\beta\right) &\leq E_0 \left[\exp\left(-c_\alpha(1 - \delta_0) \left(\frac{yn^\beta}{\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*}}\right)^\alpha\right) \right] \\ &+ P_0\left(\|\mathbb{1}_{\bar{\mathcal{D}}_{b+\epsilon}} l_n\|_{\alpha^*} \geq An^\zeta\right), \end{aligned} \quad (9.16)$$

where $(1 - \delta_0) = (1 - \epsilon_0)^\alpha(1 + \epsilon_0)^{1-\alpha}$, which can be made as close as 1, as one wishes. Now, it is easy to conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n^\zeta} \log \mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta) &\leq -\inf_{\xi > 0} \left\{ c_\alpha \left(\frac{y}{\xi}\right)^\alpha + \mathcal{I}(\alpha^*)\xi \right\} \\ &= -(\alpha + 1)c_\alpha^{\frac{1}{\alpha+1}} \left(\frac{y\mathcal{I}(\alpha^*)}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}. \end{aligned} \quad (9.17)$$

9.2.2. *Lower Bound for RWRs.* We call in this section $\bar{\mathcal{D}} = \{z \in \mathbb{Z}^d : l_n(z) \geq \delta n^\zeta\}$, for a fixed but small δ . Since, we have assumed the η -variables to have a bell-shaped distribution, we have according to Lemma 2.1 of Asselah and Castell (2007a),

$$\mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta) \geq \mathbf{P}(\langle \eta, \mathbb{1}_{\bar{\mathcal{D}}} l_n \rangle \geq yn^\beta). \quad (9.18)$$

Then, we condition on the random walk law, and average with respect to the η variables which we require to be large on each site of $\bar{\mathcal{D}}$. Recall now that we can assume $|\bar{\mathcal{D}}| \leq 1/\delta^2$ by (2.22) (for δ small enough). We use (9.18) to deduce for any $\epsilon > 0$

$$\begin{aligned} \mathbf{P}(\langle \eta, l_n \rangle \geq yn^\beta) &\geq E_0 \left[Q \left[\min_{z \in \bar{\mathcal{D}}} \eta(z) \geq \epsilon n^\zeta, \langle \eta, \mathbb{1}_{\bar{\mathcal{D}}} l_n \rangle \geq yn^\beta \right] \right] \\ &\geq E_0 \left[\sup_{x(i), i \in \bar{\mathcal{D}}} \left\{ C^{|\bar{\mathcal{D}}|} \exp\left(- (1 + \epsilon)c_\alpha^{\frac{1}{\alpha+1}} \sum_{i \in \bar{\mathcal{D}}} x(i)^\alpha\right) : \langle x, \mathbb{1}_{\bar{\mathcal{D}}} l_n \rangle \geq yn^\beta \right\} \right] \\ &\geq C^{1/\delta^2} E_0 \left[\mathbb{1}\left\{|\bar{\mathcal{D}}| \leq \frac{1}{\delta^2}\right\} \exp\left(-c_\alpha(1 + \epsilon) \left(\frac{yn^\beta}{\|\mathbb{1}_{\bar{\mathcal{D}}} l_n\|_{\alpha^*}}\right)^\alpha\right) \right] \\ &\geq C^{1/\delta^2} \exp\left(-c_\alpha(1 + \epsilon) \left(\frac{yn^\beta}{\xi^* n^\zeta}\right)^\alpha\right) P_0\left(\|\mathbb{1}_{\bar{\mathcal{D}}} l_n\|_{\alpha^*} \geq \xi^* n^\zeta, |\bar{\mathcal{D}}| \leq \frac{1}{\delta^2}\right), \end{aligned}$$

where ξ^* realizes the infimum in (9.17). Now, as ϵ is sent to 0 after n is sent to infinity, we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n^\zeta} \log P(\langle \eta, l_n \rangle \geq yn^\beta) \geq -(\alpha + 1)c_\alpha^{\frac{1}{\alpha+1}} \left(\frac{y\mathcal{I}(\alpha^*)}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}. \quad (9.19)$$

10. Appendix.

10.1. *Proof of Lemma 4.1.* Fix $\mathbf{k} \in V_L(\Lambda', n)$. By Chebychev's inequality, for any $\lambda > 0$

$$P\left(\sum_{i=1}^{|\mathbf{k}|} \mathbb{1}_{\{|S_{T^{(i)}} - S_{T^{(i-1)}}| > \sqrt{L}, T^{(|\mathbf{k}|)} < \infty\}} \geq \epsilon\sqrt{n}\right) \leq e^{-\lambda\epsilon\sqrt{n}} \times E\left[\prod_{i=1}^{|\mathbf{k}|} \exp\left(\lambda \mathbb{1}_{\{|S_{T^{(i)}} - S_{T^{(i-1)}}| > \sqrt{L}\}}\right)\right]. \quad (10.1)$$

Now, by using the strong Markov's property, and induction, we bound the right hand side of (10.1) by

$$e^{-\lambda\epsilon\sqrt{n}} \left(\sup_{z \in \Lambda' \cup \{0\}} E_z \left[\exp\left(\lambda \mathbb{1}_{\{|S_T| > \sqrt{L}, T < \infty\}}\right) \right] \right)^{|\mathbf{k}|}. \quad (10.2)$$

Now,

$$\begin{aligned} E_z[\exp(\lambda \mathbb{1}_{\{|S_T| > \sqrt{L}, T < \infty\}})] &\leq 1 + (e^\lambda - 1)P_z(|S_T| > \sqrt{L}, T < \infty) \\ &\leq 1 + (e^\lambda - 1)P_z\left(\cup\left\{T(\xi) < \infty : \xi \in \Lambda; |\xi - z| > \sqrt{L}\right\}\right) \\ &\leq 1 + (e^\lambda - 1)|\Lambda| \sup\left\{P_z(T(\xi) < \infty); |\xi - z| > \sqrt{L}, \xi \in \Lambda\right\} \\ &\leq 1 + (e^\lambda - 1)\frac{\bar{c}|\Lambda|}{L^{d/2-1}} \\ &\leq \exp\left((e^\lambda - 1)\frac{\bar{c}|\Lambda|}{L^{d/2-1}}\right). \end{aligned} \quad (10.3)$$

Now, since $|\mathbf{k}| \leq c_0\sqrt{n}$, we have

$$P\left(\sum_{i=1}^{|\mathbf{k}|} \mathbb{1}_{\{|S_{T^{(i)}} - S_{T^{(i-1)}}| > \sqrt{L}, T^{(|\mathbf{k}|)} < \infty\}} \geq \epsilon\sqrt{n}\right) \leq \exp\left(-\sqrt{n}\left(\lambda\epsilon - (e^\lambda - 1)\frac{c_0\bar{c}|\Lambda|}{L^{d/2-1}}\right)\right).$$

Thus, for any $\epsilon > 0$, we can choose L large enough so that the result holds. \blacksquare

10.2. *Proof of Lemma 4.8.* We first introduce a fixed scale, $l_0 \in \mathbb{N}$, to be adjusted later as a function of $|\Lambda|$, and assume that $|x - y| \geq 4|\Lambda|l_0$. Indeed, the case $|x - y| \leq 4|\Lambda|l_0$ is easy to treat since $P_x(S_T = y) > 0$ implies the existence of a path from x to y avoiding Λ ; it is then easy to see that since Λ is finite, the length of the shortest path joining x and y and avoiding Λ can be bounded by a constant depending only on $|\Lambda|$. Forcing the walk to follow this path costs only a positive constant which depends on $|\Lambda|$.

We introduce two sets of concentric shells around x and y : for $i = 1, \dots, |\Lambda| - 1$

$$C_i = B(x, (2i + 2)l_0) \setminus B(x, 2il_0), \quad \text{and} \quad C_0 = B(x, 2l_0), \quad (10.4)$$

and similarly $\{D_i, i = 0, \dots, |\Lambda|\}$ are centered around y , and for all i, j $C_i \cap D_j = \emptyset$. There is necessarily $i, j \leq |\Lambda|$ such that

$$C_i \cap \Lambda = \emptyset, \quad \text{and} \quad D_j \cap \Lambda = \emptyset. \quad (10.5)$$

Define now two stopping times corresponding to exiting $mid-C_i$

$$\sigma_i = \inf \{n \geq 0 : S_n \notin B(x, (2i+1)l_0)\},$$

and entering $mid-D_j$

$$\tau_j = \inf \{n \geq 0 : S_n \in B(y, (2j+1)l_0)\}. \quad (10.6)$$

Note that when $\sigma_i < \infty$ and $\tau_j < \infty$, we have $\text{dist}(S_{\sigma_i}, \Lambda) \geq l_0$, and $\text{dist}(S_{\tau_j}, \Lambda) \geq l_0$. We show that for any L we can find ϵ_L (going to 0 as $L \rightarrow \infty$), such that

$$P_x(T(\mathcal{S}) < T < \infty, S_T = y) \leq \frac{\epsilon_L}{2} P_x(S_T = y). \quad (10.7)$$

Note that (10.7) implies that for ϵ_L small enough

$$P_x(S_T = y) \leq \frac{1}{1 - \epsilon_L/2} P_x(T < T(\mathcal{S}), S_T = y) \leq e^{\epsilon_L} P_x(T < T(\mathcal{S}), S_T = y). \quad (10.8)$$

To show (10.7), we condition the flight $\{S_0 = x, S_T = y\}$ on its values at σ_i and τ_j

$$P_x(S_T = y) \geq \sum_{z \in C_i} P_x(S_{\sigma_i} = z, \sigma_i < T) P_z(\tau_j < T) \inf_{z' \in D_j} P_{z'}(S_T = y). \quad (10.9)$$

Note that if $P_x(S_T = y) > 0$, there is necessarily a path from D_j to y which avoids Λ so that, there is a constant c_0 (depending only on l_0) such that

$$\inf_{z' \in D_j} P_{z'}(S_T = y) > c_0. \quad (10.10)$$

We need to estimate $P_z(\tau_j < T)$. First, by classical estimates (see Proposition 2.2.2 of Lawler, 1991), there are $c_1, c_2 > 0$ such that when $|x - y| \geq 4l_0|\Lambda|$, and $z \in C_i$

$$\frac{c_2 \text{cap}(D_j)}{|z - y|^{d-2}} \leq P_z(\tau_j < \infty) \leq \frac{c_1 \text{cap}(D_j)}{|z - y|^{d-2}}. \quad (10.11)$$

We establish now that if we choose l_0 so that

$$l_0^{d-2} \geq 2|\Lambda| \frac{c_1 c_G 2^{d-2}}{c_2}, \quad \text{then} \quad P_z(\tau_j < T) \geq \frac{1}{2} P_z(\tau_j < \infty). \quad (10.12)$$

Since $\text{dist}(z, \Lambda) > l_0$

$$\begin{aligned} P_z(T < \tau_j < \infty) &\leq \sum_{\xi \in \Lambda \setminus D_0} P_z(S_T = \xi, T < \tau_j < \infty) P_\xi(\tau_j < \infty) \\ &\leq |\Lambda| \sup_{\xi \in \Lambda \setminus D_0} \{P_z(T(\xi) < \infty) P_\xi(\tau_j < \infty)\}. \end{aligned} \quad (10.13)$$

We use again estimate (10.11) to obtain

$$P_z(T < \tau_j < \infty) \leq c_1 c_G |\Lambda| \sup_{\xi \in \Lambda \setminus D_0} \left\{ \frac{1}{|z - \xi|^{d-2}} \times \frac{\text{cap}(D_j)}{|\xi - y|^{d-2}} \right\}. \quad (10.14)$$

Now, for $\xi \in \Lambda \setminus D_0$, we have $\min(|z - \xi|, |\xi - y|) > l_0$, and on the other side the triangle inequality yields $\max(|z - \xi|, |\xi - y|) > \frac{|z - y|}{2}$. Thus, we obtain

$$P_z(T < \tau_j) \leq \frac{c_1 c_G 2^{d-2}}{l_0^{d-2}} |\Lambda| \frac{\text{cap}(D_j)}{|z - y|^{d-2}}$$

$$\begin{aligned}
&\leq \frac{c_1 c_G 2^{d-2}}{c_2} \frac{|\Lambda|}{l_0^{d-2}} \frac{c_2 \text{cap}(D_j)}{|z-y|^{d-2}} \\
&\leq \frac{c_1 c_G 2^{d-2}}{c_2} \frac{|\Lambda|}{l_0^{d-2}} P_z(\tau_j < \infty). \tag{10.15}
\end{aligned}$$

This implies (10.12).

Now, for any $z \in C_i$, by conditioning on $S_{T(\mathcal{S})}$, we obtain

$$P_z(T(\mathcal{S}) < T < \infty, S_T = y) \leq E_z [\mathbf{1}\{T(\mathcal{S}) < T < \infty\} P_{S_{T(\mathcal{S})}}(S_T = y)] \leq \frac{c_G}{L^{d-2}}. \tag{10.16}$$

Thus, for any $z \in C_i$,

$$P_z(\tau_j < T) \inf_{z' \in D_j} P_{z'}(S_T = y) \geq c_0 \frac{c_2 \text{cap}(D_j)}{2|z-y|^{d-2}} \geq \frac{P_z(T(\mathcal{S}) < T < \infty, S_T = y)}{\epsilon_L/2}, \tag{10.17}$$

with (recalling that $|x-y| \geq 4|\Lambda|l_0$ and $|x-y| \leq \sqrt{L}$), with a constant $C(\Lambda) > 0$

$$\epsilon_L = \frac{4c_G|z-y|^{d-2}}{c_0 c_2 \text{cap}(D_j) L^{d-2}} \leq \frac{2^d c_G}{c_0 c_2 \text{cap}(D_j)} \left(\frac{|x-y|}{L}\right)^{d-2} \leq C(\Lambda) \left(\frac{1}{\sqrt{L}}\right)^{d-2}. \tag{10.18}$$

Now, after summing over $z \in C_i$, we obtain (10.7). \blacksquare

10.3. *Proof of Lemma 4.9.* We consider two cases: (i) $\sqrt{L} < |x-y| \leq \kappa L$ where κ is a small parameter, and (ii) $|x-y| > \kappa L$.

Also, we denote by $C(\lambda)$ a positive constant which depend only on $|\Lambda|$. We might use the same name in different places.

Case (i). We use the same steps as in the previous proof up to (10.17) where we replace $|z-y|$ by $2|x-y|$, and obtain

$$P_x(S_T = y) \geq \frac{c_0}{2} \frac{c_2 \text{cap}(D_j)}{2^{d-2}|x-y|^{d-2}}. \tag{10.19}$$

Now, (10.16) implies that if

$$\kappa^{d-2} \leq \frac{c_0 c_2 \text{cap}(D_j)}{2^d c_G}, \quad \text{then} \quad P_x(S_T = y) \leq 2P_x(T < T(\mathcal{S}), S_T = y). \tag{10.20}$$

Case (ii). First note that

$$P_x(S_T = y) \leq P_x(T(y) < \infty) \leq \frac{c_G}{|x-y|^{d-2}}. \tag{10.21}$$

Now, set $L' = \kappa L$, and note that $\text{diam}(\mathcal{C})$ is a multiple (depending only on Λ) times L' . Now, a way of realizing $\{S_T = y, T < T(\mathcal{S})\}$ is to go through a finite number of adjacent spheres of diameter L' . From a hitting point on one sphere, we force the walk to exit only from a tiny fraction of the surface of the next sphere, until we reach the last sphere, say on z^* , for which it is easy to show that there are two universal positive constants c, c' such that

$$P_{z^*}(S_T = y, T < T(\mathcal{S})) \geq c P_{z^*}(T(y) < \infty) \geq c' \frac{\tilde{c}_G}{|x-y|^{d-2}}. \tag{10.22}$$

Note that when starting on x , the probability of exiting $B(x, |x-y|)$ through site y is of order of the surface $|x-y|^{1-d}$, and this is much smaller of $P_x(T(y) < \infty)$ which should be close to $P_x(S_T = y)$ in cases where all other points of Λ be very far

from x, y . Thus, we have to consider more paths than $\{S_{T(B(x,|x-y|)^c)} = y, S_0 = x\}$. By Lemma 3.1 and Remark 3.2, there is a finite sequence x_1, \dots, x_k (not necessarily in \mathcal{C}) such that $L'/2 \leq |x_{i+1} - x_i| \leq L'$ and such that $B(x_i, L) \subset \mathcal{S}(\mathcal{C})$.

$$\delta = \frac{1}{4|\Lambda|^{\frac{1}{d-1}}}, \quad Q_i = \{z : |z - x_i| = |x_{i+1} - x_i|\}, \quad \text{and} \quad \Sigma_i = Q_i \cap B(x_{i+1}, \frac{L'}{4}). \quad (10.23)$$

Note that $|\Sigma_i|$ is of order $(\frac{L'}{4})^{d-1}$. We can throw $|\Lambda|$ points on Σ_i , say at a distance of at least $\delta L'$, and one of them, say y_i^* , necessarily satisfies

$$B(y_i^*, \delta L') \cap \Lambda = \emptyset, \quad \text{and set} \quad B_i^* = B(y_i^*, \frac{\delta L'}{2}) \cap \Sigma_i. \quad (10.24)$$

Now, when the walk starts on x_{i+1} , it exits from any point $z \in Q_{i+1}$ with roughly the same chances (see i.e. Lemma 1.7.4 of Lawler, 1991), so that there is c_S such that for $i \geq 0$,

$$P_{x_{i+1}}(S_{H_i} = z) \geq \frac{c_S}{|x_{i+2} - x_{i+1}|^{d-1}}, \quad \text{where} \quad H_i = T(Q_{i+1}). \quad (10.25)$$

By Harnack's inequality (see Theorem 1.7.2 of Lawler, 1991), for any $z \in B_i^*$

$$P_z(S_{H_i} \in B_{i+1}^*) \geq \frac{c_S |B_{i+1}^*|}{(2L)^{d-1}} \quad (10.26)$$

Now, there is $\chi > 0$ such that

$$|B_i^*| \geq \chi(\delta L')^{d-1},$$

which yields

$$P_z(S_{H_i} \in B_{i+1}^*) \geq c_S \chi \left(\frac{\delta \kappa}{2}\right)^{d-1}. \quad (10.27)$$

Note that it costs more to hit Λ before Q_{i+1}^c . Indeed,

$$\begin{aligned} P_z(S_{H_i} \in B_{i+1}^*, T < H_i) &\leq \sum_{\xi \in \Lambda} P_z(T(\xi) < \infty) P_\xi(H_i < \infty) \\ &\leq \sup_{\xi \in \Lambda} \frac{c_G |\Lambda|}{|z - \xi|^{d-2}} \times \frac{c_1 \text{cap}(B_{i+1}^*)}{|\xi - y_{i+1}^*|^{d-2}}. \end{aligned} \quad (10.28)$$

By definition, $\text{cap}(B_{i+1}^*) \leq |B_{i+1}^*| \leq (\delta L')^{d-1}$. Now, z and y_{i+1}^* are chosen in such a way that $\min(|z - \xi|, |\xi - y_{i+1}^*|) \geq L'/2$ so that for a positive constant c_2 ,

$$P_z(S_{H_i} \in B_{i+1}^*, T < H_i) \leq \frac{c_S c_1 |\Lambda| (\delta L')^{d-1}}{(L'/2)^{2d-4}} \leq \frac{c_2 |\Lambda| (\delta)^{d-1}}{(L')^{d-3}}. \quad (10.29)$$

Since in $d \geq 5$, we have $d - 3 > 0$, L can be chosen large enough so that

$$P_z(S_{H_i} \in B_{i+1}^*, H_i < T) \geq \frac{1}{2} P_z(S_{H_i} \in B_{i+1}^*) \quad (10.30)$$

Now, we define θ_k as the time-translation of k units of a random walk trajectory, and $\tilde{H}_i = H_i \circ \theta_{H_{i-1}}$. The following scenario produces $\{S_T = y, T < T(\mathcal{S})\}$:

$$\bigcap_{i=1}^k \left\{ S_{\tilde{H}_i} \in B_{i+1}^*, \tilde{H}_i < T \circ \theta_{H_{i-1}} \right\} \cap \left\{ S_{T \circ \theta_{H_k}} = y, T \circ \theta_{H_k} < T(\mathcal{S}) \circ \theta_{H_k} \right\} \quad (10.31)$$

By using the strong Markov's property, and (10.30), we obtain

$$P_x(S_T = y, T < T(\mathcal{S})) \geq \left(\frac{c_S \chi}{2} \left(\frac{\delta \kappa}{2} \right)^{d-1} \right)^k \inf_{z \in B_k^*} P_z(S_T = y, T < T(\mathcal{S})). \quad (10.32)$$

In the last term in (10.32), note that for any $z \in B_k^*$, $L'/2 \leq |z - y| \leq L'$ so that we are in the situation of Case(i), where inequality (10.20), and (10.19) yields

$$P_z(S_T = y, T < T(\mathcal{S})) \geq \frac{1}{2} P_z(S_T = y) \geq \frac{c}{|z - y|^{d-2}}.$$

Since Lemma 3.1 establishes that for some constant $C(\Lambda) > 0$, $\text{diam}(\mathcal{C}) \leq C(\Lambda)L$, and $|z - y| \geq \frac{\kappa}{2}L$, we have for a constant $C(\Lambda)$

$$P_x(S_T = y, T < T(\mathcal{S})) \geq \frac{C(\Lambda)}{|x - y|^{d-2}} \geq \frac{C(\Lambda)}{c_G} P_x(S_T = y). \quad \blacksquare$$

10.4. *Proof of Lemma 4.10.* We start with shorthand notations $\mathcal{S}_1 = \mathcal{S}(\mathcal{C})$ and $\tilde{\mathcal{S}}_1 = \mathcal{S}(\mathcal{L}(\mathcal{C}))$, and we define

$$\mathcal{S}_2 = \{z : \text{dist}(z, \mathcal{C}) = 2 \max(\text{diam}(\mathcal{C}), L)\},$$

and $\tilde{\mathcal{S}}_2$ is similar to \mathcal{S}_2 but $\mathcal{L}(\mathcal{C})$ is used instead of \mathcal{C} in its definition.

First, we obtain an upper bound for the weights of paths joining y to x by conditioning over hitting sites on \mathcal{S}_2 and \mathcal{S}_1 , and by using the strong Markov's property

$$\begin{aligned} P_y(S_T = x) &= \sum_{z_1 \in \mathcal{S}_1} E_y \left[\mathbb{1}_{\{T(\mathcal{S}_2) < T\}} P_{S_T(\mathcal{S}_2)}(S_{T(\mathcal{S}_1)} = z_1, T(\mathcal{S}_1) < T) \right] P_{z_1}(S_T = x) \\ &\leq P_y(T(\mathcal{S}_2) < \infty) \sum_{z_1 \in \mathcal{S}_1} \left(\sup_{z \in \tilde{\mathcal{S}}_2} P_z(S_{T(\mathcal{S}_1)} = z_1) \right) P_{z_1}(S_T = x) \end{aligned} \quad (10.33)$$

We need to compare (10.33) with the corresponding decomposition for trajectories starting on y with $\{S_T = \tilde{x}\}$, where we set $\tilde{x} = \mathcal{L}(x)$ for simplicity,

$$\begin{aligned} P_y(S_T = \tilde{x}) &= \sum_{\tilde{z}_1 \in \tilde{\mathcal{S}}_1} E_y \left[\mathbb{1}_{\{T(\tilde{\mathcal{S}}_2) < T\}} P_{S_T(\tilde{\mathcal{S}}_2)}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1, T(\tilde{\mathcal{S}}_1) < T) \right] P_{\tilde{z}_1}(S_T = \tilde{x}) \\ &\geq P_y(T(\tilde{\mathcal{S}}_2) < T) \sum_{\tilde{z}_1 \in \tilde{\mathcal{S}}_1} \inf_{\tilde{z} \in \tilde{\mathcal{S}}_2} P_{\tilde{z}}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1, T(\tilde{\mathcal{S}}_1) < T) P_{\tilde{z}_1}(S_T = \tilde{x}). \end{aligned} \quad (10.34)$$

We now bound each term in (10.33) by the corresponding one in (10.34).

About $P_{z_1}(S_T = x)$. From (3.3) of Lemma 3.1, $\mathcal{S}_2 \cap \Lambda = \mathcal{C}$. By the same reasoning as in the proof of Lemma 4.9, there is a constant C_0 such that for any $z_1 \in \mathcal{S}_1$

$$P_{z_1}(S_T = x) \leq C_0 P_{z_1}(S_T = x, T < T(\mathcal{S}_2)). \quad (10.35)$$

As long as we consider paths from \mathcal{S}_1 to x which do not escape \mathcal{S}_2 , we can transport them, using translation invariance of the law of random walk

$$P_{z_1}(S_T = \tilde{x}, T < T(\tilde{\mathcal{S}}_2)) = P_{z_1}(S_T = x, T < T(\mathcal{S}_2)), \quad (10.36)$$

and by using (10.35) and (10.36), we finally obtain

$$P_{z_1}(S_T = x) \leq C_0 P_{z_1}(S_T = \tilde{x}, T < T(\tilde{\mathcal{S}}_2)) \leq C_0 P_{\tilde{z}_1}(S_T = \tilde{x}). \quad (10.37)$$

About $P_y(T(\mathcal{S}_2) < \infty)$. By Proposition 2.2.2 of Lawler (1991), there are c_1, c_2 positive constants such that

$$\frac{c_2 \text{cap}(\mathcal{S}_2)}{|y-x|^{d-2}} \leq P_y(T(\mathcal{S}_2) < \infty) \leq \frac{c_1 \text{cap}(\mathcal{S}_2)}{|y-x|^{d-2}}, \quad (10.38)$$

and (10.38) holds also with a tilda over x and \mathcal{S}_2 . Since $|y-\tilde{x}| \leq 2|y-x|$ by (3.17), we have

$$P_y(T(\mathcal{S}_2) < \infty) \leq \frac{c_1}{c_2} 2^{d-2} P_y(T(\tilde{\mathcal{S}}_2) < \infty). \quad (10.39)$$

We need now to check that paths reaching $\tilde{\mathcal{S}}_2$ from y have good chances not to meet any sites of Λ . In other words, we need

$$P_y(T(\tilde{\mathcal{S}}_2) < \infty) \leq 2P_y(T(\tilde{\mathcal{S}}_2) < T). \quad (10.40)$$

The argument is similar to the one showing $P_z(\tau_j) \leq 2P_z(\tau_j < T)$ in (10.12) of the proof of Lemma 4.9. We omit to reproduce it. Thus, from (10.40) and (10.39),

$$P_y(T(\mathcal{S}_2) < \infty) \leq \frac{2^{d-1}c_1}{c_2} P_y(T(\tilde{\mathcal{S}}_2) < T). \quad (10.41)$$

We show that starting from $\tilde{z} \in \tilde{\mathcal{S}}_2$, a walk has good chances of hitting $\tilde{\mathcal{S}}_1$ before Λ , as we show (10.40), and here again we omit the argument showing that for any $\tilde{z}_1 \in \tilde{\mathcal{S}}_1$

$$P_{\tilde{z}}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1) \leq 2P_{\tilde{z}}(T(\tilde{\mathcal{S}}_1) < T, S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1). \quad (10.42)$$

About the supremum in (10.33). Now, by Harnack's inequality for the discrete Laplacian (see Theorem 1.7.2 of Lawler, 1991), there is $c_H > 0$ independent of n such that for any $z_2, z'_2 \in \mathcal{S}_2$, and any $z_1 \in \mathcal{S}_1$

$$P_{z_2}(S_{T(\mathcal{S}_1)} = z_1) \leq c_H P_{z'_2}(S_{T(\mathcal{S}_1)} = z_1). \quad (10.43)$$

Now, using (10.42), and the obvious fact

$$P_{z'_2}(S_{T(\mathcal{S}_1)} = z_1) = P_{\mathcal{L}(z'_2)}(S_{T(\tilde{\mathcal{S}}_1)} = \mathcal{L}(z_1)),$$

we obtain for any $z_1 \in \mathcal{S}_1$

$$\begin{aligned} \sup_{z \in \mathcal{S}_2} P_z(S_{T(\mathcal{S}_1)} = z_1) &\leq c_H \inf_{\tilde{z} \in \tilde{\mathcal{S}}_2} P_{\tilde{z}}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1) \\ &\leq 2c_H \inf_{\tilde{z} \in \tilde{\mathcal{S}}_2} P_{\tilde{z}}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1, T(\tilde{\mathcal{S}}_1) < T). \end{aligned} \quad (10.44)$$

Starting with (10.33), and combining (10.37), (10.46), and (10.44), we obtain

$$\begin{aligned} P_y(S_T = x) &\leq P_y(T(\mathcal{S}_2) < \infty) \sum_{z_1 \in \mathcal{S}_1} \left(\sup_{z \in \mathcal{S}_2} P_z(S_{T(\mathcal{S}_1)} = z_1) \right) P_{z_1}(S_T = x) \\ &\leq \frac{2^{d-1}c_1}{c_2} P_y(T(\tilde{\mathcal{S}}_2) < T) \sum_{\tilde{z}_1 \in \tilde{\mathcal{S}}_1} 2c_H \inf_{\tilde{z} \in \tilde{\mathcal{S}}_2} P_{\tilde{z}}(S_{T(\tilde{\mathcal{S}}_1)} = \tilde{z}_1, T(\tilde{\mathcal{S}}_1) < T) \\ &\quad \times C_0 P_{\tilde{z}_1}(S_T = \tilde{x}) \leq P_y(S_T = \mathcal{L}(x)). \end{aligned}$$

■

10.5. *Proof of Lemma 4.12.* We only prove the first inequality in (4.20), the second is similar. The proof uses arguments used in the proof of Lemma 4.9, and Lemma 4.10. Namely, consider $x, x' \in \mathcal{C}$, and draw shells $\{C_k\}$ and $\{D_k\}$ as in (10.4) but around x and x' respectively. Note that here $C_k \cap D_{k'}$ may not be empty. Also, choose i and j such that condition (10.5) holds. Then, we decompose $\{S_T = x\}$ by conditioning on \mathcal{S}_1 as in (10.33). On the term $P_{z_1}(S_T = x)$ we use the following rough bound

$$P_{z_1}(S_T = x) \leq P_{z_1}(T(x) < \infty) \leq \frac{c_d}{|z_1 - x|^{d-2}}. \quad (10.45)$$

We now use the obvious observation that $2|z_1 - x| \geq |z_1 - x'|$. Indeed, $|z_1 - x| \geq \text{diam}(\mathcal{C}) \geq |x - x'|$ implies that $2|z_1 - x| \geq |z_1 - x| + |x - x'| \geq |z_1 - x'|$ by the triangle inequality. Thus there are a constant c_3 such that for the hitting time τ_j defined in (10.6)

$$P_{z_1}(\tau_j < \infty) \geq \frac{c_2 \text{cap}(D_j)}{|z_1 - x'|^{d-2}} \geq \frac{c_3}{|z_1 - x|^{d-2}}. \quad (10.46)$$

From (10.33) and (10.46), we have

$$P_y(S_T = x) \leq \frac{c_d}{c_3} \sum_{z_1 \in \mathcal{S}_1} P_y(T(\mathcal{S}_1) < T, S_{T(\mathcal{S}_1)} = z_1) P_{z_1}(\tau_j < \infty) \quad (10.47)$$

By argument (10.15), and the choice of l_0 in (10.12), we have $2P_{z_1}(\tau_j < T) \geq P_{z_1}(\tau_j < \infty)$. Finally, from D_j to x' , there is a path avoiding $\Lambda' \setminus \{x'\}$ which cost a bounded amount depending only on l_0 . ■

10.6. *Proof of Corollary 4.13.* Note that by Lemma 4.12, we have

$$P_x(S_T = y) \leq C_I P_x(S_T = y'). \quad (10.48)$$

Now, $P_x(S_T = y') = P_{y'}(S_T = x)$, and we use again Lemma 4.12

$$P_{y'}(S_T = x) \leq C_I P_{y'}(S_T = x') \implies P_x(S_T = y) \leq C_I^2 P_{x'}(S_T = y'). \quad (10.49)$$

■

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