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A strictly stationary, "causal," 5–tuplewise independent counterexample to the central limit theorem

Richard C. Bradley

Department of Mathematics, Indiana University, Bloomington, Indiana 47405, U.S.A.
E-mail address: bradleyr@indiana.edu

Abstract. A strictly stationary sequence of random variables is constructed with the following properties: (i) the random variables take the values -1 and $+1$ with probability $1/2$ each, (ii) every five of the random variables are independent of each other, (iii) the sequence is "causal" in a certain sense, (iv) the sequence has a trivial double tail σ-field, and (v) regardless of the normalization used, the partial sums do not converge to a (nondegenerate) normal law. The example has some features in common with a recent construction (for an arbitrary fixed positive integer N), by Alexander Pruss and the author, of a strictly stationary N-tuplewise independent counterexample to the central limit theorem.

1. Introduction

For a given integer $N \geq 2$ and a given sequence $X := (X_k, k \in \mathbf{Z})$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, the random variables $X_k, k \in \mathbf{Z}$ are said to be "N-tuplewise independent" if for every choice of N distinct integers $k(1), k(2), \ldots, k(N)$, the random variables $X_{k(1)}, X_{k(2)}, \ldots, X_{k(N)}$, are independent. For $N = 2$ (resp. $N = 3$), the word "N-tuplewise" is also expressed as "pairwise" (resp. "triplewise").

Etemadi (1981) proved a strong law of large numbers for sequences of pairwise independent, identically distributed random variables with finite absolute first moment. Janson (1988) showed with several classes of counterexamples that for strictly stationary sequences of pairwise independent, nondegenerate, square-integrable random variables, the Central Limit Theorem (henceforth abbreviated CLT) need not hold. Subsequently, Bradley (1989, Theorem 1) constructed another such counterexample, a 3-state one that has the additional property of satisfying

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the absolute regularity (weak Bernoulli) condition. (The definition of that condition will be given later in this Introduction.) Yet another counterexample was constructed by Cuesta and Matrán (1991, Section 2.3), a construction based on elementary number-theoretic properties of addition on $\{0, 1, \ldots, p-1\}$ mod p, where p is a prime number.

For an arbitrary fixed integer $N \geq 3$, Pruss (1998) constructed a (not strictly stationary) sequence of bounded, nondegenerate, N-tuplewise independent, identically distributed random variables for which the CLT fails to hold. In that paper, Pruss left open the question whether, for any integer $N \geq 3$, a strictly stationary counterexample exists. For $N = 3$, Bradley (2007b, Theorem 1) answered that question affirmatively by showing that the counterexample in Bradley (1989, Theorem 1) alluded to above is in fact triplewise independent. Recently, Bradley and Pruss (2009) have answered that question affirmatively for arbitrary $N \geq 3$, with a (strictly stationary) counterexample adapted from the (nonstationary) one in Pruss (1998).

In a similar spirit, for an arbitrary integer $N \geq 2$, Flaminio (1993) constructed a (nondegenerate) strictly stationary, finite-state, N-tuplewise independent random sequence $X := (X_k, k \in \mathbf{Z})$ which also has zero entropy and is mixing (in the ergodic-theoretic sense). That paper explicitly left open the question of whether those examples satisfy the CLT.

In this paper here, a (nondegenerate) strictly stationary, 5-tuplewise independent counterexample to the CLT will be constructed which is finite-state and has the extra property of being "causal" and therefore "Bernoulli." The interest in the property of "Bernoulli," for (finite-state) N-tuplewise independent counterexamples to the CLT, was suggested to the author by Jon Aaronson and Benjamin Weiss. There does not seem to be a visible way of constructing such a (strictly stationary) finite-state Bernoulli counterexample which is N-tuplewise independent for any given $N \geq 6$. The techniques in the example given here do not appear to adapt effectively to $N \geq 6$; and there is no visible way of adapting the recent example of Bradley and Pruss (2009) alluded to above, into one that is (finite-state and) Bernoulli. The example given here will also have the further property of possessing a trivial double tail σ-field. (The terms "causal," "Bernoulli," and "double tail σ-field" will be defined below.)

The main result and discussions. Before the result is stated, some notations will be needed.

Let $\mathbf{N}$ denote the set of all positive integers. Let $\mathcal{R}$ denote the Borel σ-field on the real number line $\mathbf{R}$. The notation $\Rightarrow$ will mean convergence in distribution.

Suppose $X := (X_k, k \in \mathbf{Z})$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. For each positive integer n, define the partial sum

$$S_n := S(X, n) := X_1 + X_2 + \cdots + X_n. \tag{1.1}$$

Also, for $-\infty \leq J \leq L \leq \infty$, let $\mathcal{F}_J^L$ denote the σ-field $\subset \mathcal{F}$ generated by the random variables X_k, $J \leq k \leq L$ ($k \in \mathbf{Z}$). The "double tail σ-field" of the sequence X is

$$\mathcal{T}_{\text{double}}(X) := \bigcap_{n \in \mathbf{N}} (\mathcal{F}_{-\infty}^{-n} \vee \mathcal{F}_n^{\infty}). \tag{1.2}$$

A σ-field $\mathcal{A} \subset \mathcal{F}$ is said to be "trivial" if $P(A) = 0$ or 1 for every $A \in \mathcal{A}$.

Here is our main result:

Theorem 1.1. *There exists a strictly stationary sequence $X := (X_k,\ k \in \mathbf{Z})$ of random variables (on some probability space $(\Omega, \mathcal{F}, P)$) with the following six properties:*

(A) *The random variables X_k take just the values -1 and 1, with $P(X_0 = -1) = P(X_0 = 1) = 1/2$ (and hence $EX_0 = 0$ and $EX_0^2 = 1$).*

(B) *For every five distinct integers $k(1)$, $k(2)$, $k(3)$, $k(4)$,* and *$k(5)$, the five random variables $X_{k(1)}$, $X_{k(2)}$, $X_{k(3)}$, $X_{k(4)}$,* and *$X_{k(5)}$ are independent.*

(C) *There exist a sequence $(\eta_k, k \in \mathbf{Z})$ of independent, identically distributed real-valued random variables (on $(\Omega, \mathcal{F}, P)$), and a Borel function $f : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \cdots \to \{-1, 1\}$, such that for every $k \in \mathbf{Z}$,*

$$X_k = f(\eta_k, \eta_{k-1}, \eta_{k-2}, \dots) \quad \text{a.s.} \tag{1.3}$$

(D) *The double tail σ-field of X is trivial (that is, $P(A) = 0$ or 1 for every $A \in \mathcal{T}_{\text{double}}(X)$).*

(E) *One has that* $\limsup_{n\to\infty} E(S_n/\sqrt{n}\,)^6 < 15$.

(F) *For every infinite set $Q \subset \mathbf{N}$, there exist an infinite set $T \subset Q$ and a nondegenerate, non-normal probability measure μ on $(\mathbf{R}, \mathcal{R})$ such that $S_n/\sqrt{n} \Rightarrow \mu$ as $n \to \infty$, $n \in T$.*

Here are some comments on the various properties in this theorem — starting with properties (A), (B), (E), and (F), the ones most closely tied to the central limit question.

First, by property (A), in our look at the central limit question in connection with this example, the natural normalization of the partial sums is $S_n/\sqrt{n}$.

Property (B) in Theorem 1.1 is of course 5-tuplewise independence. For $N \geq 6$, the question of possible existence of a similar, strictly stationary, N-tuplewise independent counterexample — including properties (C) and (D) — remains open. The techniques in the construction for Theorem 1.1 do not appear to extend effectively to $N \geq 6$. As a comparison, for a given arbitrary fixed positive integer N, the paper of Bradley and Pruss (2009) gives a construction of a strictly stationary sequence in which (i) the random variables are uniformly distributed on the interval $[-\sqrt{3}, \sqrt{3}\,]$ (and hence have mean 0 and variance 1), (ii) the sequence satisfies N-tuplewise independence, and (iii) the sequence satisfies property (F) (as well as a variant of property (E)) in Theorem 1.1; that sequence satisfies ergodicity (as was shown in Bradley and Pruss, 2009), but does not satisfy property (C) (suitably reformulated) or property (D) in Theorem 1.1.

Property (E) may seem rather pointless at first. However, property (F) is an elementary consequence of properties (A), (B), and (E) together with the fact that a $N(0, 1)$ random variable Z satisfies $EZ^6 = 15$. (The argument will be given in detail in Section 9 below. An analogous argument, involving moments of a high even order, was used by Bradley and Pruss, 2009.)

In property (F), the probability measure μ may depend on the set Q.

(As a comparison, in a couple of the pairwise independent counterexamples alluded to above — one of those in Janson (1988) and the one in Cuesta and Matrán (1991, Section 2.3) — the partial sums, appropriately normalized, converge in distribution to a nondegenerate, non-normal law as $n \to \infty$ along the entire sequence of positive integers.)

The formulation of property (F) may seem somewhat awkward. However, properties (E) and (F) indirectly give the following information:

(i) The family of distributions of the random variables $(S_n/\sqrt{n},\ n \in \mathbf{N})$ is tight.
(ii) There does not exist an infinite set $Q \subset \mathbf{N}$ such that $S_n/\sqrt{n}$ converges to 0 (or to any other constant) in probability as $n \to \infty,\ n \in Q$.
(iii) There does not exist an infinite set $Q \subset \mathbf{N}$ such that $S_n/\sqrt{n}$ converges in distribution to a (nondegenerate) normal law as $n \to \infty,\ n \in Q$.
(iv) By (F) and the Theorem of Types (see e.g. Billingsley, 1995, Theorem 14.2), there do not exist an infinite set $Q \subset \mathbf{N}$ and real numbers $a_n,\ b_n,\ n \in Q$, with $b_n \to \infty$ as $n \to \infty,\ n \in Q$, such that $(S_n - a_n)/b_n$ converges in distribution to a nondegenerate normal law as $n \to \infty,\ n \in Q$.

One can describe property (C) by saying that the random sequence X is "causal." Such uses of that term are well known in the literature. See e.g. Brockwell and Davis (1991) for its use in the context of linear models in time series analysis.

Properties (C) and (D) are motivated partly by the general question of what properties from ergodic theory can help insure that a CLT holds. In order to elaborate on that, we will need to give some more definitions and background information.

The "Bernoulli" property. Suppose A is nonempty finite set, and $X := (X_k, k \in \mathbf{Z})$ is a nondegenerate strictly stationary sequence of random variables taking their values in A. This sequence X is said to be "Bernoulli" if, without changing its distribution (on $A^{\mathbf{Z}}$), it can be represented as a stationary coding of a finite-state i.i.d. sequence — that is, if X can be represented in the form

$$X_k = h\Big((\dots, Y_{k-1}, Y_k), (Y_{k+1}, Y_{k+2}, \dots)\Big) \tag{1.4}$$

for $k \in \mathbf{Z}$, where $Y := (Y_k, k \in \mathbf{Z})$ is a sequence of independent, identically distributed random variables taking their values in a finite set B, and $h : B^{\mathbf{Z}} \to A$ is a Borel function. (Here, a given "two sided" sequence $b := (b_k, k \in \mathbf{Z})$ of elements of B is written as $((\dots, b_{-1}, b_0), (b_1, b_2, \dots))$ as a convenient way to avoid ambiguity.) This is one of numerous equivalent ways of formulating the class of (strictly stationary, finite-state) random sequences that are "Bernoulli." For a list of some others, see e.g. Shields (1996, p. 235, lines 12-15).

By a special case of a classic theorem of Ornstein (see Ornstein, 1970a, p. 350, line 11, with reference to Ornstein, 1970b), properties ((A) and) (C) in Theorem 1.1 imply the Bernoulli property. That is:

Remark 1.2. Automatically, the random sequence X in Theorem 1.1 is Bernoulli.

In ergodic theory, it is well known that a nondegenerate strictly stationary, finite-state sequence with zero entropy is not Bernoulli; see e.g. Petersen (1989, section 6.4). It follows that the examples of Flaminio (1993) alluded to above are not Bernoulli. Consequently, even if those examples of Flaminio turn out to be counterexamples to the CLT (apparently still an open question), Theorem 1.1 (with Remark 1.2) still gives new information (beyond the examples given by Flaminio (1993) and by Bradley and Pruss, 2009) in that it provides a 5-tuplewise independent counterexample which has the additional property of being Bernoulli.

Two strong mixing conditions. Next, we would like to use Theorem 1.1 to obtain some perspective on a classic CLT (stated in Theorem 1.4 below) involving the Rosenblatt (1956) "strong mixing" condition. This will require some more definitions and background information.

Again suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of (real-valued, not necessarily finite-state) random variables on a probability space $(\Omega, \mathcal{F}, P)$.

For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, define the measures of dependence

$$\begin{aligned}\alpha(\mathcal{A}, \mathcal{B}) &:= \sup_{A \in \mathcal{A},\ B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)| \qquad \text{and} \\ \beta(\mathcal{A}, \mathcal{B}) &:= \sup\ (1/2) \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|\end{aligned}$$

where the latter supremum is taken over all pairs of finite partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for each i and $B_j \in \mathcal{B}$ for each j. (The factor of $1/2$ in the definition of $\beta(\mathcal{A}, \mathcal{B})$ is of no significance, but has become standard in the literature.) For each positive integer n, define the dependence coefficients

$$\begin{aligned}\alpha(n) &:= \alpha(X, n) &:= \alpha(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty}) \quad \text{and} \\ \beta(n) &:= \beta(X, n) &:= \beta(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty}).\end{aligned}$$

By strict stationarity, $\alpha(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) = \alpha(n)$ and $\beta(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) = \beta(n)$ for every integer j. The (strictly stationary) sequence X is said to satisfy the Rosenblatt (1956) "strong mixing" condition, or "α-mixing," if $\alpha(n) \to 0$ as $n \to \infty$; and it is said to satisfy the "absolute regularity" (Volkonskiĭ and Rozanov, 1959) condition if $\beta(n) \to 0$ as $n \to \infty$. For the first of those conditions, the term "α-mixing" will be used here in order to avoid ambiguity from conflicting uses of the phrase "strong mixing" in the literature.

Obviously absolute regularity implies α-mixing. For strictly stationary, finite-state sequences, one also has the following:

(i) First, α-mixing does not imply absolute regularity (see e.g. Bradley, 2007a, Vol. 1, Theorem 9.10(II)).

(ii) The "weak Bernoulli" condition, defined and studied by Friedman and Ornstein (1970), is equivalent to absolute regularity, and (as was shown in that paper) it implies the Bernoulli property.

(iii) It is unknown (an open problem posed by Donald Ornstein in the 1970s) whether α-mixing implies the Bernoulli property.

(iv) The Bernoulli property does not imply α-mixing (and hence also does not imply absolute regularity); that was shown by Smorodinsky (1971).

The strictly stationary, 3-state, triplewise independent, absolutely regular counterexample (to the CLT) developed in Bradley (1989, 2007b), alluded to above, is in two respects "optimal" under absolute regularity: First, with its random variables being bounded (in fact, finite-state), its "mixing rate" $\beta(n) = O(1/n)$ (as $n \to \infty$) is essentially as rapid as possible (it cannot satisfy $\beta(n) = o(1/n)$ or even $\alpha(n) = o(1/n)$), by a CLT of Merlevède and Peligrad (2000). Second, as was pointed out in Bradley (2007b, section 1) with a brief explanation, if a given strictly stationary sequence of nondegenerate, square-integrable random variables satisfies both α-mixing (or absolute regularity) and 4-tuplewise independence, then it satisfies the CLT. (This fact is an elementary corollary, via a truncation argument, of the CLT under α-mixing given in Theorem 1.4 below.) As a consequence:

Remark 1.3. The random sequence X in Theorem 1.1 cannot satisfy α-mixing (or absolute regularity).

In the latter part of the book Bradley (2007a, Vol. 3), there is a detailed presentation of a large collection of strictly stationary, absolutely regular (but in most cases not pairwise independent or finite-state) counterexamples to the CLT, including examples of Davydov (1973), and also including a (slightly embellished) presentation of the well known example of Herrndorf (1983) in which the random variables are uncorrelated.

The double tail σ-field. Let us digress to take a quick look at some of the ways property (D) in Theorem 1.1 fits in with the various dependence conditions above, for strictly stationary sequences.

(i) First, it is well known and elementary (see e.g. Bradley, 2007a, Vol. 1, Proposition 5.17) that absolute regularity implies a trivial double tail σ-field.

(ii) Next, α-mixing does not imply a trivial double tail σ-field. See for example the counterexample in Bradley (1986) or Bradley (2007a, Vol. 2, Theorem 24.14) with real state space, or the finite-state counterexample constructed by Burton et al. (1996). Each of those examples is "bilaterally deterministic" (that is, the entire random sequence is, modulo null-sets, measurable with respect to its double tail σ-field).

(iii) In the finite-state case, the Bernoulli property does not imply a trivial double tail σ-field. Ornstein and Weiss (1975) showed instead that within the class of strictly stationary, finite-state random sequences that are Bernoulli, the ones that are also bilaterally deterministic are in a certain sense "ubiquitous."

(iv) In the finite-state case, a trivial double tail σ-field does not imply α-mixing (see e.g. Bradley, 2007a, Vol. 1, Theorem 9.11(II)), and hence also does not imply absolute regularity.

A classic CLT under α-mixing. Now let us take a quick look at the following classic theorem:

Theorem 1.4. *Suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of (real-valued) random variables such that $EX_0 = 0$, $EX_0^2 < \infty$, $\sigma_n^2 := ES_n^2 \to \infty$ as $n \to \infty$, and $\alpha(n) \to 0$ as $n \to \infty$. Then the following two conditions (I), (II) are equivalent:*

(I) *The family of random variables $(S_n^2/\sigma_n^2,\ n \in \mathbf{N})$ is uniformly integrable.*

(II) *$S_n/\sigma_n \Rightarrow N(0,1)$ as $n \to \infty$.*

Even if the assumption of α-mixing were omitted altogether, (II) implies (I), by a well known, elementary argument (see e.g. Denker, 1986 or Mori and Yoshihara, 1986 or Bradley, 2007a, Vol. 1, top half of p. 38). The interest here in Theorem 1.4 is the fact that under all of the given assumptions, (I) implies (II). That fact was shown by Cogburn (1960, Theorem 13) (with α-mixing replaced by a similar but technically weaker condition), a reference that did not seem to be well known for a long time; and its proof was also given by Denker (1986) and by Mori and Yoshihara (1986). (A proof of Theorem 1.4 is spelled out in generous detail in Bradley, 2007a, Vol. 1, Theorem 1.19.)

By properties (A), (B), and (E) in Theorem 1.1, the sequence X in Theorem 1.1 satisfies $EX_0 = 0$, $EX_0^2 < \infty$, $\sigma_n^2 := ES_n^2 = n$ for $n \in \mathbf{N}$, as well as the uniform integrability of $(S_n^2/n,\ n \in \mathbf{N})$. Hence by properties (A), (C), (D), and (F) in Theorem 1.1, together with Remark 1.2, one has the following:

Remark 1.5. In Theorem 1.4 (for the assertion that (I) implies (II)), even if the random variables X_k are finite-state and 5-tuplewise independent, the assumption of α-mixing cannot be replaced by either (or both) of the assumptions that

(i) X is Bernoulli (or even that Property (C) in Theorem 1.1 holds),
(ii) X has a trivial double tail σ-field.

Suppose the hypothesis of Theorem 1.4 holds, along with the extra assumptions that (i) $\sigma_n^2 = n \cdot h(n)$ where $h : (0, \infty) \to (0, \infty)$ is slowly varying at ∞, and (ii) $\limsup_{n\to\infty} \|S_n\|_2/[(\pi/2)^{1/2}E|S_n|\,] \le 1$. Then the CLT (specifically, conclusion (II) of Theorem 1.4) holds. That is a result of Dehling et al. (1986, Theorem 4). (For an exposition of their argument in generous detail, see Bradley, 2007a, Vol. 2, Theorem 17.11.) It seems to be an open question whether that still holds if the assumption of α-mixing is replaced by (say) properties (B), (C), and (D) in Theorem 1.1. Because of the absolute-value signs, the quantities $E|S_n|$ seem to be hard to estimate effectively for the construction given below for Theorem 1.1.

The rest of this paper is devoted to the proof of Theorem 1.1 Here is how that proof will be organized, in Sections 2 through 10:

- §2 Some basic notations
- §3 Some special probability measures on $\{-1, 1\}^m$ with m being powers of 6
- §4 A particular class of functions on certain infinite sequences of vectors
- §5 A special Markov chain (based on Section 4) and a related random sequence
- §6 Scaffolding: primarily a random field $(W_k^{(n)},\ n \in \mathbf{N},\ k \in \mathbf{Z})$ based on Section 5
- §7 More scaffolding: including the random sequence X for Theorem 1.1
- §8 More scaffolding: random sequences $(X_k^{(n)}, k \in \mathbf{Z})$ and $(Y_k^{(n)}, k \in \mathbf{Z})$ for $n \in \mathbf{N}$
- §9 The proofs of most properties in Theorem 1.1
- §10 Proof of property (D) in Theorem 1.1

2. Notations and conventions

This section gives some specific notations and conventions that will be used throughout this paper.

The cardinality of a set S will be denoted $\operatorname{card} S$.

For a given probability space $(\Omega, \mathcal{F}, P)$, the indicator function (on Ω) of a given event A will be denoted $I(A)$, and the σ–field ($\subset \mathcal{F}$) generated by a given collection $(V_j,\ j \in J)$ of random variables (where J is an index set) will be denoted $\sigma(V_j, j \in J)$.

A "left-infinite sequence" (of elements of some set) is a family of elements $(x_k, k \le j)$ where j is an integer (and k is restricted to integers). Often $j = 0$ in that context.

Notations 2.1.

(A) For typographical convenience, for a given nonnegative integer n, whenever the integer 6^n appears in a subscript or superscript, we shall use the notation

$$\text{sxtp}(n) := 6^n. \tag{2.1}$$

(The letters "sxtp" are an abbreviation of the word "sextuple.")

(B) For a given $n \in \mathbf{N}$ and a given vector $a := (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$, the sum and product of the elements will be denoted by

$$\text{sum}(a) := \sum_{k=1}^{n} a_k \quad \text{and} \quad \text{prod}(a) := \prod_{k=1}^{n} a_k. \tag{2.2}$$

(C) Suppose $m \in \mathbf{N}$, and for each $u \in \{1, 2, \dots, 6\}$, $v_u := (v_{u,1}, v_{u,2}, \dots, v_{u,m})$ is a vector with m coordinates. Then the notation $\langle v_1, v_2, v_3, v_4, v_5, v_6\rangle$ will mean the vector $w := (w_1, w_2, \dots, w_{6m})$ with $6m$ coordinates such that for each $u \in \{1, 2, \dots, 6\}$ and each $j \in \{1, 2, \dots, m\}$, $w_{(u-1)m+j} = v_{u,j}$. That is, in the vector $\langle v_1, v_2, v_3, v_4, v_5, v_6\rangle$, with $6m$ coordinates, the first m coordinates are (in order) the coordinates of v_1, the next m coordinates are those of v_2, and so on, with the last m coordinates being those of v_6.

(D) On the family of all nonempty finite subsets of $\mathbf{Z}$, the following partial ordering will be used: For such sets A and B, the notation $A < B$ (or $B > A$) means that $\max A < \min B$ (equivalently, $a < b$ for all $a \in A$, $b \in B$).

(E) Suppose S is a nonempty finite set of integers. If $(a_k,\ k \in S)$ is a family of elements of some set A, then the notation a_S denotes the vector defined by

$$a_S := \big(a_{s(1)}, a_{s(2)}, \dots, a_{s(n)}\big) \tag{2.3}$$

where $n = \text{card}\, S$ and $s(1) < s(2) < \cdots < s(n)$ are *in increasing order* the elements of S. Similarly, if $(X_k,\ k \in S)$ is a family of random variables on a probability space $(\Omega, \mathcal{F}, P)$, then the notation X_S denotes the random vector defined by

$$X_S := \big(X_{s(1)}, X_{s(2)}, \dots, X_{s(n)}\big) \tag{2.4}$$

where the $s(j)$'s are as above. The vector a_S in (2.3) and the random vector X_S in (2.4) will also be expressed respectively as $(a_{s(i)},\ 1 \le i \le n)$ and $(X_{s(i)},\ 1 \le i \le n)$.

(F) (i) A set $S \subset \mathbf{Z}$ is said to be "doubly infinite" if it contains infinitely many negative integers and infinitely many positive integers.

(ii) If $(a_k,\ k \in \mathbf{Z})$ is a ("two-sided") sequence of elements of some set A, and S is a doubly infinite subset of $\mathbf{Z}$, then the notation $(a_j,\ j \in S)$ refers to the sequence $(a_{s(\ell)},\ \ell \in \mathbf{Z})$ where

$$\cdots < s(-2) < s(-1) < s(0) \le 0 < 1 \le s(1) < s(2) < s(3) < \dots$$

and $S = \{\dots, s(-1), s(0), s(1), \dots\}$.

(G) If $(a_k,\ k \in \mathbf{Z})$ is a ("two-sided") sequence of elements of some set A, and S is a subset of $\mathbf{Z}$ which is *both infinite and bounded above,* then a convention "opposite" to that of sections (E) and (F) will be used: The notation $(a_j,\ j \in S)$ refers to the sequence $(a_{s(0)}, a_{s(-1)}, a_{s(-2)}, \dots)$ where $s(0) > s(-1) > s(-2) > \dots$ and $S = \{s(0), s(-1), s(-2), \dots\}$.

(H) Suppose $\mathcal{E} = \{E_1, E_2, E_3, \dots\}$ is a nonempty family of finitely many or countably many sets $E_i \subset \mathbf{Z}$. (The sets E_i themselves can be countable or finite, or even empty, and they need not be disjoint.) Then define the notation

$$\text{union } \mathcal{E} := E_1 \cup E_2 \cup E_3 \cup \dots \ ,$$

the union of all sets $\in \mathcal{E}$.

Definition 2.2.

(A) For each $n \in \{0, 1, 2, \dots\}$, we shall define a function

$$\psi_n : \{0, 1, 2, 3, 4, 5, 6\}^{\mathbf{N}} \longrightarrow \{0, 1, 2, \dots\}.$$

Suppose $a := (a_0, a_1, a_2, \dots)$ is a sequence of elements of $\{0, 1, \dots, 6\}$. If the set $\{k \geq 0 : a_k = 1\}$ is infinite, then define the nonnegative integers $\psi_0(a),\ \psi_1(a),\ \psi_2(a),\ \dots$ (uniquely) by the conditions

$$0 \leq \psi_0(a) < \psi_1(a) < \psi_2(a) < \dots \quad \text{and}$$

$$\{k \geq 0 : a_k = 1\} = \{\psi_0(a), \psi_1(a), \psi_2(a), \dots\}.$$

If instead the set $\{k \geq 0 : a_k = 1\}$ is finite, then $\psi_n(a) := 0$ for all $n \geq 0$. (This last sentence will be an irrelevant formality.)

(B) *Remark.* In that definition (both cases), if $a_0 = 1$ then $\psi_0(a) = 0$.

Definition 2.3.

(A) A ("two-sided") sequence $w := (w_k,\ k \in \mathbf{Z})$ of elements of $\{0, 1, 2, 3, 4, 5, 6\}$ satisfies "Condition $\mathcal{S}$" if the following three statements hold:

(i) For each $i \in \{1, 2, 3, 4, 5, 6\}$ the set $\{k \in \mathbf{Z} : w_k = i\}$ is doubly infinite. (That is not required for $i = 0$.)

(ii) For each $i \in \{1, 2, 3, 4, 5\}$ and each $\ell \in \mathbf{Z}$ such that $w_\ell = i$, either (a) $w_{\ell+1} = i + 1$ or (b) for some $m \geq 2$, one has that $w_{\ell+1} = w_{\ell+2} = \dots = w_{\ell+m-1} = 0$ and $w_{\ell+m} = i + 1$.

(iii) For each $\ell \in \mathbf{Z}$ such that $w_\ell = 6$, either (a) $w_{\ell+1} = 1$ or (b) for some $m \geq 2$, one has that $w_{\ell+1} = w_{\ell+2} = \dots = w_{\ell+m-1} = 0$ and $w_{\ell+m} = 1$.

That is, the sequence w satisfies Condition $\mathcal{S}$ if "from time immemorial," the non-zero elements in w cycle through $1, 2, \dots, 6$ in order, over and over again, with perhaps some 0's in between.

(B) *Remark.* Suppose $w := (w_k,\ k \in \mathbf{Z})$ is a sequence of elements of $\{0, 1, \dots, 6\}$ that satisfies Condition $\mathcal{S}$. Then for any two integers J and L such that $J \leq L$, one has that

$$\begin{aligned} &\text{card}\{k \in \mathbf{Z} : J \leq k \leq L \ \text{ and } \ w_k = 1\} \\ &\qquad \leq 1 + (1/6) \cdot \text{card}\{k \in \mathbf{Z} : J \leq k \leq L \ \text{ and } \ w_k \neq 0\}. \end{aligned} \tag{2.5}$$

The point is that between any two "consecutive 1's," there are (exactly) five non-zero elements $(2, 3, 4, 5, 6$, each once). Hence, if the left side of (2.5) equals ℓ for some $\ell \geq 2$, then the set in the right side of (2.5) has at least $\ell + 5(\ell - 1)$ elements, and thus (2.5) holds. Equation (2.5) holds trivially when its left side is 0 or 1.

(C) *Remark.* Suppose $S := \{\dots, s(-1), s(0), s(1), \dots\}$ is a doubly infinite set of integers where $\dots < s(-1) < s(0) < s(1) < \dots$. Suppose $w := (w_k,\ k \in \mathbf{Z})$ is a ("two-sided") sequence of elements of $\{0, 1, \dots, 6\}$ such that (i) $w_k = 0$ for all $k \in \mathbf{Z} - S$, and (ii) the ("two-sided") sequence

$(\ldots, w_{s(-1)}, w_{s(0)}, w_{s(1)}, \ldots)$ satisfies Condition $\mathcal{S}$. Then the entire sequence w satisfies Condition $\mathcal{S}$.

(In words, if a given two-sided sequence satisfies Condition $\mathcal{S}$, and one sticks some zeros between its entries, the resulting new sequence still satisfies Condition $\mathcal{S}$.)

Notations 2.4.

(A) The 6×6 identity matrix will be denoted I_6.

(B) The transpose of a vector v will be denoted v^t.

(C) The elements of $\{0,1\}^6$ will be represented as "row" vectors. Suppose m is a positive integer, and for each $i \in \{1, 2, \ldots, m\}$, $\alpha_i := (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i6})$ is an element of $\{0,1\}^6$. Then for the $6 \times m$ matrix whose columns are the transposes of the α_i's respectively, we shall use the following notation:

$$\left[\alpha_1^t \mid \alpha_2^t \mid \cdots \mid \alpha_m^t\right] := \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \alpha_{16} & \alpha_{26} & \cdots & \alpha_{m6} \end{bmatrix}. \tag{2.6}$$

(D) A ("left-infinite") sequence $(\beta_0, \beta_{-1}, \beta_{-2}, \ldots)$ of elements of $\{0,1\}^6$ is said to be "back-standard" if

$$\left[\beta_{\ell-5}^t \mid \beta_{\ell-4}^t \mid \cdots \mid \beta_\ell^t\right] = I_6 \tag{2.7}$$

holds for infinitely many integers $\ell \le 0$. A ("two-sided") sequence $(\beta_k, k \in \mathbf{Z})$ of elements of $\{0,1\}^6$ is said to be "two-sided standard" if the set $\{\ell \in \mathbf{Z} : (2.7) \text{ holds}\}$ is doubly infinite (see Section 2.1(F)).

(E) In the proofs of lemmas, we shall often use the notation

$$\beta_j^\ell := (\beta_j, \beta_{j+1}, \ldots, \beta_\ell) \tag{2.8}$$

when $j \le \ell$ are integers and $\beta_k \in \{0,1\}^6$ for each $k \in \{j, \ldots, \ell\}$, and also the ("reverse") notation

$$\beta_{-\infty}^\ell := (\beta_\ell, \beta_{\ell-1}, \beta_{\ell-2}, \ldots) \tag{2.9}$$

when $\ell \in \mathbf{Z}$ and $\beta_k \in \{0,1\}^6$ for each $k \le \ell$.

Notations 2.5.

(A) Suppose that on a probability space $(\Omega, \mathcal{F}, P)$, η is a random variable (or random vector, etc.) taking its values in a measurable space $(A, \mathcal{A})$. The distribution (or "law") of η (on $(A, \mathcal{A})$) will be denoted $\mathcal{L}(\eta)$. For any event F such that $P(F) > 0$, the conditional distribution of η given F will be denoted $\mathcal{L}(\eta \mid F)$.

(B) Suppose m is a positive integer and λ is a probability measure on $\{-1,1\}^m$. Then $\lambda^{[6]}$ will denote the six-fold "product measure" of λ. That is, $\lambda^{[6]}$ is the distribution (on $\{-1,1\}^{6m}$) of a $\{-1,1\}^{6m}$–valued random vector

$$\langle Z_1, Z_2, \ldots, Z_6 \rangle$$

(in Notation 2.1(C)) where $Z_1, Z_2, \ldots, Z_6$ are six independent $\{-1,1\}^m$–valued random vectors, each having distribution λ.

(C) Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. For two events A and B, the notation $A \doteq B$ will mean that $P(A \triangle B) = 0$, where $\triangle$ denotes the symmetric difference. If A is an event and $\mathcal{B}$ is a σ-field $\subset \mathcal{F}$, then the notation $A \dot{\in} \mathcal{B}$ will mean that there exists an event $B \in \mathcal{B}$ such that $A \doteq B$. If $\mathcal{A}$ and $\mathcal{B}$ are

σ–fields $\subset \mathcal{F}$, the notation $\mathcal{A} \mathrel{\dot{\subset}} \mathcal{B}$ will mean that for every $A \in \mathcal{A}$, one has that $A \mathrel{\dot{\in}} \mathcal{B}$.

In arguments below, we shall sometimes simply show $\mathcal{A} \mathrel{\dot{\subset}} \mathcal{B}$ with a quick verification, in place of $\mathcal{A} \subset \mathcal{B}$, when that latter literal inclusion is not needed and requires a longer argument.

Definition 2.6.

(A) Suppose that on a probability space $(\Omega, \mathcal{F}, P)$, $Y := (Y_k,\ k \in \mathbf{Z})$ is a sequence of random variables taking their values in a measurable space $(A, \mathcal{A})$ and $Z := (Z_k,\ k \in \mathbf{Z})$ is a sequence of random variables taking their values in a measurable space $(B, \mathcal{B})$. The ordered pair (Y, Z) is said to satisfy "Condition $\mathcal{M}$" if the following two conditions are satisfied:

(i) The sequence Z is strictly stationary.

(ii) There exists a measurable function $f : B^{\mathbf{N}} \to A$ (that is, $f^{-1}(H) \in \mathcal{B}^{\mathbf{N}}$ for every $H \in \mathcal{A}$, for the "infinite product" σ–field $\mathcal{B}^{\mathbf{N}} := \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \dots)$ such that for each $k \in \mathbf{Z}$, $Y_k = f(Z_k, Z_{k-1}, Z_{k-2}, \dots)$ a.s.

(B) *Remark.* Obviously, if (Y, Z) satisfies Condition $\mathcal{M}$, then (i) the sequence Y is strictly stationary, and (ii) $\sigma(Y) \mathrel{\dot{\subset}} \sigma(Z)$ (see section 2.5(C)).

Notations 2.7. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

(A) An ordered triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of σ–fields $\subset \mathcal{F}$ is a "Markov triplet" if one has that for all $A \in \mathcal{A}$ and all $C \in \mathcal{C}$, $P(A \cap C|\mathcal{B}) = P(A|\mathcal{B}) \cdot P(C|\mathcal{B})$ a.s.

A "restricted" version of (A) will also be needed for some random sequences that are not Markov chains but have some "limited" Markov properties:

(B) Suppose $B \in \mathcal{F}$, and suppose $\mathcal{A}$ and $\mathcal{C}$ are σ–fields $\subset \mathcal{F}$. The ordered triplet $(\mathcal{A}, B, \mathcal{C})$ is a "restricted Markov triplet" if (i) $P(B) > 0$, and (ii) for all $A \in \mathcal{A}$ and all $C \in \mathcal{C}$, $P(A \cap C|B) = P(A|B) \cdot P(C|B)$.

(C) *Remark.* If $(\mathcal{A}, B, \mathcal{C})$ is a restricted Markov triplet, $A \in \mathcal{A}$, $P(A \cap B) > 0$, and $C \in \mathcal{C}$, then (by a trivial calculation) $P(C|A \cap B) = P(C|B)$.

(D) *Remark.* Suppose $\mathcal{G}$ and $\mathcal{H}$ are independent σ–fields $\subset \mathcal{F}$. Suppose $G \in \mathcal{G}$, and $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ–fields $\subset \mathcal{G}$, and $(\mathcal{G}_1, G, \mathcal{G}_2)$ is a restricted Markov triplet. Suppose $H \in \mathcal{H}$, and $\mathcal{H}_1$ and $\mathcal{H}_2$ are σ–fields $\subset \mathcal{H}$, and $(\mathcal{H}_1, H, \mathcal{H}_2)$ is a restricted Markov triplet. Then $(\mathcal{G}_1 \vee \mathcal{H}_1,\ G \cap H,\ \mathcal{G}_2 \vee \mathcal{H}_2)$ is a restricted Markov triplet.

Proof of (D). Of course $P(G \cap H) = P(G) \cdot P(H) > 0$. By a simple calculation, if $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$, $H_1 \in \mathcal{H}_1$, and $H_2 \in \mathcal{H}_2$, then

$$
\begin{aligned}
&P(G_1 \cap G_2 \cap H_1 \cap H_2 \mid G \cap H) = P(G_1 \cap G_2 \mid G) \cdot P(H_1 \cap H_2 \mid H) \\
&= P(G_1|G) \cdot P(G_2|G) \cdot P(H_1|H) \cdot P(H_2|H) \\
&= P(G_1 \mid G \cap H) \cdot P(G_2 \mid G \cap H) \cdot P(H_1 \mid G \cap H) \cdot P(H_2 \mid G \cap H).
\end{aligned}
$$

Thus under the probability measure $Q(\,.\,) := P(\,.\,|G \cap H)$ on $(\Omega, \mathcal{F})$, the four σ–fields $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are independent, and hence (under Q) the σ–fields $\mathcal{G}_1 \vee \mathcal{H}_1$ and $\mathcal{G}_2 \vee \mathcal{H}_2$ are independent. Thus (D) holds.

Remark 2.8. (a standard trivial but useful fact). Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, $C = \bigcup_i C_i$ where $C_1, C_2, C_3, \dots$ is a finite or countable sequence of (pairwise) disjoint events, $P(C_i) > 0$ for at least one i, D is an event, $p \in [0, 1]$, and $P(D|C_i) = p$ (resp. $\geq p$ resp. $\leq p$) for every i such that $P(C_i) > 0$. Then $P(D|C) = p$ (resp. $\geq p$ resp. $\leq p$).

3. Some key probability measures

This section is devoted to the definitions and key properties of certain discrete probability measures that will play a pervasive role in the construction for Theorem 1.1.

In connection with the design of error-correcting codes, the book by MacWilliams and Sloane (1977) includes an extensive treatment of constructions, for a given positive integer k, of "big" k-tuplewise independent random vectors from "small" ones. The material here in Section 3 fits into that general framework (in a somewhat hidden way). However, the details will need to be spelled out here, in order to facilitate the proofs of certain properties (in particular, the bounds on sixth moments of partial sums) in Theorem 1.1.

The main tool in this section is a well known, elementary "parity" trick, built into Definitions 3.1 and 3.3 below. The same general type of "parity" trick was used (with different distributions) by Pruss (1998) and by Bradley and Pruss (2009) in the constructions in those papers, and was also used (for different purposes) in the book MacWilliams and Sloane (1977) alluded to above.

In this section, for convenience, for a given $n \in \mathbf{N}$ and a given element $x \in \{-1,1\}^{\mathrm{sxtp}(n)}$ (see (2.1)), x will be represented as $(x_0, x_1, \dots, x_{\mathrm{sxtp}(n)-1})$ (instead of $x_1, x_2, \dots, x_{\mathrm{sxtp}(n)})$).

Definition 3.1. Referring to (2.2), define the set

$$\Upsilon := \{x := (x_0, x_1, \dots, x_5) \in \{-1,1\}^6 : \mathrm{prod}(x) = -1\}.$$

That is, Υ is the set of all 6-tuples of -1's and $+1$'s with an odd number of -1's. For a given $x := (x_0, x_1, \dots, x_5) \in \Upsilon$, $\mathrm{sum}(x)$ (see (2.2)) is -4 (resp. 0 resp. 4) if exactly 5 (resp. 3 resp. 1) of the x_i's are -1.

Let ν^{key} denote the uniform probability measure on Υ. That is, $\nu^{\mathrm{key}}(\{x\}) = 1/32$ for each $x \in \Upsilon$.

Remark 3.2. If $V := (V_0, V_1, \dots, V_5)$ is a $\{-1,1\}^6$–valued random vector with distribution ν^{key} (thus $V \in \Upsilon$ a.s.), then by trivial arguments, the following statements hold:

(A) The distribution of the random vector $-V$ is ν^{key}.
(B) For any permutation σ of $\{0,1,\dots,5\}$, the distribution of the random vector $(V_{\sigma(0)}, V_{\sigma(1)}, \dots, V_{\sigma(5)})$ is ν^{key}.
(C) For each $i \in \{0,1,\dots,5\}$, $P(V_i = -1) = P(V_i = 1) = 1/2$.
(D) For every set $S \subset \{0,1,\dots,5\}$ such that card $S = 5$, the random variables V_i, $i \in S$ are independent.
(E) $\mathrm{prod}(V) = -1$ a.s. (see (2.2)).
(F) $P(\mathrm{sum}(V) = 0) = 5/8$ and $P(\mathrm{sum}(V) = -4) = P(\mathrm{sum}(V) = 4) = 3/16$ (see (2.2)).
(G) For each $i \in \{0,1,\dots,5\}$, $P(V_i = 1|\mathrm{sum}(V) = 4) = 5/6$ and $P(V_i = -1|\mathrm{sum}(V) = 4) = 1/6$, and hence $E(V_i|\mathrm{sum}(V) = 4) = 2/3$.

Definition 3.3. Refer to (2.1) and (2.2). For each $n \in \mathbf{N}$, we shall define four probability measures $\nu^{(n)}_{\mathrm{ord}}$, $\nu^{(n)}_{\mathrm{cen}}$, $\nu^{(n)}_{\mathrm{fri}}$, and $\nu^{(n)}_{\mathrm{pos}}$ on the set $\{-1,1\}^{\mathrm{sxtp}(n)}$, such that the following holds:

$$\text{If } W := \big(W_0, W_1, \dots, W_{\mathrm{sxtp}(n)-1}\big) \quad \text{is a } \{-1,1\}^{\mathrm{sxtp}(n)}\text{–valued random vector with distribution } \nu^{(n)}_{\mathrm{pos}}, \text{then } \mathrm{sum}(W) = 4^n \text{ a.s.} \tag{3.1}$$

(The subscripts "ord", "cen," "fri," and "pos" are respectively abbreviations of the words "ordinary" "center," "fringe," and "positive." The relevance of those subscripts will become clear below.) For $n = 0$, we shall also define two probability measures $\nu^{(0)}_{\mathrm{fri}}$ and $\nu^{(0)}_{\mathrm{pos}}$ on the set $\{-1,1\}$, such that (3.1) holds for $n = 0$. The definition is recursive and is as follows:

Start with $n = 0$. On the set $\{-1,1\}$, define the probability measures $\nu^{(0)}_{\mathrm{fri}}$ and $\nu^{(0)}_{\mathrm{pos}}$ by $\nu^{(0)}_{\mathrm{fri}}(\{-1\}) = \nu^{(0)}_{\mathrm{fri}}(\{1\}) = 1/2$ and $\nu^{(0)}_{\mathrm{pos}}(\{1\}) = 1$. Trivially (3.1) holds for $n = 0$.

Now suppose $n \geq 0$, and suppose the probability measure $\nu^{(n)}_{\mathrm{pos}}$ on $\{-1,1\}^{\mathrm{sxtp}(n)}$ has already been defined such that (3.1) holds. Let

$$W^{(i)} := (W^{(i)}_0, W^{(i)}_1, \dots, W^{(i)}_{\mathrm{sxtp}(n)-1}),$$

$i \in \{0,1,\dots,5\}$ be six independent $\{-1,1\}^{\mathrm{sxtp}(n)}$–valued random vectors, each having distribution $\nu^{(n)}_{\mathrm{pos}}$. Let $V := (V_0, V_1, \dots, V_5)$ be a $\{-1,1\}^6$-valued random vector which is independent of the family $(W^{(i)}, 0 \leq i \leq 5)$ and has distribution ν^{key} (see Definition 3.1). Let $Z := (Z_0, Z_1, \dots, Z_{\mathrm{sxtp}(n+1)-1})$ be the $\{-1,1\}^{\mathrm{sxtp}(n+1)}$-valued random vector defined as follows:

$$\forall\, i \in \{0,1,\dots,5\},\ \forall\, j \in \{0,1,\dots,6^n - 1\}, \quad Z_{i\cdot\mathrm{sxtp}(n)+j} := V_i \cdot W^{(i)}_j. \tag{3.2}$$

Then (see (2.2), (3.1), and (3.2)) with probability 1,

$$\mathrm{sum}(Z) = \sum_{i=0}^{5} \sum_{j=0}^{\mathrm{sxtp}(n)-1} V_i W^{(i)}_j = (\mathrm{sum}(V)) \cdot 4^n, \tag{3.3}$$

and hence (see Remark 3.2(F)) $\mathrm{sum}(Z)$ takes the value 0 resp. -4^{n+1} resp. 4^{n+1} with probability 5/8 resp. 3/16 resp. 3/16.

Let $\nu^{(n+1)}_{\mathrm{ord}}$ denote the distribution on $\{-1,1\}^{\mathrm{sxtp}(n+1)}$ of the random vector Z. Let $\nu^{(n+1)}_{\mathrm{cen}}$ resp. $\nu^{(n+1)}_{\mathrm{fri}}$ resp. $\nu^{(n+1)}_{\mathrm{pos}}$ denote the conditional distribution on $\{-1,1\}^{\mathrm{sxtp}(n+1)}$ of the random vector Z given the event $\{\mathrm{sum}(Z) = 0\}$ resp. $\{|\mathrm{sum}(Z)| = 4^{n+1}\}$ resp. $\{\mathrm{sum}(Z) = 4^{n+1}\}$. Thus (3.1) holds with n replaced by $n + 1$. This completes the recursive definition.

Remark 3.4. For each $n \in \mathbf{N}$, $\nu^{(n)}_{\mathrm{ord}} = (5/8)\nu^{(n)}_{\mathrm{cen}} + (3/8)\nu^{(n)}_{\mathrm{fri}}$.

Proof. If Z is a $\{-1,1\}^{\mathrm{sxtp}(n)}$–valued random vector with distribution $\nu^{(n)}_{\mathrm{ord}}$, and $B \subset \{-1,1\}^{\mathrm{sxtp}(n)}$, then (see the comments after (3.3), but with $n+1$ replaced by n)

$$\begin{aligned}
\nu^{(n)}_{\mathrm{ord}}(B) &= P(Z \in B) \\
&= P(Z \in B | \mathrm{sum}(Z) = 0) \cdot P(\mathrm{sum}(Z) = 0) \\
&\quad + P(Z \in B |\, |\mathrm{sum}(Z)| = 4^n) \cdot P(|\mathrm{sum}(Z)| = 4^n) \\
&= \nu^{(n)}_{\mathrm{cen}}(B) \cdot (5/8) + \nu^{(n)}_{\mathrm{fri}}(B) \cdot (3/8).
\end{aligned}$$

The various distributions in Definition 3.3 have pervasive symmetries. We will need later on, and will verify in the next two sections, only a couple of mild aspects or manifestations of those symmetries.

Remark 3.5.

(A) For a given $n \geq 0$, if the random vectors $W^{(i)}$, $i \in \{0, 1, \dots, 5\}$, V, and Z are as in Definition 3.3, then the distribution of the random collection $(V, W^{(0)}, W^{(1)}, \dots, W^{(5)})$ is (by independence) a product measure on $\{-1,1\}^6 \times (\{-1,1\}^{\text{sxtp}(n)})^6$, and (by Remark 3.2(A)) is the same as that of $(-V, W^{(0)}, W^{(1)}, \dots, W^{(5)})$, and hence by (3.2) the distribution on $\{-1,1\}^{\text{sxtp}(n+1)}$ of the random vector $-Z$ is the same as that of Z.

(B) By an elementary argument, it follows that for any given $n \in \mathbf{N}$, the distributions $\nu_{\text{ord}}^{(n)}$, $\nu_{\text{cen}}^{(n)}$, and $\nu_{\text{fri}}^{(n)}$ on $\{-1,1\}^{\text{sxtp}(n)}$ satisfy the symmetry conditions $\nu_{\text{ord}}^{(n)}(\{-x\}) = \nu_{\text{ord}}^{(n)}(\{x\})$, $\nu_{\text{cen}}^{(n)}(\{-x\}) = \nu_{\text{cen}}^{(n)}(\{x\})$, and $\nu_{\text{fri}}^{(n)}(\{-x\}) = \nu_{\text{fri}}^{(n)}(\{x\})$. For $n = 0$, this holds trivially for $\nu_{\text{fri}}^{(0)}$.

(C) Here for convenient reference are a few other features and elementary consequences of Definition 3.3, for a given $n \geq 0$ and a given $x \in \{-1,1\}^{\text{sxtp}(n)}$: (i) If $\text{sum}(x) = 4^n$ then $\nu_{\text{fri}}^{(n)}(\{-x\}) = \nu_{\text{fri}}^{(n)}(\{x\}) = (1/2)\nu_{\text{pos}}^{(n)}(\{x\})$. (ii) If $\nu_{\text{pos}}^{(n)}(\{x\}) > 0$ then $\text{sum}(x) = 4^n$. (iii) If $\nu_{\text{fri}}^{(n)}(\{x\}) > 0$ then $\text{sum}(x) = -4^n$ or 4^n. (iv) If ($n \geq 1$ and) $\nu_{\text{cen}}^{(n)}(\{x\}) > 0$ then $\text{sum}(x) = 0$. (v) If ($n \geq 1$ and) $\nu_{\text{ord}}^{(n)}(\{x\}) > 0$ then $\text{sum}(x) = -4^n$, 0, or 4^n.

Lemma 3.6. *Suppose $n \geq 0$. Suppose $Y := (Y_0, Y_1, \dots, Y_{\text{sxtp}(n)-1})$ is a $\{-1,1\}^{\text{sxtp}(n)}$–valued random vector whose distribution is $\nu_{\text{pos}}^{(n)}$. Then $EY_k = (2/3)^n$ for every $k \in \{0, 1, \dots, 6^n - 1\}$.*

Proof. Lemma 3.6 holds trivially for $n = 0$. Now for induction, suppose it holds for a given $n \geq 0$. Let the random vectors $W^{(i)}$, $i \in \{0, 1, \dots, 5\}$, V, and Z be as in Definition 3.3. By (3.3), the events $\{\text{sum}(Z) = 4^{n+1}\}$ and $\{\text{sum}(V) = 4\}$ are identical (modulo a null set). By (3.2), Remark 3.2(G), and our induction hypothesis, for each $i \in \{0, 1, \dots, 5\}$ and each $j \in \{0, 1, \dots, 6^n - 1\}$,

$$E\Big(Z_{i\cdot\text{sxtp}(n)+j} \,\Big|\, \text{sum}(Z) = 4^{n+1}\Big) = E\left(V_i \cdot W_j^{(i)} \,\Big|\, \text{sum}(V) = 4\right)$$
$$= EW_j^{(i)} \cdot E(V_i | \text{sum}(V) = 4) = (2/3)^n \cdot (2/3) = (2/3)^{n+1}.$$

That is, $E(Z_k | \text{sum}(Z) = 4^{n+1}) = (2/3)^{n+1}$ for every $k \in \{0, 1, \dots, 6^{n+1} - 1\}$. By Definition 3.3 itself, Lemma 3.6 holds for $n + 1$. That completes the induction step and the proof.

Lemma 3.7. *Suppose $n \geq 0$. Suppose $Y := (Y_0, Y_1, \dots, Y_{\text{sxtp}(n+1)-1})$ is a $\{-1,1\}^{\text{sxtp}(n+1)}$–valued random vector with distribution $(\nu_{\text{fri}}^{(n)})^{[6]}$ (section 2.5(B)). Suppose $Z := (Z_0, Z_1, \dots, Z_{\text{sxtp}(n+1)-1})$ is a $\{-1,1\}^{\text{sxtp}(n+1)}$–valued random vector with the distribution $\nu_{\text{ord}}^{(n+1)}$. Then the following three statements hold:*

(A) *For every set $S \subset \{0, 1, \dots, 6^{n+1} - 1\}$ such that* card $S = 5$, *the random vectors Y_S and Z_S (see (2.4)) have the same distribution on $\{-1,1\}^5$.*

(B) *For every nonempty set $Q \subset \{0, 1, \dots, 6^{n+1} - 1\}$, one has that*

$$E\left(\sum_{k \in Q} Y_k\right)^6 \geq E\left(\sum_{k \in Q} Z_k\right)^6. \tag{3.4}$$

(C) *Also,*

$$E(\mathrm{sum}(Y))^6 = E(\mathrm{sum}(Z))^6 + 720 \cdot 4^{6n}. \tag{3.5}$$

Proof. Suppose $n \geq 0$. Without loss of generality, to prove Lemma 3.7, we shall construct particular convenient random vectors Y and Z with the required distributions, and then prove statements (A), (B), and (C) for those two random vectors.

Let the random vectors $W^{(i)}$, $i \in \{0, 1, \dots, 5\}$, V, and Z be as in Definition 3.3. In that context, let U_i, $i \in \{0, 1, \dots, 5\}$ be independent, identically distributed $\{-1, 1\}$–valued random variables with $P(U_i = -1) = P(U_i = 1) = 1/2$, with the family $(U_i,\ 0 \leq i \leq 5)$ being independent of the family $(V;\ W^{(i)},\ 0 \leq i \leq 5)$. Define the $\{-1, 1\}^{\mathrm{sxtp}(n+1)}$–valued random vector $Y := (Y_0, Y_1, \dots, Y_{\mathrm{sxtp}(n+1)-1})$ as follows:

$$\begin{aligned} &\forall\, i \in \{0, 1, \dots, 5\}, \forall\, j \in \{0, 1, \dots, 6^n - 1\}, \\ &\qquad\qquad Y_{i \cdot \mathrm{sxtp}(n)+j} := U_i \cdot W_j^{(i)}. \end{aligned} \tag{3.6}$$

By Remark 3.5(B)(C), Definition 3.3, and a simple argument, for each $i \in \{0, 1, \dots, 5\}$, the distribution on $\{-1, 1\}^{\mathrm{sxtp}(n)}$ of the random vector $(U_i \cdot W_j^{(i)},\ 0 \leq j \leq 6^n - 1)$ (see the sentence after (2.4)) is $\nu_{\mathrm{fri}}^{(n)}$. From this and the definition of $\nu^{(n+1)}$ in Definition 3.3, the random vectors Y and Z constructed here have the distributions specified in the statement of Lemma 3.7.

Just for this proof, define for each $i \in \{0, 1, \dots, 5\}$ the set

$$K_i = K_i^{(n)} := \{k \in \mathbf{Z} : i \cdot 6^n \leq k \leq (i+1) \cdot 6^n - 1\}. \tag{3.7}$$

These sets $K_0, K_1, \dots, K_5$ form a partition of the set $\{0, 1, \dots, 6^{n+1} - 1\}$. As a consequence of Remark 3.2(C)(D) and the above conditions, for any set $\Lambda \subset \{0, 1, \dots, 5\}$ with $\mathrm{card}\, \Lambda = 5$, the random family $(V_i,\ i \in \Lambda;\ W^{(i)},\ i \in \Lambda)$ has the same distribution—a ten-fold product measure on $\{-1, 1\}^5 \times (\{-1, 1\}^{\mathrm{sxtp}(n)})^5$ — as the random family $(U_i,\ i \in \Lambda;\ W^{(i)},\ i \in \Lambda)$. Hence by (3.2) and (3.6), for every set $\Lambda \subset \{0, 1, \dots, 5\}$ with $\mathrm{card}\, \Lambda = 5$ (see section 2.5(A) and the sentence after (2.4)),

$$\mathcal{L}\Bigl(Y_k,\ k \in \bigcup_{i \in \Lambda} K_i\Bigr) = \mathcal{L}\Bigl(Z_k,\ k \in \bigcup_{i \in \Lambda} K_i\Bigr) \tag{3.8}$$

(an equality of distributions on $\{-1, 1\}^{5 \cdot \mathrm{sxtp}(n)}$).

Now if $S \subset \{0, 1, \dots, 6^{n+1} - 1\}$ is such that $\mathrm{card}\, S = 5$, then $S \subset \bigcup_{i \in \Lambda} K_i$ for some set $\Lambda \subset \{0, 1, \dots, 5\}$ with $\mathrm{card}\, \Lambda \leq 5$. Hence statement (A) in Lemma 3.7 follows from (3.8).

Proof of statements (B) and (C). For any nonempty set $Q \subset \{0, 1, \dots, 6^{n+1} - 1\}$,

$$\begin{aligned} &E\Bigl(\sum_{k \in Q} Y_k\Bigr)^6 - E\Bigl(\sum_{k \in Q} Z_k\Bigr)^6 \\ &= \sum_{k(0) \in Q} \sum_{k(1) \in Q} \cdots \sum_{k(5) \in Q} \left[E\left(\prod_{i=0}^{5} Y_{k(i)}\right) - E\left(\prod_{i=0}^{5} Z_{k(i)}\right) \right]. \end{aligned} \tag{3.9}$$

For any set $\Lambda \subset \{0, 1, \dots, 5\}$ with $\mathrm{card}\, \Lambda = 5$, and any choice of (not necessarily distinct) elements $k(0), k(1), \dots, k(5)$ of $\bigcup_{i \in \Lambda} K_i$, the term in the brackets in (3.9) equals 0 by (3.8). On the other hand, if $k(i) \in K_i$ for every $i \in \{0, 1, \dots, 5\}$, then representing $k(i) = i \cdot 6^n + j(i)$ where $j(i) \in \{0, 1, \dots, 6^n - 1\}$, one has

$$E\left(\prod_{i=0}^{5} Y_{k(i)}\right) = \left[\prod_{i=0}^{5} EU_i\right] \cdot \left[\prod_{i=0}^{5} EW_{j(i)}^{(i)}\right] = 0$$

by (3.6) and the trivial fact $EU_i = 0$, and one also has

$$E\left(\prod_{i=0}^{5} Z_{k(i)}\right) = \left[E\left(\prod_{i=0}^{5} V_i\right)\right] \cdot \left[\prod_{i=0}^{5} EW_{j(i)}^{(i)}\right] = -1 \cdot [(2/3)^n]^6 = -(2/3)^{6n}$$

by (3.2), Remark 3.2(E), and Lemma 3.6. Hence by (3.9) for any nonempty set $Q \subset \{0, 1, \dots, 6^{n+1} - 1\}$,

$$[\text{Left side of (3.9)}] = (2/3)^{6n} \cdot 6! \cdot \prod_{i=0}^{5} \operatorname{card}(Q \cap K_i). \tag{3.10}$$

Statement (B) in Lemma 3.7 follows. Since card $K_i = 6^n$ for each i, statement (C) follows from (3.10) with $Q = \{0, 1, \dots, 6^{n+1} - 1\}$ itself. That completes the proof.

4. Some particular functions

A key "building block" in the construction of the random sequence X in Theorem 1.1 will be a particular strictly stationary, finite-state, irreducible, aperiodic Markov chain. It will be given a particular, explicit representation as a "causal moving function" (à la eq. (1.3)) of an i.i.d. finite-state sequence. This representation will be spelled out in detail in Section 4 here and Section 5 together (the Markov chain itself will be identified in Lemma 5.3), in order to facilitate transparent proofs (in Sections 7 and 10 respectively) of property (C) and (especially) property (D) in Theorem 1.1. The particular form of the representation that will be used here is an old one that goes back several decades; its origin is hard to trace. Its spirit goes back at least to a paper of Rosenblatt (1960) in which it is shown that every strictly stationary, countable-state, irreducible, aperiodic Markov chain can be represented as a "causal moving function" of an i.i.d. sequence.

Definition 4.1. For each $n \in \mathbf{N}$, we shall define a function $h_n : (\{0,1\}^6)^n \to \{1, 2, 3, 4, 5, 6\}$. The definition will be recursive and is as follows:

First, define the function $h_1 : \{0,1\}^6 \to \{1, 2, \dots, 6\}$ as follows: For $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_6) \in \{0,1\}^6$,

$$h_1(\alpha) := \begin{cases} 6 \text{ if } \alpha_1 = 0 \\ 1 \text{ if } \alpha_1 = 1. \end{cases} \tag{4.1}$$

Now suppose $n \ge 2$, and the function $h_{n-1} : (\{0,1\}^6)^{n-1} \to \{1, 2, \dots, 6\}$ has already been defined. Define the function $h_n : (\{0,1\}^6)^n \to \{1, 2, \dots, 6\}$ as follows:

Suppose that for each $i \in \{1, 2, \dots, n\}$, $\beta_i := (\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,6}) \in \{0,1\}^6$. If $h_{n-1}(\beta_1, \beta_2, \dots, \beta_{n-1}) = j \in \{1, 2, 3, 4, 5\}$, then

$$h_n(\beta_1, \beta_2, \dots, \beta_n) := \begin{cases} j & \text{if } \beta_{n,j+1} = 0 \\ j+1 & \text{if } \beta_{n,j+1} = 1. \end{cases} \tag{4.2}$$

If instead $h_{n-1}(\beta_1, \beta_2, \ldots, \beta_{n-1}) = 6$, then

$$h_n(\beta_1, \beta_2, \ldots, \beta_n) := \begin{cases} 6 & \text{if } \beta_{n,1} = 0 \\ 1 & \text{if } \beta_{n,1} = 1. \end{cases} \tag{4.3}$$

That completes the recursive definition of the functions h_n.

Lemma 4.2. *Suppose $\ell \in \{-5, -6, -7, \ldots\}$, and $\beta_\ell, \beta_{\ell+1}, \ldots, \beta_{-1}$ are each an element of $\{0,1\}^6$, and that*

$$\left[\beta_{-5}^t \mid \beta_{-4}^t \mid \beta_{-3}^t \mid \beta_{-2}^t \mid \beta_{-1}^t\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

(the first five columns of the identity matrix I_6 — see Section 2.4(A)(B)(C)). Then

$$h_{-\ell}(\beta_\ell, \beta_{\ell+1}, \ldots, \beta_{-1}) = 5. \tag{4.5}$$

Proof. Consider first the case $\ell = -5$. By (4.4), (4.1), and then four applications of (4.2), one obtains $h_1(\beta_{-5}) = 1$, $h_2(\beta_{-5}, \beta_{-4}) = 2$, $h_3(\beta_{-5}, \beta_{-4}, \beta_{-3}) = 3$, $h_4(\beta_{-5}^{-2}) = 4$ (we start using the Notations 2.4(E)), and finally $h_5(\beta_{-5}^{-1}) = 5$, which is (4.5).

Now consider the case where $\ell \le -6$. We shall give the argument for the case where

$$h_{-5-\ell}(\beta_\ell, \beta_{\ell+1}, \ldots, \beta_{-6}) = 3. \tag{4.6}$$

The argument is similar for the other possible values $(1, 2, 4, 5$, and $6)$ for the right hand side of (4.6).

Now $\beta_{-5,4} = 0$ by (4.4), and hence $h_{-4-\ell}(\beta_\ell^{-5}) = 3$ by (4.6) and (4.2). Next, $\beta_{-4,4} = 0$ by (4.4), and hence $h_{-3-\ell}(\beta_\ell^{-4}) = 3$ now follows from (4.2). Similarly $\beta_{-3,4} = 0$ and hence $h_{-2-\ell}(\beta_\ell^{-3}) = 3$. Next, $\beta_{-2,4} = 1$ by (4.4), and hence $h_{-1-\ell}(\beta_\ell^{-2}) = 4$ by (4.2). Finally, $\beta_{-1,5} = 1$, and hence (4.5) now follows from (4.2). That completes the proof of Lemma 4.2.

In Definitions 4.3 and 4.5 below, two closely related functions on $(\{0,1\}^6)^{\mathbf{N}}$ will be defined. Because of the way those functions will be used, "left-infinite" sequences will be used in their definitions.

Definition 4.3. Define the function $g_{\text{basic}}(\{0,1\}^6)^{\mathbf{N}} \to \{1,2,3,4,5,6\}$ as follows:

Suppose $\beta_0, \beta_{-1}, \beta_{-2}, \ldots$ each $\in \{0,1\}^6$.

If the sequence $(\beta_0, \beta_{-1}, \beta_{-2}, \ldots)$ is not back-standard (see Section 2.4 (A) (B) (C) (D)), then define $g_{\text{basic}}(\beta_0, \beta_1, \beta_{-2}, \ldots) := 6$.

Now suppose instead that the sequence $(\beta_0, \beta_{-1}, \beta_{-2}, \ldots)$ is back-standard (again see Section 2.4). Let L denote the greatest integer $\in \{0, -1, -2, \ldots\}$ such that $[\beta_{L-5}^t \mid \beta_{L-4}^t \mid \cdots \mid \beta_L^t] = I_6$.

(i) If $L = 0$ (that is, if $[\beta_{-5}^t \mid \beta_{-4}^t \mid \ldots |\beta_0^t] = I_6$), then define $g_{\text{basic}}(\beta_0,\ \beta_{-1},\ \beta_{-2},\ \ldots) := 6$.

(ii) If instead $L \le -1$, then referring to Definition 4.1, define

$$g_{\text{basic}}(\beta_0, \beta_{-1}, \beta_{-2}, \ldots) := h_{-L}(\beta_{L+1}, \beta_{L+2}, \ldots, \beta_0). \tag{4.7}$$

Lemma 4.4. *Suppose $(\beta_0, \beta_{-1}, \beta_{-2}, \ldots)$ is a ("left-infinite") back-standard sequence of elements of $\{0,1\}^6$ (see Section 2.4(D)).*

(A) *If* $g_{\text{basic}}(\beta_{-1}, \beta_{-2}, \beta_{-3}, \dots) = j \in \{1, 2, 3, 4, 5\}$, *then*

$$g_{\text{basic}}(\beta_0, \beta_{-1}, \beta_{-2}, \dots) = \begin{cases} j & \text{if } \beta_{0,j+1} = 0 \\ j+1 & \text{if } \beta_{0,j+1} = 1. \end{cases} \tag{4.8}$$

(B) *If instead* $g_{\text{basic}}(\beta_{-1}, \beta_{-2}, \beta_{-3}, \dots) = 6$, *then*

$$g_{\text{basic}}(\beta_0, \beta_{-1}, \beta_{-2}, \dots) = \begin{cases} 6 & \text{if } \beta_{0,1} = 0 \\ 1 & \text{if } \beta_{0,1} = 1. \end{cases} \tag{4.9}$$

Proof. Let L denote the greatest *negative* integer (the integer 0 is excluded here) such that (see Section 2.4(A)(B)(C))

$$\left[\beta^t_{L-5} \mid \beta^t_{L-4} \mid \cdots \mid \beta^t_L\right] = I_6. \tag{4.10}$$

The proofs of statements (A) and (B) in Lemma 4.4 will be handled together, and will be divided into three cases. The Notations 2.4(E) (both eqs. (2.8) and (2.9)) will be used.

Case 1: $[\beta^t_{-5} \mid \beta^t_{-4} \mid \cdots \mid \beta^t_0] = I_6$. Then $\beta_{\ell,6} = 0$ for $\ell \in \{-5, -4, \dots, -1\}$, and hence by (4.10) (which implies $\beta_{L,6} = 1$) and its entire sentence, $L \le -6$ must hold. By Definition 4.3 and Lemma 4.2, $g_{\text{basic}}(\beta^{-1}_{-\infty}) = h_{-L-1}(\beta^{-1}_{L+1}) = 5$. Hence here in Case 1, the hypothesis of statement (A) (in Lemma 4.4) holds with $j = 5$ there, and statement (B) there is vacuous. Since (by the definition of I_6) $\beta_{0,6} = 1$, the right side of (4.8) equals 6. Also, by Definition 4.3(i), the left side of (4.8) equals 6. Hence (4.8) holds, and statement (A) is verified. That completes the argument for Case 1.

Case 2: $[\beta^t_{-5} \mid \beta^t_{-4} \mid \cdots \mid \beta^t_0] \ne I_6$ and $L = -1$ (redundant — see (4.10)). Then by Definition 4.3(i) (see (4.10)), $g_{\text{basic}}(\beta^{-1}_{-\infty}) = 6$, and hence the hypothesis of statement (B) (in Lemma 4.4) is satisfied, and statement (A) there is vacuous. By Definition 4.3(ii) and Definition 4.1, $g_{\text{basic}}(\beta^0_{-\infty}) = h_1(\beta_0) = 6$ resp. 1 if $\beta_{0,1} = 0$ resp. 1. Thus (4.9) holds, and statement (B) is verified. That completes the argument for Case 2.

Case 3: $[\beta^t_{-5} \mid \beta^t_{-4} \mid \cdots \mid \beta^t_0] \ne I_6$ and $L \le -2$. By Definition 4.3(ii),

$$g_{\text{basic}}(\beta^{-1}_{-\infty}) = h_{-L-1}(\beta^{-1}_{L+1}) \tag{4.11}$$

and

$$g_{\text{basic}}(\beta^0_{-\infty}) = h_{-L}(\beta^0_{L+1}). \tag{4.12}$$

If the hypothesis of statement(A) (in Lemma 4.4) holds for some $j \in \{1, 2, \dots, 5\}$, then (for that j) $h_{-L-1}(\beta^{-1}_{L+1}) = j$ by (4.11), and then $h_{-L}(\beta^0_{L+1}) = j$ resp. $j+1$ if $\beta_{0,j+1} = 0$ resp. 1 by equation (4.2) in Definition 4.1, and then (4.8) — the conclusion of statement (A) — holds by (4.12).

If instead the hypothesis of statement (B) holds, then $h_{-L-1}(\beta^{-1}_{L+1}) = 6$ by (4.11), $h_{-L}(\beta^0_{L+1}) = 6$ resp. 1 if $\beta_{0,1} = 0$ resp. 1 by (4.3) in Definition 4.1, and then (4.9) — the conclusion of statement (B) — holds by (4.12). That completes the argument for Case 3, and the proof of Lemma 4.4.

Definition 4.5. Define the function $g_{\text{spaced}}(\{0,1\}^6)^{\mathbf{N}} \to \{0, 1, 2, 3, 4, 5, 6\}$ as follows: For any given ("left-infinite") sequence $(\beta_0, \beta_{-1}, \beta_{-2}, \dots)$ of elements of

$\{0,1\}^6$, define

$$g_{\text{spaced}}(\beta_0, \beta_{-1}, \beta_{-2}, \dots) \qquad (4.13)$$
$$:= \begin{cases} g_{\text{basic}}(\beta_0, \beta_{-1}, \beta_{-2}, \dots) & \text{if } g_{\text{basic}}(\beta_0, \beta_{-1}, \beta_{-2}, \dots) \\ & \quad \neq g_{\text{basic}}(\beta_{-1}, \beta_{-2}, \beta_{-3}, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

(The subscript "spaced" in (4.13) is motivated by conclusion (A) in the next lemma—think of elements $\{1, \dots, 6\}$ "spaced apart" with 0's in between.)

Lemma 4.6. *Refer to Sections 2.4(D) and 2.3(A). Suppose $(\beta_k,\ k \in \mathbf{Z})$ is a two-sided standard sequence of elements of $\{0,1\}^6$. For each $k \in \mathbf{Z}$, define the numbers*

$$u_k := g_{\text{basic}}(\beta_k, \beta_{k-1}, \beta_{k-2}, \dots) \qquad (4.14)$$

and (see (4.13))

$$w_k := g_{\text{spaced}}(\beta_k, \beta_{k-1}, \beta_{k-2}, \dots) = \begin{cases} u_k & \text{if } u_k \neq u_{k-1} \\ 0 & \text{if } u_k = u_{k-1}. \end{cases} \qquad (4.15)$$

Then (A) the sequence $(w_k,\ k \in \mathbf{Z})$ satisfies condition $\mathcal{S}$; and (B) one has that

$$\{k \in \mathbf{Z} : w_k = 1\} = \{k \in \mathbf{Z} : u_{k-1} = 6 \ \text{ and } \ u_k = 1\}, \qquad (4.16)$$

and for each $i \in \{2,3,4,5,6\}$,

$$\{k \in \mathbf{Z} : w_k = i\} = \{k \in \mathbf{Z} : u_{k-1} = i-1 \ \textit{ and } \ u_k = i\}. \qquad (4.17)$$

Proof. By (4.14) and Lemma 4.4(A)(B), for each $k \in \mathbf{Z}$,

$$u_k - u_{k-1} \equiv 0 \text{ or } 1 \bmod 6. \qquad (4.18)$$

Hence by (4.15),

$$\{k \in \mathbf{Z} : w_k = 6\} = \{k \in \mathbf{Z} : u_{k-1} = 5 \text{ and } u_k = 6\}. \qquad (4.19)$$

Also, for each $\ell \in \mathbf{Z}$ such that $[\beta_{\ell-5} \mid \beta_{\ell-4} \mid \cdots \mid \beta_\ell] = I_6$, one has that $u_\ell = 6$ by (4.14) and Definition 4.3(i) and $u_{\ell-1} = 5$ by (4.14), Definition 4.3(ii), and Lemma 4.2. Hence (by the hypothesis of Lemma 4.6) the set in (4.19) is doubly infinite (see Section 2.1(F)). Hence also the ("larger") set $\{k \in \mathbf{Z} : u_{k-1} \neq u_k\}$ is doubly infinite.

Now suppose $k \in \mathbf{Z}$ is such that $w_k = 6$. Then $u_k = 6$ by (4.19). Now by (4.18), either $u_{k+1} = 1$ or there exists $m \geq 2$ such that $(u_{k+1}, u_{k+2}, \dots, u_{k+m}) = (6, 6, \dots, 6, 1)$. Hence by (4.15), either $w_{k+1} = 1$ or there exists $m \geq 2$ such that $(w_{k+1}, w_{k+2}, \dots, w_{k+m}) = (0, 0, \dots, 0, 1)$.

By a similar argument, if $i \in \{1,2,3,4,5\}$ and $k \in \mathbf{Z}$ is such that $w_k = i$, then either $w_{k+1} = i+1$ or there exists $m \geq 2$ such that $(w_{k+1}, \dots, w_{k+m}) = (0, 0, \dots, 0, i+1)$.

Since the set $\{k \in \mathbf{Z} : w_k = 6\}$ in (4.19) is doubly infinite (as was noted above), it now follows (by trivial induction) that for each $i \in \{1,2,3,4,5\}$, the set $\{k \in \mathbf{Z} : w_k = i\}$ is also doubly infinite. From the preceding two paragraphs, we now have that the sequence $(w_k,\ k \in \mathbf{Z})$ satisfies Condition $\mathcal{S}$ (again see Definition 2.3(A)). Equation (4.16) and (for $2 \leq i \leq 6$) equation (4.17) now follow from (4.14), (4.15), and (4.18). Lemma 4.6 is proved.

Lemma 4.7. *For each $n \in \mathbf{N}$, there exist functions $\rho_n : (\{0,1\}^6)^n \to \{1,2,3,4,5,6\}$ and $\varphi_n : (\{0,1\}^6)^n \to \{0,1,2,3,4,5,6\}$ such that following holds:*

For any ("left infinite") back-standard sequence $(\beta_n, \beta_{n-1}, \beta_{n-2}, \dots)$ of elements of $\{0,1\}^6$ (see Section 2.4(D)) such that

$$\left[\beta^t_{-5} \mid \beta^t_{-4} \mid \cdots \mid \beta^t_0\right] = I_6, \tag{4.20}$$

one has that

$$g_{\text{basic}}(\beta_n, \beta_{n-1}, \beta_{n-2}, \dots) = \rho_n(\beta_1, \beta_2, \dots, \beta_n) \tag{4.21}$$

and

$$g_{\text{spaced}}(\beta_n, \beta_{n-1}, \beta_{n-2}, \dots) = \varphi_n(\beta_1, \beta_2, \dots, \beta_n). \tag{4.22}$$

Proof. Again the Notations 2.4(E) will be used.

In the arguments below, keep in mind that the equality

$$\left[\beta^t_{\ell-5} \mid \beta^t_{\ell-4} \mid \cdots \mid \beta^t_\ell\right] = I_6 \tag{4.23}$$

cannot hold for $\ell \in \{1,2,\dots,5\}$, for that would contradict (4.20).

The first task is to define the functions ρ_n, $n \in \mathbf{N}$.

For each $n \in \{1,2,3,4,5\}$, define the function $\rho_n : (\{0,1\}^6)^n \to \{1, 2, \dots, 6\}$ by $\rho_n = h_n$ from Definition 4.1. If $1 \le n \le 5$ and (4.20) holds, then (4.21) holds by Definition 4.3(ii).

Now suppose instead that $n \ge 6$. Define the function $\rho_n(\{0,1\}^6)^n \to \{1,2,\dots,6\}$ as follows: Suppose $\beta_1, \beta_2, \dots, \beta_n$ each $\in \{0,1\}^6$. Let S denote the set of all integers $\ell \in \{6,7,\dots,n\}$ such that (4.23) holds. If that set S is empty, then define $\rho_n(\beta_1^n) := h_n(\beta_1^n)$ (from Definition 4.1). If that set S is nonempty but does not contain n, then define $\rho_n(\beta_1^n) := h_{n-L}(\beta^n_{L+1})$ where L is the greatest element of S. If $n \in S$, then define $\rho_n(\beta_1^n) := 6$. That completes the definition of ρ_n. Now using Definition 4.3(i)(ii), one can verify, case by case, that if (4.20) holds then (4.21) holds.

Our next task is to define the functions φ_n, $n \in \mathbf{N}$.

Define the function $\varphi_1 : \{0,1\}^6 \to \{0,1,\dots,6\}$ as follows:

$$\text{For } \alpha := (\alpha_1, \alpha_2, \dots, \alpha_6) \in \{0,1\}^6,$$

$$\varphi_1(\alpha) := \alpha_1 = \begin{cases} 1 & \text{if } \alpha_1 = 1 \\ 0 & \text{if } \alpha_1 = 0 \end{cases} \tag{4.24}$$

Now for each $n \ge 2$, define the function $\varphi_n : (\{0,1\}^6)^n \to \{0,1,\dots,6\}$ as follows: If $\beta_1, \beta_2, \dots, \beta_n$ each $\in \{0,1\}^6$, define

$$\varphi_n(\beta_1^n) := \begin{cases} \rho_n(\beta_1^n) & \text{if } \rho_n(\beta_1^n) \ne \rho_{n-1}(\beta_1^{n-1}) \\ 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

That completes the definition of φ_n for $n \in \mathbf{N}$.

For $n \ge 2$, if (4.20) holds, then (4.22) holds by (4.25), (4.21), and Definition 4.5. For $n = 1$, note that if (4.20) holds, then $g_{\text{basic}}(\beta^0_{-\infty}) = 6$ by Definition 4.3(i), $g_{\text{basic}}(\beta^1_{-\infty}) = 6$ resp. 1 if $\beta_{1,1} = 0$ resp. 1 by Lemma 4.4(B), $g_{\text{spaced}}(\beta^1_{-\infty}) = 0$ resp. 1 if $\beta_{1,1} = 0$ resp. 1 by Definition 4.5, and hence (4.22) holds by (4.24). That completes the proof of Lemma 4.7

Lemma 4.8. *Suppose* $(\beta_k,\ k \in \mathbf{Z})$ *is a two-sided standard sequence of elements of* $\{0,1\}^6$ *(see Section 2.4(D)). Suppose* m *is a positive integer, and for each* $u \in \{1,2,\dots,m\}$, $z_u := (0,0,0,0,0,0)$. *Then*

$$\forall\ u \in \{1,\dots,m\}, \quad g_{\text{spaced}}(z_u, z_{u-1},\dots,z_1,\beta_0,\beta_{-1},\beta_{-2},\dots) = 0; \tag{4.26}$$

and

$$\forall\ n \in \mathbf{N}, \qquad g_{\text{spaced}}(\beta_n,\beta_{n-1},\dots,\beta_1,z_m,z_{m-1},\dots,z_1,\beta_0,\beta_{-1},\beta_{-2},\dots) \\ = g_{\text{spaced}}(\beta_n,\beta_{n-1},\dots,\beta_1,\beta_0,\beta_{-1},\beta_{-2},\dots). \tag{4.27}$$

Proof. It suffices to prove this lemma for the case $m = 1$, for then the lemma as stated follows by induction on m.

Accordingly, let $z := z_1 = (0,0,0,0,0,0)$. Then

$$g_{\text{basic}}(z,\beta_0,\beta_{-1},\beta_{-2},\dots) = g_{\text{basic}}(\beta_0,\beta_{-1},\beta_{-2},\dots) \tag{4.28}$$

by Lemma 4.4 ((A) or (B), whichever applies). Hence $g_{\text{spaced}}(z,\beta_0,\beta_{-1},\beta_{-2},\dots) = 0$ by Definition 4.5, giving (4.26) (in the case $u = m = 1$). Our remaining task is to prove (4.27) (for $m = 1$).

If $g_{\text{basic}}(\beta_0,\beta_{-1},\beta_{-2},\dots) = j \in \{1,2,\dots,5\}$, then by (4.28) and Lemma 4.4(A),

$$g_{\text{basic}}(\beta_1,z,\beta_0,\beta_{-1},\beta_{-2},\dots) = g_{\text{basic}}(\beta_1,\beta_0,\beta_{-1},\beta_{-2},\dots), \tag{4.29}$$

with the common value being j resp. $j+1$ if $\beta_{1,j+1} = 0$ resp. 1. If instead $g_{\text{basic}}(\beta_0,\beta_{-1},\beta_{-2},\dots) = 6$, then (4.29) holds similarly by (4.28) and Lemma 4.4(B).

Starting with (4.29) and applying induction on n, using Lemma 4.4(A)(B) one obtains that

$$\forall\ n \in \mathbf{N}, \qquad g_{\text{basic}}(\beta_n,\beta_{n-1},\dots,\beta_1,z,\beta_0,\beta_{-1},\beta_{-2},\dots) \\ = g_{\text{basic}}(\beta_n,\beta_{n-1},\dots,\beta_1,\beta_0,\beta_{-1},\beta_{-2},\dots). \tag{4.30}$$

By (4.28), (4.30), and Definition 4.5, one has that for all $n \in \mathbf{N}$, (4.30) holds with the subscript "basic" replaced on both sides by "spaced." That is, (4.27) holds for $m = 1$. That completes the proof.

5. A Markov chain and a related process

This section will build on section 4, and will give a study of two particular strictly stationary sequences — one a Markov chain, and the other closely related to it — that together will play a key role as a " building block" in the construction of the random sequence X for Theorem 1.1.

Throughout this section, the setting is a probability space $(\Omega, \mathcal{F}, P)$, rich enough to accommodate all random variables defined in Construction 5.1 below.

Construction 5.1.

(A) On the probability space $(\Omega, \mathcal{F}, P)$, let $(\xi_{k,i},\ k \in \mathbf{Z},\ i \in \{1,2,3,4,5,6\})$ be an array of independent, identically distributed $\{0,1\}$–valued random variables such that for each (k,i),

$$P(\xi_{k,i} = 0) = 5/8 \quad \text{and} \quad (\xi_{k,i} = 1) = 3/8. \tag{5.1}$$

For each $k \in \mathbf{Z}$, define the random vector

$$\xi_k := (\xi_{k,1},\xi_{k,2},\xi_{k,3},\xi_{k,4},\xi_{k,5},\xi_{k,6}) \tag{5.2}$$

(with the above $\xi_{k,i}$'s). Then $\xi := (\xi_k,\, k \in \mathbf{Z})$ is a sequence of independent, identically distributed $\{0,1\}^6$–valued random vectors.

(For technical convenience, we assume that the random variables $\xi_{k,i}$ are all defined in $\{0,1\}$ at *every* $\omega \in \Omega$, not just on a set of probability 1.)

(B) Refer to Definitions 4.3 and 4.5. Define the sequence $U := (U_k,\, k \in \mathbf{Z})$ of $\{1,2,3,4,5,6\}$–valued random variables, and the sequence of $W := (W_k,\, k \in \mathbf{Z})$ of $\{0,1,2,3,4,5,6\}$–valued random variables as follows: For each $k \in \mathbf{Z}$,

$$U_k := g_{\text{basic}}(\xi_k, \xi_{k-1}, \xi_{k-2}, \dots) \quad \text{and} \tag{5.3}$$

$$\begin{aligned} W_k \quad &:= g_{\text{spaced}}(\xi_k, \xi_{k-1}, \xi_{k-2}, \dots) \\ &= U_k \cdot I(U_k \neq U_{k-1}) \end{aligned} \tag{5.4}$$

where the second equality in (5.4) comes from (5.3) and Definition 4.5.

(C) For convenient reference, here is a list (with some redundancy) of some basic properties of these random variables. First,

$$\begin{aligned} \forall\, k \in \mathbf{Z},\ \forall\, \omega \in \Omega, \quad U_k(\omega) &\in \{1,2,3,4,5,6\} \text{ and} \\ W_k(\omega) &\in \{0,1,2,3,4,5,6\} \end{aligned} \tag{5.5}$$

by (5.3), (5.4), and Definitions 4.3 and 4.5. Next,

$$\forall\, k \in \mathbf{Z},\ \forall\, \omega \in \Omega,\ U_k(\omega) - U_{k-1}(\omega) \equiv 0 \text{ or } 1 \bmod 6. \tag{5.6}$$

For any $\omega \in \Omega$ such that the ("left-infinite") sequence $(\xi_j(\omega),\ \xi_{j-1}(\omega),\ \xi_{j-2}(\omega), \dots)$ of elements of $\{0,1\}^6$ is back-standard (see Section 2.4(D)) for some (hence every) $j \in \mathbf{Z}$, (5.6) holds by (5.3), (5.5), and Lemma 4.4. For all other $\omega \in \Omega$, (5.6) holds trivially (with $U_k(\omega) = 6$ for all $k \in \mathbf{Z}$) by (5.3) and the third paragraph of Definition 4.3. Also, referring to (5.5), one has that for each $k \in \mathbf{Z}$, by (5.3), (5.4), (5.6), and Definition 4.5,

$$\begin{aligned} &\{W_k = 0\} = \{U_k = U_{k-1}\}; \\ &\{W_k = 1\} = \{U_k = 1\} \cap \{U_{k-1} = 6\}; \text{ and} \\ &\forall\, i \in \{2,3,4,5,6\},\ \{W_k = i\} = \{U_k = i\} \cap \{U_{k-1} = i-1\}. \end{aligned} \tag{5.7}$$

Lemma 5.2. *In the context of Construction 5.1, the following statements hold (see Section 2.4(A)(D)):*

(A) *One has that* $P([\xi^t_{-5} \mid \xi^t_{-4} \mid \cdots \mid \xi^t_0] = I_6) = (5/8)^{30} \cdot (3/8)^6$.

(B) *The sequence ξ is two-sided standard a.s.*

(C) *Defining the random variable*

$$\tau := \min\left\{n \geq 6 : \left[\xi^t_{n-5} \mid \xi^t_{n-4} \mid \cdots \mid \xi^t_n\right] = I_6\right\} \tag{5.8}$$

(note that that set is a.s. nonempty, in fact infinite, by (B)), one has that $E\tau \leq 6 \cdot (8/5)^{30} \cdot (8/3)^6$.

(D) *Suppose $\mathcal{A}$ is a σ-field ($\subset \mathcal{F}$, in the underlying probability space $(\Omega, \mathcal{F}, P)$) such that $\mathcal{A}$ is independent of the sequence ξ. Suppose*

$$\kappa := (\dots, \kappa(-1), \kappa(0), \kappa(1), \dots)$$

is a random, $\mathcal{A}$–measurable, strictly increasing sequence of integers. Then (i) $\widetilde{\xi} := (\widetilde{\xi}_j, j \in \mathbf{Z}) := (\xi_{\kappa(j)},\ j \in \mathbf{Z})$ is a sequence of independent, identically distributed $\{0,1\}^6$–valued random variables with the same marginal *distribution as that of the random variables ξ_k, $k \in \mathbf{Z}$; (ii) this sequence $\widetilde{\xi}$ is independent of the σ-field $\mathcal{A}$; (iii) this sequence $\widetilde{\xi}$ is two-sided standard a.s.; and (iv) defining the random variable $\widetilde{\tau}$ to be the analog of the right hand side of (5.8), with $\xi^t_{n-\ell}$ replaced by $\widetilde{\xi}^t_{n-\ell}$ for each $\ell \in \{0, 1, \ldots, 5\}$, one has that $E\widetilde{\tau} \le 6 \cdot (8/5)^{30} \cdot (8/3)^6$.*

Proof. Statement (A) holds trivially by (5.1) and its entire sentence (since the 6×6 identity matrix I_6 has 30 0's and 6 1's).

Next, the $\{0, 1\}$–valued random variables

$$Z_k := I\left(\left[\xi^t_{k-5} \mid \xi^t_{k-4} \mid \cdots \mid \xi^t_k\right] = I_6\right), \tag{5.9}$$

$k \in \{\ldots, -12, -6, 0, 6, 12, \ldots\}$ are independent and identically distributed with $P(Z_k = 1) = (5/8)^{30} \cdot (3/8)^6 > 0$. Hence by two applications of the strong law of large numbers, statement (B) holds.

Next, defining the random variable $T := \min\{m \ge 1 : Z_{6m} = 1\}$ (see (5.9)), one has that $\tau \le 6T$ a.s. by (5.8), and hence $E\tau \le 6 \cdot ET$. Of course T is finite a.s. and has a geometric distribution: For each $n \in \mathbf{N}$, $P(T = n) = (1-p)^{n-1}p$ where $p := (5/8)^{30} \cdot (3/8)^6$. Hence as a standard fact (see e.g.Pitman, 1993, p. 212, Example 3, Problem 1), $ET = 1/p = (8/5)^{30} \cdot (8/3)^6$. Statement (C) follows.

Proof of statement (D). Suppose first that m is a positive integer.

Next, suppose $A \in \mathcal{A}$ is such that $P(A) > 0$.

Next, suppose $j(-m), j(-m+1), \ldots, j(m)$ are $2m+1$ integers such that the event $F := \bigcap_{u=-m}^{m} \{\kappa(u) = j(u)\}$ satisfies $P(F \cap A) > 0$. Then (see Section 2.5(A)), since $F \cap A \in \mathcal{A}$,

$$\begin{aligned}
&\mathcal{L}\left(\widetilde{\xi}_{-m}, \widetilde{\xi}_{-m+1}, \ldots, \widetilde{\xi}_m \mid A \cap F\right) \\
&= \mathcal{L}\left(\xi_{\kappa(-m)}, \xi_{\kappa(-m+1)}, \ldots, \xi_{\kappa(m)} \mid A \cap F\right) \\
&= \mathcal{L}\left(\xi_{j(-m)}, \xi_{j(-m+1)}, \ldots, \xi_{j(m)} \mid A \cap F\right) \\
&= \mathcal{L}\left(\xi_{j(-m)}, \xi_{j(-m+1)}, \ldots, \xi_{j(m)}\right) \\
&= \lambda \times \cdots \times \lambda,
\end{aligned}$$

the $(2m+1)$-order product measure (here and below), where λ is the marginal distribution (on $\{0, 1\}^6$) of the ξ_κ's.

It follows from a simple application of Remark 2.8 that $\mathcal{L}(\widetilde{\xi}_{-m}, \widetilde{\xi}_{-m+1}, \ldots, \widetilde{\xi}_m \mid A) = \lambda \times \cdots \times \lambda$. By the same argument with A replaced by Ω,

$$\mathcal{L}(\widetilde{\xi}_{-m}, \widetilde{\xi}_{-m+1}, \ldots, \widetilde{\xi}_m) = \lambda \times \cdots \times \lambda.$$

Since $A \in \mathcal{A}$ (with $P(A) > 0$) was arbitrary, it now also follows that the random vector $(\widetilde{\xi}_{-m}, \widetilde{\xi}_{-m+1}, \ldots, \widetilde{\xi}_m)$ is independent of the σ–field $\mathcal{A}$.

Since $m \in \mathbf{N}$ was arbitrary, conclusions (i) and (ii) in statement (D) now follow from standard measure–theoretic arguments. Finally, conclusions (iii) and (iv) in statement (D) now follow from an application of statements (B) and (C) to the sequence $\widetilde{\xi}$. This completes the proof of statement (D), and of Lemma 5.2

Lemma 5.3. *The random sequence U in Construction 5.1(B) (equation (5.3)) is a strictly stationary, irreducible, aperiodic Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$, with invariant marginal distribution given by*

$$P(U_0 = j) = 1/6 \quad \forall\, j \in \{1, 2, 3, 4, 5, 6\}, \tag{5.10}$$

and with one-step transition probability matrix $(p_{ij},\ 1 \le i,j \le 6)$ *(where* $p_{ij} = P(U_1 = j \mid U_0 = i)$*) given by*

$$\begin{aligned} p_{ii} &:= 5/8 \quad \textit{for } i \in \{1,2,3,4,5,6\}, \\ p_{i,i+1} &:= 3/8 \quad \textit{for } i \in \{1,2,3,4,5\}, \\ p_{61} &:= 3/8, \quad \textit{and} \\ p_{ij} &:= 0 \quad \textit{for all other ordered pairs}(i,j). \end{aligned} \tag{5.11}$$

Proof. Since the sequence $\xi := (\xi_k,\ k \in \mathbf{Z})$ of $\{0,1\}^6$–valued random variables is i.i.d., the strict stationarity of the sequence U is a standard consequence of (5.3). Also, the random variables U_k, $k \in \mathbf{Z}$ take their values in the set $\{1,2,\dots,6\}$ by e.g. (5.5).

Our next task is to show the following: If i and j are each an element of $\{1,2,\dots,6\}$, K is an integer, $A \in \sigma(U_K, U_{K-1}, U_{K-2}, \dots)$ is an event, and $P(A \cap \{U_K = i\}) > 0$, then

$$\begin{aligned} &P(U_{K+1} = j \mid A \cap \{U_K = i\}) \\ &\qquad = P\left(U_{K+1} = j \mid U_K = i\right) = p_{ij} \end{aligned} \tag{5.12}$$

where p_{ij} is as defined in (5.11). Once that is established, it will follow that the sequence U is a Markov chain with one-step transition probability matrix (5.11). From that matrix, it will then be easy to see that the Markov chain U is irreducible (by the second and third lines in (5.11)) and aperiodic (by the first line in (5.11)). Also, it is easy to show that the (unique) invariant distribution on the state space $\{1,2,\dots,6\}$ for the one-step transition probability matrix (5.11) is the uniform distribution on $\{1,2,\dots,6\}$ (as in (5.10)). Thus, once (5.12) is verified, the proof of Lemma 5.3 will be complete.

We shall verify (5.12) for the case where $K = 0$, $i \in \{1,2,3,4,5\}$, and $j = i+1$. By similar arguments and strict stationarity, one can verify (5.12) in the other cases.

Suppose $A \in \sigma(U_0, U_{-1}, U_{-2}, \dots)$ and $P(A \cap \{U_0 = i\}) > 0$. For any $\omega \in \Omega$ such that $U_0(\omega) = i$, one has by (5.3) and Lemma 4.4(A) that

$$U_1(\omega) = \begin{cases} i & \text{if } \xi_{1,i+1}(\omega) = 0 \\ i+1 & \text{if } \xi_{1,i+1}(\omega) = 1. \end{cases} \tag{5.13}$$

Also, the event $A \cap \{U_0 = i\}$ is a member of the σ-field $\sigma(\xi_0, \xi_{-1}, \xi_{-2}, \dots)$ (see (5.3) again) and is therefore independent of the ($\{0,1\}^6$–valued) random variable ξ_1. Hence by (5.13) and (5.1),

$$\begin{aligned} P(U_1 = i+1 \mid A \cap \{U_0 = i\}) &= P(\xi_{1,i+1} = 1 \mid A \cap \{U_0 = i\}) \\ &= P(\xi_{1,i+1} = 1) = 3/8. \end{aligned} \tag{5.14}$$

Applying the same argument with A replaced by Ω, one has that $P(U_1 = i+1 \mid U_0 = i) = 3/8$. Since $p_{i,i+1} := 3/8$ by (5.11), one now has by (5.14) that (5.12) holds (for the case $K = 0$, $i \in \{1,2,\dots,5\}$, and $j = i+1$). That completes the proof of Lemma 5.3.

Lemma 5.4. *The random sequence* W *in Construction 5.1(B) (equation (5.4)) is strictly stationary and has the following properties:*

(A) *The sequence* W *satisfies Condition* $\mathcal{S}$ *a.s. (see Definition 2.3(A)).*

(B) *The (marginal) distribution of* W_0 *(on* $\{0,1,2,3,4,5,6\}$*) is given by* $P(W_0 = 0) = 5/8$ *and* $P(W_0 = i) = 1/16$ *for all* $i \in \{1,2,3,4,5,6\}$.

(C) *For each* $j \in \mathbf{Z}$*, the* $\{0,1\}$*–valued random variable* $I(W_j \neq 0)$ *is independent of the* σ*–field* $\sigma(U_k, W_k,\ k \leq j-1)$.

(D) *The* $\{0,1\}$*–valued random variables* $I(W_k \neq 0)$*,* $k \in \mathbf{Z}$ *are independent and identically distributed, each taking the value 0 resp. 1 with probability* $5/8$ *resp.* $3/8$*. In fact for each* $j \in \mathbf{Z}$*, the random sequence* $(I(W_k \neq 0),\ k \in \{j, j+1, j+2, \dots\})$ *is independent of the* σ*–field* $\sigma(U_k, W_k,\ k \leq j-1)$.

The redundancies here (statement (C) is a special case of (D), and $\sigma(U_k, W_k,\ k \leq j-1) = \sigma(U_k,\ k \leq j-1)$ by (5.4)) are for convenient reference.

Proof. Since the sequence $\xi := (\xi_k,\ k \in \mathbf{Z})$ of $\{0,1\}^6$–valued random variables is i.i.d., the strict stationarity of the sequence W is a standard consequence of (5.4). Also, statement (D) (in Lemma 5.4) follows from statements (B) and (C), strict stationarity, and an elementary induction argument. Our remaining task is to prove statements (A), (B), and (C).

Statement (A) holds by Lemma 5.2(B), equation (5.4), and Lemma 4.6(A).

To prove statement (B), first note that (see (5.5)) $P(W_0 = 0) = 1 - \sum_{i=1}^{6} P(W_0 = i)$. Hence to prove (B) it suffices to prove that $P(W_0 = i) = 1/16$ for each $i \in \{1,2,\dots,6\}$. But that holds by (5.7) and Lemma 5.3; for example, for $i = 6$ one thereby has

$$\begin{aligned} P(W_0 = 6) = P(U_0 = 6,\, U_{-1} = 5) = P(U_{-1} = 5) \cdot P(U_0 = 6 \mid U_{-1} = 5) \\ = (1/6) \cdot (3/8) = 1/16. \end{aligned}$$

Thus statement (B) holds.

Proof of statement (C). The argument is the same for any $j \in \mathbf{Z}$. We shall give it for $j = 1$. By (5.7), $\sigma(U_k, W_k,\ k \leq 0) = \sigma(U_k,\ k \leq 0)$. Our task is to prove that the $\{0,1\}$–valued random variable $I(W_1 \neq 0)$ is independent of the σ-field $\sigma(U_k,\ k \leq 0)$.

Let $m \in \mathbf{N}$ be arbitrary but fixed. By a standard measure–theoretic argument, it suffices to show that the random variable $I(W_1 \neq 0)$ is independent of the random vector $(U_0, U_{-1}, U_{-2}, \dots, U_{-m})$.

Referring to (5.5), suppose $i_0, i_1, \dots, i_m$ are each an element of $\{1,\ 2,\ \dots, 6\}$, and that the event $F := \bigcap_{k=0}^{m} \{U_{-k} = i_k\}$ satisfies $P(F) > 0$. It suffices to show that

$$P(I(W_1 \neq 0) = 0 \mid F) = P(I(W_1 \neq 0) = 0). \tag{5.15}$$

(For then by taking complements, one also obtains $P(I(W_1 \neq 0) = 1 \mid F) = P(I(W_1 \neq 0) = 1)$.) The right side of (5.15) is of course simply $P(W_1 = 0)$, which is $5/8$ by statement (B). By (5.7) and Lemma 5.3, the left side of (5.15) is simply

$$\begin{aligned} P(W_1 = 0 \mid F) = P(U_1 = U_0 \mid F) = P(U_1 = i_0 \mid F) \\ = P(U_1 = i_0 \mid U_0 = i_0) = 5/8. \end{aligned}$$

Thus (5.15) holds. That completes the proof of statement (C), and of Lemma 5.4

Lemma 5.5. *Refer to Construction 5.1(B), Lemma 5.4(A)(B), and Definition 2.3(A). Define the random variable*

$$T := \min\{k \in \mathbf{N} : W_k = 1\}. \tag{5.16}$$

Then $E(T \mid W_0 = 1) = 16$ *and* $\mathrm{Var}(T \mid W_0 = 1) = 80/3$.

Proof. For convenience, for each $k \ge 1$, define the $\{0,1\}$–valued random variable $V_k := I(W_k \ne 0)$. By Lemma 5.4(D) (both sentences of it), conditional on the event $\{W_0 = 1\}$, the random variables $V_1, V_2, V_3, \dots$ are independent and identically distributed, each taking the value 0 resp. 1 with probability 5/8 resp. 3/8. By Lemma 5.4(A) (see Definition 2.3(A)) and (5.16), for P–a.e. $\omega \in \{W_0 = 1\}$,

$$T(\omega) = \min\left\{ n \in \mathbf{N} : \sum_{k=1}^{n} V_k(\omega) = 6 \right\}.$$

Hence conditional on the event $\{W_0 = 1\}$, the random variable T has the "negative binomial" distribution with parameters $r = 6$ and $p = 3/8$. Hence by standard elementary calculations (see e.g.Pitman, 1993, pp. 213–215, Example 4, Problem 2), $E(T \mid W_0 = 1) = r/p = 16$ and $\mathrm{Var}(T \mid W_0 = 1) = r(1-p)/p^2 = 80/3$. Thus Lemma 5.5 holds.

Lemma 5.6. *Refer to Section 2.7(B) and the sequence W in Construction 5.1(B). For any given $J \in \mathbf{Z}$, the ordered triplet*

$$(\sigma(W_k, k \le J),\ \{W_J = 1\},\ \sigma(W_k, k \ge J)) \tag{5.17}$$

is a restricted Markov triplet.

Proof. By strict stationarity (see Lemma 5.4), it suffices to give the argument for $J = 0$.

Refer to Section 2.7(A). We shall use some standard elementary properties of Markov triplets (see e.g. Bradley, 2007a, Vol. 1, Appendix, Section A701). By Lemma 5.3,

$$(\sigma(U_k, k \le -2),\ \sigma(U_{-1}, U_0),\ \sigma(U_k, k \ge 1))$$

is a Markov triplet. Hence so is

$$(\sigma(U_k, k \le 0),\ \sigma(U_{-1}, U_0),\ \sigma(U_k, k \ge -1)).$$

Hence (trivially) by (5.7), so is

$$(\sigma(W_k, k \le 0),\ \sigma(U_{-1}, U_0),\ \sigma(W_k, k \ge 0)). \tag{5.18}$$

Now by (5.7), $\{W_0 = 1\} = \{U_0 = 1\} \cap \{U_{-1} = 6\}$, which is an atom of the σ–field $\sigma(U_{-1}, U_0)$. Hence from the Markov triplet (5.18), one has that (5.17) is (for $J = 0$) a restricted Markov triplet. That completes the proof.

6. Scaffolding (part 1)

Throughout the rest of this paper, the context will be a particular probability space $(\Omega, \mathcal{F}, P)$, rich enough to accommodate all random variables defined henceforth. With or without explicit mention, many of those "random variables" will be random vectors or random sequences, taking their values in spaces such as $(\{0,1\}^6)^{\mathbf{N}}$ or $\{-1,1\}^{\mathrm{sxtp}(n)}$.

Sections 6 (here), 7, and 8 will provide "scaffolding" that will be used for the definition (in Section 7) of the random sequence X for Theorem 1.1 and for the proofs of the various properties of X stated in that theorem.

Construction 6.1.

(A) On our probability space $(\Omega, \mathcal{F}, P)$, let $\xi_{ki}^{(n)}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$, $i \in \{1,2,3,4,5,6\}$ be an array of independent, identically distributed $\{0,1\}$–valued random variables such that for each (n,k,i),

$$P\left(\xi_{ki}^{(n)} = 0\right) = 5/8 \quad \text{and} \quad P\left(\xi_{ki}^{(n)} = 1\right) = 3/8. \tag{6.1}$$

From those random variables, for convenient reference, for each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, define the $\{0,1\}^6$–valued random variable (random vector)

$$\xi_k^{(n)} := \left(\xi_{k1}^{(n)}, \xi_{k2}^{(n)}, \dots, \xi_{k6}^{(n)}\right). \tag{6.2}$$

Of course those random variables $\xi_k^{(n)}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$ are independent and have the same distribution (a product measure) on $\{0,1\}^6$. From those random variables, again for convenient reference, for each $n \in \mathbf{N}$, define the random sequence $\xi^{(n)} := (\xi_k^{(n)}, k \in \mathbf{Z})$.

(B) For each $n \in \mathbf{N}$, the random sequence $\xi^{(n)}$ here has (trivially) the same distribution (on $(\{0,1\}^6)^{\mathbf{Z}}$) as the random sequence ξ in Construction 5.1. Henceforth, *theorems involving the random sequence ξ in Section 5 will be applied freely to each of the random sequences $\xi^{(n)}$, $n \in \mathbf{N}$ here.*

(C) Referring to (6.2), for each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, define the $(\{0,1\}^6)^n$–valued random variable

$$\overline{\xi}_k^{(n)} := \left(\xi_k^{(1)}, \xi_k^{(2)}, \dots, \xi_k^{(n)}\right). \tag{6.3}$$

Of course, for any particular $n \in \mathbf{N}$, those random variables $\overline{\xi}_k^{(n)}$, $k \in \mathbf{Z}$ are independent and identically distributed. For convenient reference, for each $n \in \mathbf{N}$, define the resulting random sequence $\overline{\xi}^{(n)} := (\overline{\xi}_k^{(n)}, k \in \mathbf{Z})$.

(D) Again referring to (6.2), for each $k \in \mathbf{Z}$, define the $(\{0,1\}^6)^{\mathbf{N}}$–valued random variable

$$\overline{\xi}_k^{(\infty)} := \left(\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}, \dots\right). \tag{6.4}$$

Of course those random variables $\overline{\xi}_k^{(\infty)}$ are independent and identically distributed. Define the resulting random sequence $\overline{\xi}^{(\infty)} := (\overline{\xi}_k^{(\infty)}, k \in \mathbf{Z})$.

Construction 6.2. For each $n \in \mathbf{N}$, we shall define a sequence $W^{(n)} := (W_k^{(n)}, k \in \mathbf{Z})$ of $\{0,1,2,3,4,5,6\}$–valued random variables such that (see Section 2.6 and Construction 6.1(C))

$$\text{the ordered pair } (W^{(n)}, \overline{\xi}^{(n)}) \text{ satisfies Condition } \mathcal{M}. \tag{6.5}$$

Also, for each $n \in \mathbf{N}$, we shall define an event $\Omega_{\text{good}}^{(n)}$ such that

$$P\left(\Omega_{\text{good}}^{(n)}\right) = 1 \tag{6.6}$$

and (see Section 2.3)

$$\begin{gathered}\text{for each } \omega \in \Omega_{\text{good}}^{(n)}, \text{ the sequence } W^{(n)}(\omega) \\ \text{of elements of } \{0,1,\dots,6\} \text{ satisfies condition } \mathcal{S}.\end{gathered} \tag{6.7}$$

Also, for each $n \in \mathbf{N}$, we shall define nonnegative integer–valued random variables $\Psi(n,k,j)$, $k \in \mathbf{Z}$, $j \in \{0,1,2,\dots\}$, such that for each $j \in \{0,1,2,\dots\}$, defining the random sequence $\Psi(n,j) := (\Psi(n,k,j),\ k \in \mathbf{Z})$, one has that (see Section 2.6 again)

$$\text{the ordered pair } (\Psi(n,j),\, W^{(n)}) \text{ satisfies Condition } \mathcal{M}. \tag{6.8}$$

The definition will be recursive — starting with just $W^{(1)}$ and $\Omega^{(1)}_{\text{good}}$, and then (for the recursion step) set up with $\Psi(n,j)$ defined together with $W^{(n+1)}$ and $\Omega^{(n+1)}_{\text{good}}$.

Initial step. First, define the sequence $W^{(1)} := (W^{(1)}_k,\ k \in \mathbf{Z})$ of $\{0,1,\dots,6\}$–valued random variables as follows: For each $k \in \mathbf{Z}$ (see (6.2) and Definition 4.5),

$$W^{(1)}_k := g_{\text{spaced}}\left(\xi^{(1)}_k, \xi^{(1)}_{k-1}, \xi^{(1)}_{k-2}, \dots\right). \tag{6.9}$$

Obviously (see Sections 6.1(B) and 2.6) equation (6.5) holds for $n = 1$.

Also, let $\Omega^{(1)}_{\text{good}}$ denote the set of all $\omega \in \Omega$ such that the sequence $\xi^{(1)}(\omega) := (\xi^{(1)}_k(\omega),\ k \in \mathbf{Z})$ of elements of $\{0,1\}^6$ (see the very end of Construction 6.1(A)) is two-sided standard (see Notations 2.4(D)). Equation (6.6) holds for $n = 1$ by Lemma 5.2(B), and equation (6.7) holds for $n = 1$ by Lemma 4.6(A).

Recursion step. Now suppose $n \in \mathbf{N}$, and the event $\Omega^{(n)}_{\text{good}}$ and the sequence $W^{(n)} := (W^{(n)}_k,\ k \in \mathbf{Z})$ of $\{0,1,\dots,6\}$–valued random variables are already defined and satisfy (6.5), (6.6), and (6.7).

(A) For each $k \in \mathbf{Z}$ and each $j \in \{0,1,2,\dots\}$, define the nonnegative integer–valued random variable $\Psi(n,k,j)$ by (see Definition 2.2)

$$\Psi(n,k,j) := \psi_j\left(W^{(n)}_k, W^{(n)}_{k-1}, W^{(n)}_{k-2}, \dots\right). \tag{6.10}$$

Now by (6.5) and Definition 2.6, the sequence $W^{(n)}$ is strictly stationary. Hence by (6.10), equation (6.8) holds for every $j \geq 0$.

(B) Next, define the sequence $W^{(n+1)} := (W^{(n+1)}_k,\ k \in \mathbf{Z})$ of $\{0,1,\dots,6\}$–valued random variables as follows: For each $k \in \mathbf{Z}$ and each $\omega \in \Omega$, referring to Definition 4.5 and (6.10),

$$W^{(n+1)}_k(\omega) := \begin{cases} 0 & \text{if } W^{(n)}_k(\omega) \neq 1 \\ g_{\text{spaced}}\Big(\xi^{(n+1)}_k(\omega), \xi^{(n+1)}_{k-\Psi(n,k,1)(\omega)}(\omega), & \\ \qquad \xi^{(n+1)}_{k-\Psi(n,k,2)(\omega)}(\omega), & \\ \qquad \xi^{(n+1)}_{k-\Psi(n,k,3)(\omega)}(\omega), \dots\Big) & \text{if } W^{(n)}_k(\omega) = 1. \end{cases} \tag{6.11}$$

Verification of (6.5) *with n replaced by $n+1$.* For each $j \geq 0$, each $k \in \mathbf{Z}$, and each $\omega \in \Omega$, by (6.10),

$$\xi^{(n+1)}_{k-\Psi(n,k,j)(\omega)}(\omega) = \sum_{u=0}^{\infty} \xi^{(n+1)}_{k-u}(\omega) \cdot I\left(\Psi(n,k,j) = u\right)(\omega). \tag{6.12}$$

Now by (6.8), (6.5), and Definition 2.6 (and a trivial argument), for each $j \geq 0$, the ordered pair $(\Psi(n,j), \overline{\xi}^{(n)})$ satisfies Condition $\mathcal{M}$. Hence by (6.12) and Construction 6.1(C), for each $j \geq 0$, the ordered pair

$$\left(\left(\xi^{(n+1)}_{k-\Psi(n,k,j)},\ k \in \mathbf{Z}\right), \overline{\xi}^{(n+1)}\right)$$

satisfies Condition $\mathcal{M}$. Hence by (6.5) for the given n, (6.11), and Construction 6.1(C), equation (6.5) holds with n replaced by $n+1$.

(C) Our final task in the recursion step is to define the event $\Omega_{\text{good}}^{(n+1)}$ and then to verify (6.6) and (6.7) with n replaced by $n+1$. To facilitate this, we shall define a couple of other random sequences.

Define the random strictly increasing sequence $\kappa^{(n)} := (\kappa(n,j),\ j \in \mathbf{Z})$ of integers as follows: Referring to (6.6) and (6.7) (for the given n) and to Section 2.3, for each $\omega \in \Omega_{\text{good}}^{(n)}$, define the integers $\kappa(n,j)(\omega)$, $j \in \mathbf{Z}$ (uniquely) by

$$\begin{aligned} \cdots < \kappa(n,-2)(\omega) < \kappa(n,-1)(\omega) < \kappa(n,0)(\omega) \le 0 \\ < 1 \le \kappa(n,1)(\omega) < \kappa(n,2)(\omega) < \kappa(n,3)(\omega) < \dots \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} \left\{k \in \mathbf{Z} : W_k^{(n)}(\omega) = 1\right\} = \\ \{\dots, \kappa(n,-1)(\omega), \kappa(n,0)(\omega), \kappa(n,1)(\omega), \dots\}. \end{aligned} \tag{6.14}$$

(The sequence $\kappa^{(n)}$ can be left undefined on the null set $(\Omega_{\text{good}}^{(n)})^c$.) By (6.5) and (6.13)–(6.14), $\sigma(\kappa^{(n)}) \dot\subset \sigma(\overline{\xi}^{(n)})$, and hence by Construction 6.1(A)(C), the sequence $\kappa^{(n)}$ is independent of the sequence $\xi^{(n+1)}$. Hence by Lemma 5.2(D) (and Section 6.1(B)),

$$\begin{aligned} \text{the sequence } (\xi_{\kappa(n,j)}^{(n+1)},\ j \in \mathbf{Z}) \text{ is i.i.d. with the} \\ \text{same marginal distribution as the } \xi_k\text{'s in Construction 5.1;} \end{aligned} \tag{6.15}$$

$$\begin{aligned} \text{the sequence } (\xi_{\kappa(n,j)}^{(n+1)}, j \in \mathbf{Z}) \\ \text{is independent of the } \sigma\text{-field} \sigma(\overline{\xi}^{(n)}); \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \text{the sequence } (\xi_{\kappa(n,j)}^{(n+1)}, j \in \mathbf{Z}) \text{ is} \\ \text{two-sided standard a.s. (see Section 2.4(D))}\ . \end{aligned} \tag{6.17}$$

Let $\Omega_{\text{good}}^{(n+1)}$ denote the set of all $\omega \in \Omega_{\text{good}}^{(n)}$ such that the sequence $(\xi_{\kappa(n,j)(\omega)}^{(n+1)}(\omega),\ j \in \mathbf{Z})$ of elements of $\{0,1\}^6$ is two-sided standard. Then by (6.17), equation (6.6) holds with n replaced by $n+1$.

Verification of (6.7) *with n replaced by $n+1$.* Refer to (6.10), (6.13)–(6.14), and Definition 2.2. For each $\omega \in \Omega_{\text{good}}^{(n+1)}$, each $\ell \in \mathbf{Z}$, and each $j \ge 0$,

$$\Psi(n, \kappa(n,\ell)(\omega), j)(\omega) = \kappa(n,\ell)(\omega) - \kappa(n,\ell-j)(\omega)$$

and hence

$$\kappa(n,\ell)(\omega) - \Psi(n, \kappa(n,\ell)(\omega), j)(\omega) = \kappa(n,\ell-j)(\omega).$$

Hence for each $\omega \in \Omega_{\text{good}}^{(n+1)}$ and each $\ell \in \mathbf{Z}$, by (6.11) and (6.14),

$$W_{\kappa(n,\ell)(\omega)}^{(n+1)}(\omega) = g_{\text{spaced}}\left(\xi_{\kappa(n,\ell)(\omega)}^{(n+1)}(\omega), \xi_{\kappa(n,\ell-1)(\omega)}^{(n+1)}(\omega), \xi_{\kappa(n,\ell-2)(\omega)}^{(n+1)}(\omega), \dots\right). \tag{6.18}$$

By (6.18), the definition of $\Omega_{\text{good}}^{(n+1)}$, and Lemma 4.6(A), one has that for each $\omega \in \Omega_{\text{good}}^{(n+1)}$, the sequence $(W_{\kappa(n,\ell)(\omega)}^{(n+1)}(\omega),\ \ell \in \mathbf{Z})$ of elements of $\{0, 1, \dots, 6\}$ satisfies Condition $\mathcal{S}$. Also, by (6.11) and (6.14), for each $\omega \in \Omega_{\text{good}}^{(n+1)}$, one has that

$$\begin{aligned} W_k^{(n+1)}(\omega) = 0 \\ \text{for all } k \in \mathbf{Z} - \{\dots, \kappa(n, -1)(\omega), \kappa(n, 0)(\omega), \kappa(n, 1)(\omega), \dots\}. \end{aligned} \tag{6.19}$$

Hence by Remark 2.3(C), equation (6.7) holds with n replaced by $n + 1$. This completes the recursion step and Construction 6.2.

Construction 6.3. This builds on the material in Constructions 6.1 and 6.2

(A) Referring to the events $\Omega_{\text{good}}^{(n)}$, $n \in \mathbf{N}$ in Construction 6.2, define the event

$$\Omega_0 := \bigcap_{n=1}^{\infty} \Omega_{\text{good}}^{(n)}. \tag{6.20}$$

Then by (6.6),

$$P(\Omega_0) = 1. \tag{6.21}$$

Also, by (6.7), (6.14), and the definitions of the events $\Omega_{\text{good}}^{(n)}$, $n \in \mathbf{N}$ in Construction 6.2, one has the following:

(i) For each $\omega \in \Omega_0$ and each $n \in \mathbf{N}$, the sequence $(W_k^{(n)}(\omega),\ k \in \mathbf{Z})$ of elements of $\{0, 1, \dots, 6\}$ satisfies Condition $\mathcal{S}$. (Again recall Definition 2.3(A).)

(ii) For each $\omega \in \Omega_0$, the sequence $(\xi_k^{(1)}(\omega),\ k \in \mathbf{Z})$ of elements of $\{0, 1\}^6$ is two-sided standard.

(iii) For each $\omega \in \Omega_0$ and each $n \in \mathbf{N}$, the sequence $(\xi_j^{(n+1)}(\omega),\ j \in \{k \in \mathbf{Z} : W_k^{(n)}(\omega) = 1\})$ (see Section 2.1(F)(ii)) is two-sided standard.

(B) *Remark.* From (6.11), one has that for a given $n \in \mathbf{N}$, $k \in \mathbf{Z}$, and $\omega \in \Omega$, if $W_k^{(n)}(\omega) \neq 1$ then $W_k^{(n+1)}(\omega) = 0$. Thus for a given $k \in \mathbf{Z}$ and $\omega \in \Omega$, the sequence $(W_k^{(1)}(\omega), W_k^{(2)}(\omega), W_k^{(3)}(\omega), \dots)$ of elements of $\{0, 1, \dots, 6\}$ will have one of the following five forms:

(i) $(0, 0, 0, \dots)$,

(ii) $(i, 0, 0, 0, \dots)$ where $i \in \{2, 3, 4, 5, 6\}$,

(iii) $(1, 1, \dots, 1, 0, 0, 0, \dots)$,

(iv) $(1, 1, \dots, 1, i, 0, 0, 0, \dots)$ where $i \in \{2, 3, 4, 5, 6\}$,

(v) $(1, 1, 1, \dots)$.

(In (iii) and (iv), the number of 1's can be any positive integer.)

(C) *Remark.* Recall the first sentence of Remark (B) above. By Remark 2.3(B) and remark (i) in Section (A) above, one has the following: For any $n \in \mathbf{N}$, any $\omega \in \Omega_0$ (see (6.20)), and any pair of integers J and L such that $J \leq L$, one has that

$$\begin{aligned} &\text{card}\{k \in \mathbf{Z} :\ J \leq k \leq L \ \text{ and } \ W_k^{(n+1)}(\omega) = 1\} \\ &\leq 1 + (1/6) \cdot \text{card}\{k \in \mathbf{Z} :\ J \leq k \leq L \ \text{ and } \ W_k^{(n+1)}(\omega) \neq 0\} \\ &\leq 1 + (1/6) \cdot \text{card}\{k \in \mathbf{Z} :\ J \leq k \leq L \ \text{ and } \ W_k^{(n)}(\omega) = 1\}. \end{aligned} \tag{6.22}$$

(D) *Remark.* For each $n \in \mathbf{N}$, each $k \in \mathbf{Z}$ and each $\omega \in \Omega_0$,

$$\begin{aligned}\Psi(n,k,0)(\omega) &= \min\left\{i \geq 0 : W^{(n)}_{k-i}(\omega) = 1\right\} \\ &\leq \min\left\{i \geq 0 : W^{(n+1)}_{k-i}(\omega) = 1\right\} = \Psi(n+1,k,0)(\omega)\end{aligned} \tag{6.23}$$

by (6.10) and Definition 2.2, since (again see Remark (B) above) the second set in (6.23) is a subset of the first; equivalently

$$k - \Psi(n,k,0)(\omega) \geq k - \Psi(n+1,k,0)(\omega). \tag{6.24}$$

(E) Referring to (6.9) and (6.11), for each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, define the $\{0,1,\dots,6\}^n$–valued random variable

$$\overline{W}^{(n)}_k := \left(W^{(1)}_k, W^{(2)}_k, \dots, W^{(n)}_k\right). \tag{6.25}$$

Accordingly, for each $n \in \mathbf{N}$, define the random sequence $\overline{W}^{(n)} := (\overline{W}^{(n)}_k,\ k \in \mathbf{Z})$. By (6.5) and Definition 2.6(A) (and Construction 6.1(C)), for each $n \in \mathbf{N}$,

$$\text{The ordered pair } (\overline{W}^{(n)}, \overline{\xi}^{(n)}) \text{ satisfies Condition } \mathcal{M}. \tag{6.26}$$

(F) *Remark.* Of course the random sequence $W^{(1)}$ in (6.9) has the same distribution (on $\{0,1,\dots,6\}^{\mathbf{Z}}$) as the random sequence W in Construction 5.1(B), by (6.9), (5.4), and Construction 6.1(B).

(G) *Remark.* For each $n \in \mathbf{N}$, the random sequence $(W^{(n+1)}_{\kappa(n,j)},\ j \in \mathbf{Z})$ (recall (6.20), (6.21), (6.13), (6.14), and (6.18)) has the same distribution (on $\{0,1,\dots,6\}^{\mathbf{Z}}$) as the random sequence $W := (W_j,\ j \in \mathbf{Z})$ in Construction 5.1(B). (This holds by (6.15), (6.18), (5.4), and a standard measure–theoretic argument.)

(H) *Remark.* For each $n \in \mathbf{N}$, the random sequence $(W^{(n+1)}_{\kappa(n,j)},\ j \in \mathbf{Z})$ (recall (G) above) is independent of the sequence $\overline{\xi}^{(n)}$, by (6.18) and (6.16).

Lemma 6.4.

(A) *For each $i \in \{1,2,\dots,6\}$, $P(W^{(1)}_0 = i) = 1/16$.*

(B) *If $n \in \mathbf{N}$ and $A \in \sigma(\overline{W}^{(n)})$ (see section 6.3(E)) are such that $P(A \cap \{W^{(n)}_0 = 1\}) > 0$, then for each $i \in \{1,2,\dots,6\}$, $P(W^{(n+1)}_0 = i \mid A \cap \{W^{(n)}_0 = 1\}) = 1/16$.*

(C) *For each $n \in \mathbf{N}$ and each $i \in \{1,2,\dots,6\}$, $P(W^{(n)}_0 = i) = 16^{-n}$.*

(D) *Suppose $n \in \mathbf{N}$. Suppose S is a nonempty finite set $\subset \mathbf{Z}$. Suppose $A \in \sigma(\overline{W}^{(n)})$. Suppose also that the event*

$$F := A \bigcap \left[\bigcap_{k\in S}\left\{W^{(n)}_k = 1\right\}\right] \tag{6.27}$$

satisfies $P(F) > 0$. Then conditional on F, the $\{0,1\}$–valued random variables $I(W^{(n+1)}_k \neq 0)$, $k \in S$ are independent and identically distributed, each taking the value 0 resp. 1 with probability 5/8 resp. 3/8.

Proof. Statement (A) holds by Lemma 5.4(B) and Remark 6.3(F).

To prove statement (B), note first that by (6.13)–(6.14), $\kappa(n,0)(\omega) = 0$ for each $\omega \in \Omega_0 \cap \{W^{(n)}_0 = 1\}$ (recall (6.20)–(6.21)). By (6.25), (6.26), and the hypothesis of statement (B), $A \cap \{W_0^{(n)} = 1\} \in \sigma(\overline{W}^{(n)}) \dot{\subset} \sigma(\overline{\xi}^{(n)})$. Hence for a given $i \in \{1, 2, \dots, 6\}$, by Remark 6.3(H), Remark 6.3(G), and Lemma 5.4(B),

$$P\left(W_0^{(n+1)} = i \,\middle|\, A \cap \{W_0^{(n)} = 1\}\right) = P\left(W_{\kappa(n,0)}^{(n+1)} = i \,\middle|\, A \cap \{W_0^{(n)} = 1\}\right)$$
$$= P\left(W_{\kappa(n,0)}^{(n+1)} = i\right) = P(W_0 = i) = 1/16$$

(where W_0 is as in Construction 5.1(B)). Thus statement (B) holds.

Statement (C) holds for $n = 1$ by statement (A). Also, for a given $n \in \mathbf{N}$ for which Statement (C) holds, and a given $i \in \{1, 2, \dots, 6\}$, one has that $\{W_0^{(n+1)} = i\} \subset \{W_0^{(n)} = 1\}$ by Remark 6.3(B), and hence by statement (B) (in Lemma 6.4),

$$P(W_0^{(n+1)} = i) = P(W_0^{(n)} = 1) \cdot P(W_0^{(n+1)} = i \mid W_0^{(n)} = 1)$$
$$= P(W_0^{(n)} = 1) \cdot 1/16 = (1/16)^{n+1}.$$

Now statement (C) holds for all $n \in \mathbf{N}$ by induction.

Proof of (D). Let $m :=$ card S. Let λ denote the "Bernoulli" probability measure on $\{0, 1\}$ given by $\lambda(\{0\}) = 5/8$ and $\lambda(\{1\}) = 3/8$. In the argument below, the notation $\lambda \times \cdots \times \lambda$ will mean the m–fold product measure on $\{0, 1\}^m$.

Let the elements of S be denoted by $s(i)$, $1 \le i \le m$ where $s(1) < s(2) < \cdots < s(m)$. Refer to (6.13)–(6.14) and to (6.20)–(6.21). For each m–tuple $j := (j(1), j(2), \dots, j(m))$ of integers such that $j(1) < j(2) < \cdots < j(m)$, define the event

$$F(j) := F \bigcap \Omega_0 \bigcap \left[\bigcap_{u=1}^{m} \{\kappa(n, j(u)) = s(u)\} \right].$$

Those events $F(j)$ are (pairwise) disjoint (and some of them will be empty), and by (6.27) and (6.13)–(6.14) their union is $F \cap \Omega_0 \doteq F$. Also, for each such j, one has that $F(j) \dot{\in} \sigma(\overline{W}^{(n)}) \dot{\subset} \sigma(\overline{\xi}^{(n)})$ by (6.25), (6.26), and the hypothesis of statement (D) (since by (6.13)–(6.14) the random variables $\kappa(n, i)$, $i \in \mathbf{Z}$ are $\sigma(W^{(n)})$–measurable modulo the null-set Ω_0^c).

Hence by Remark 6.3(H), Remark 6.3(G), and Lemma 5.4(D) (section 2.5(A)), for each such m-tuple j,

$$\mathcal{L}\left(I(W_{s(1)}^{(n+1)} \ne 0), I(W_{s(2)}^{(n+1)} \ne 0), \dots, I(W_{s(m)}^{(n+1)} \ne 0) \,\middle|\, F(j)\right)$$
$$= \mathcal{L}\left(I(W_{\kappa(n,j(1))}^{(n+1)} \ne 0), I(W_{\kappa(n,j(2))}^{(n+1)} \ne 0), \dots, I(W_{\kappa(n,j(m))}^{(n+1)} \ne 0) \,\middle|\, F(j)\right)$$
$$= \mathcal{L}\left(I(W_{\kappa(n,j(1))}^{(n+1)} \ne 0), I(W_{\kappa(n,j(2))}^{(n+1)} \ne 0), \dots, I(W_{\kappa(n,j(m))}^{(n+1)} \ne 0)\right)$$
$$= \lambda \times \cdots \times \lambda.$$

Hence by Remark 2.8,

$$\mathcal{L}\left(I(W_{s(1)}^{(n+1)} \ne 0), I(W_{s(2)}^{(n+1)} \ne 0), \dots, I(W_{s(m)}^{(n+1)} \ne 0) \,\middle|\, F\right) = \lambda \times \cdots \times \lambda.$$

Thus statement (D) holds. Lemma 6.4 is proved.

Lemma 6.5. *Suppose $n \in \mathbf{N}$. Then for each integer J, the ordered triplet*

$$\left(\sigma(\overline{W}_k^{(n)}, k \le J),\ \{W_J^{(n)} = 1\},\ \sigma(\overline{W}_k^{(n)}, k \ge J + 1)\right)$$

is a restricted Markov triplet (see Section 2.7(B)).

The slight "asymmetry" in this statement is just a matter of convenience. This statement is slightly stronger than what will be needed in its application later on.

Proof. For $n = 1$, this holds by Lemma 5.6 and Remark 6.3(F).

Now for induction, suppose N is a positive integer and Lemma 6.5 holds for $n = N$. Our task is to show that it holds for $n = N + 1$.

Of course by (6.26) (with $n = N+1$) and Definition 2.6(A), the sequence $\overline{W}^{(N+1)}$ is strictly stationary. Hence it suffices to carry out the argument in the case where ($n = N + 1$ and) $J = 0$.

Refer to (6.13)–(6.14) for $n = N$. For convenient notation, define the sequence $Z^* := (Z_j^*,\ j \in \mathbf{Z})$ of $\{0, 1, \dots, 6\}$–valued random variables as follows: For each $j \in \mathbf{Z}$

$$Z_j^* := W_{\kappa(N,j)}^{(N+1)}. \tag{6.28}$$

By the induction hypothesis,

$$\Big(\sigma(\overline{W}_k^{(N)}, k \le 0),\ \{W_0^{(N)} = 1\},\ \sigma(\overline{W}_k^{(N)}, k \ge 1)\Big)$$

is a restricted Markov triplet. By (6.28), Remark 6.3(G), and Lemma 5.6,

$$\big(\sigma(Z_j^*, j \le 0),\ \{Z_0^* = 1\},\ \sigma(Z_j^*, j \ge 0)\big)$$

is a restricted Markov triplet. Hence by (6.26), Remark 6.3(H), and Remark 2.7(D),

$$\begin{aligned}\big(\sigma(\overline{W}_k^{(N)}, k \le 0) \vee \sigma(Z_j^*, j \le 0),\ \{W_0^{(N)} = 1\} \cap \{Z_0^* = 1\},&\\ \sigma(\overline{W}_k^{(N)}, k \ge 1) \vee \sigma(Z_j^*, j \ge 1)\big)&\end{aligned} \tag{6.29}$$

is a restricted Markov triplet.

Next, by (6.13)–(6.14) and a simple argument,

$$\sigma(\kappa(N,j), j \le 0) \dot\subset \sigma(W_k^{(N)}, k \le 0) \quad \text{and} \tag{6.30}$$

$$\sigma(\kappa(N,j), j \ge 1) \dot\subset \sigma(W_k^{(N)}, k \ge 1). \tag{6.31}$$

Of course by (6.13)–(6.14), (6.28), and (6.11), for a given $k \le 0$ and a given $\omega \in \Omega_0$ (see (6.20)–(6.21)),

$$W_k^{(N+1)}(\omega) = \begin{cases} 0 & \text{if } k \notin \{\kappa(N,0)(\omega), \kappa(N,-1)(\omega), \kappa(N,-2)(\omega), \dots\} \\ Z_j^*(\omega) & \text{if } k = \kappa(N,j)(\omega) \text{ for some } j \le 0 \end{cases} \tag{6.32}$$

Hence by (6.30) (and (6.25)),

$$\begin{aligned}\sigma(W_k^{(N+1)}, k \le 0) &\dot\subset \sigma(\kappa(N,j), j \le 0) \vee \sigma(Z_j^*, j \le 0)\\ &\dot\subset \sigma(\overline{W}_k^{(N)}, k \le 0) \vee \sigma(Z_j^*, j \le 0).\end{aligned} \tag{6.33}$$

Hence by (6.25) again,

$$\sigma(\overline{W}_k^{(N+1)}, k \le 0) \dot\subset \sigma(\overline{W}_k^{(N)}, k \le 0) \vee \sigma(Z_j^*, j \le 0). \tag{6.34}$$

Next, as an analog of (6.32), one has that for $k \ge 1$ and $\omega \in \Omega_0$, $W_k^{(N+1)}(\omega) = 0$ resp. $Z_j^*(\omega)$ if $k \notin \{\kappa(N,1)(\omega), \kappa(N,2)(\omega), \kappa(N,3)(\omega), \dots\}$ resp. $k = \kappa(N,j)(\omega)$ for some $j \ge 1$. Then by (6.31), one obtains an analog of (6.33) with the inequalities $k \le 0$ and $j \le 0$ replaced by $k \ge 1$ and $j \ge 1$. Thereby one obtains the following analog of (6.34):

$$\sigma(\overline{W}_k^{(N+1)}, k \ge 1) \dot\subset \sigma(\overline{W}_k^{(N)}, k \ge 1) \vee \sigma(Z_j^*, j \ge 1). \tag{6.35}$$

Now by Remark 6.3(B) and (6.13)–(6.14) and a simple argument,

$$\{W_0^{(N+1)} = 1\} \subset \{W_0^{(N)} = 1\} \doteq \{\kappa(N,0) = 0\}.$$

As a trivial consequence, by (6.28) for $j = 0$,

$$\begin{aligned}\{W_0^{(N+1)} = 1\} &\doteq \{W_0^{(N+1)} = 1\} \cap \{W_0^{(N)} = 1\} \cap \{\kappa(N,0) = 0\} \\ &= \{W_{\kappa(N,0)}^{(N+1)} = 1\} \cap \{W_0^{(N)} = 1\} \cap \{\kappa(N,0) = 0\} \\ &= \{Z_0^* = 1\} \cap \{W_0^{(N)} = 1\} \cap \{\kappa(N,0) = 0\} \\ &\doteq \{Z_0^* = 1\} \cap \{W_0^{(N)} = 1\}.\end{aligned} \tag{6.36}$$

Now by (6.34), (6.35), (6.36), and the entire sentence containing (6.29), one has that

$$\Big(\sigma(\overline{W}_k^{(N+1)}, k \le 0),\ \{W_0^{(N+1)} = 1\},\ \sigma(\overline{W}_k^{(N+1)}, k \ge 1)\Big)$$

is a restricted Markov triplet. That completes the induction step and the proof of Lemma 6.5.

Lemma 6.6. *Suppose $n \in \mathbf{N}$. Referring to section 6.3(A) (equations (6.20) and (6.21) and statement (i)), Definition 2.3(A), and Lemma 6.4(C), define the positive integer–valued random variable*

$$T^{(n)} := \min\{k \in \mathbf{N} : W_k^{(n)} = 1\}. \tag{6.37}$$

Then

$$E\left(T^{(n)} \,\middle|\, W_0^{(n)} = 1\right) = 16^n, \tag{6.38}$$

$$\mathrm{Var}\left(T^{(n)} \,\middle|\, W_0^{(n)} = 1\right) \le 16^{2n}, \quad \textit{and (hence)} \tag{6.39}$$

$$E\left((T^{(n)})^2 \,\middle|\, W_0^{(n)} = 1\right) \le 2 \cdot 16^{2n}. \tag{6.40}$$

Proof. For $n = 1$, this holds by Lemma 5.5 and Remark 6.3(F). Now for induction suppose $N \in \mathbf{N}$ and Lemma 6.6 holds for $n = N$. Our task is to show that it holds for $n = N + 1$.

By (6.13)–(6.14),

$$T^{(N)} = \kappa(N,1) \quad \text{a.s.} \tag{6.41}$$

and also

$$\left\{W_0^{(N)} = 1\right\} \doteq \{\kappa(N,0) = 0\}. \tag{6.42}$$

As in the proof of Lemma 6.5, define the sequence $Z^* := (Z_j^*,\ j \in \mathbf{Z})$ of $\{0,1,\dots,6\}$–valued random variables by (6.28). Then (6.36) holds. We shall refer to both (6.28) and (6.36) freely below.

By (6.28), Remark 6.3(G), and Lemma 5.4(A), the sequence Z^* satisfies Condition $\mathcal{S}$ a.s. (see Definition 2.3(A) again). Accordingly, define the positive integer–valued random variable M as follows:

$$M := \min\{j \in \mathbf{N} : Z_j^* = 1\}. \tag{6.43}$$

By (6.28), Remark 6.3(G), and Lemma 5.5,

$$E(M \mid Z_0^* = 1) = 16 \quad \text{and} \quad \mathrm{Var}(M \mid Z_0^* = 1) = 80/3. \tag{6.44}$$

Next, by (6.28), Remark 6.3(H), and (6.5), the sequences $W^{(N)}$ and Z^* are independent of each other. It follows from (6.36), (6.43), and a simple calculation that $E(M \mid Z_0^* = 1) = E(M \mid W_0^{(N+1)} = 1)$ and $\mathrm{Var}(M \mid Z_0^* = 1) = \mathrm{Var}(M \mid W_0^{(N+1)} = 1)$. Hence by (6.44),

$$E(M \mid W_0^{(N+1)} = 1) = 16 \quad \text{and} \quad \mathrm{Var}(M \mid W_0^{(N+1)} = 1) = 80/3. \tag{6.45}$$

Now for every $\omega \in \Omega_0$ (see (6.20)–(6.21) again), one has the following: For any $j \in \{1, 2, \dots, M(\omega) - 1\}$ (if $M(\omega) \geq 2$), $W^{(N+1)}_{\kappa(N,j)(\omega)}(\omega) = Z_j^*(\omega) \neq 1$ by (6.28) and (6.43). For any $k \in \mathbf{Z} - \{\kappa(N,j)(\omega) : j \in \mathbf{Z}\}$, $W_k^{(N+1)}(\omega) = 0$ by (6.13)–(6.14) and Remark 6.3(B). Hence (for $\omega \in \Omega_0$), for all $k \in \{1, 2, \dots, \kappa(N, M(\omega))(\omega) - 1\}$, $W_k^{(N+1)}(\omega) \neq 1$. Also (for $\omega \in \Omega_0$), $W^{(N+1)}_{\kappa(N,M(\omega))(\omega)}(\omega) = Z^*_{M(\omega)}(\omega) = 1$ by (6.28) and (6.43).

By the preceding two sentences (see (6.20)–(6.21) again) and (6.37),

$$T^{(N+1)} = \kappa(N, M) \ \text{ a.s.} \tag{6.46}$$

Now let us look at the random variables $\kappa(N,j) - \kappa(N,j-1)$, $j \in \mathbf{N}$. By Lemma 6.5 and the strict stationarity of the sequence $W^{(N)}$ (recall (6.5) and Remark 2.6(B)), one has the following: For any positive integers J and ℓ, and any event $A \in \sigma(W_k^{(N)}, k \leq J-1)$ such that $P(A \cap \{W_J^{(N)} = 1\}) > 0$, one has that (see Remark 2.7(C))

$$\begin{aligned}
&P\left(W_{J+i}^{(N)} \neq 1 \ \forall\, i \in \{1, \dots, \ell-1\} \quad \text{and} \quad W_{J+\ell}^{(N)} = 1 \,\middle|\, A \cap \{W_J^{(N)} = 1\}\right) \\
&\quad = P\left(W_{J+i}^{(N)} \neq 1 \ \forall\, i \in \{1, \dots, \ell-1\} \quad \text{and} \quad W_{J+\ell}^{(N)} = 1 \,\middle|\, W_J^{(N)} = 1\right) \\
&\quad = P\left(W_i^{(N)} \neq 1 \ \forall\, i \in \{1, \dots, \ell-1\} \quad \text{and} \quad W_\ell^{(N)} = 1 \,\middle|\, W_0^{(N)} = 1\right).
\end{aligned}$$

(Of course for $\ell = 1$, omit the phrases $W_{J+i}^{(N)} \neq 1$ resp. $W_i^{(N)} \neq 1 \ \forall\, i \in \{1, \dots, \ell - 1\}$.) Hence (see (6.13)–(6.14) again), by a standard induction argument, one can show that conditional on the event $\{W_0^{(N)} = 1\}$, the random variables $\kappa(N,1)$ (or $\kappa(N,1) - \kappa(N,0)$ — see (6.42)), $\kappa(N,2) - \kappa(N,1)$, $\kappa(N,3) - \kappa(N,2)$, $\kappa(N,4) - \kappa(N,3), \dots$ are independent and identically distributed, with (see (6.42), (6.41), and (6.13)–(6.14))

$$\begin{aligned}
\mathcal{L}(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N)} = 1) \quad &= \mathcal{L}(\kappa(N,1) \mid W_0^{(N)} = 1) \\
&= \mathcal{L}(T^{(N)} \mid W_0^{(N)} = 1).
\end{aligned}$$

Hence by (6.36), the sentence after (6.44) (recall that $\sigma(\kappa(N,i)) \dot{\subset} \sigma(W^{(N)})$ for $i \in \mathbf{Z}$ by (6.13)–(6.14)), and a standard trivial calculation, conditional on the event $\{W_0^{(N+1)} = 1\}$, the random variables $\kappa(N,j) - \kappa(N,j-1)$, $j \in \mathbf{N}$ are independent and identically distributed, with

$$\mathcal{L}(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) \tag{6.47}$$

$$= \mathcal{L}(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N)} = 1) = \mathcal{L}(T^{(N)} \mid W_0^{(N)} = 1). \tag{6.48}$$

Hence by the induction hypothesis of (6.38) and (6.39) for $n = N$,

$$E(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) = 16^N \quad \text{and} \tag{6.49}$$

$$\mathrm{Var}(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) \leq 16^{2N}. \tag{6.50}$$

Now recall again from (6.13)–(6.14) that $\sigma(\kappa(N,i), i \in \mathbf{Z}) \dot{\subset} \sigma(W^{(N)})$. By that fact, (6.43), and the sentence after (6.44), and equation (6.36), together with a standard simple argument, conditional on the event $\{W_0^{(N+1)} = 1\}$, the random variable M is independent of the random sequence $(\kappa(N,j) - \kappa(N,j-1),\ j \in \mathbf{N})$. Hence by (6.46), (6.45), (6.49), (6.50), the entire sentence containing (6.47), and a well known elementary calculation (see e.g.Feller, 1968, p. 301, Exercise 1), one has that (recall (6.42) and (6.36))

$$\begin{aligned}
&E(T^{(N+1)} \mid W_0^{(N+1)} = 1) = E(\kappa(N,M) \mid W_0^{(N+1)} = 1) \\
&= E\left(\sum_{j=1}^{M} [\kappa(N,j) - \kappa(N,j-1)] \,\middle|\, W_0^{(N+1)} = 1 \right) \\
&= E(M \mid W_0^{(N+1)} = 1) \cdot E(\,\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) \\
&= 16 \cdot 16^N = 16^{N+1}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Var}(T^{(N+1)} \mid W_0^{(N+1)} = 1) = \mathrm{Var}(\kappa(N,M) \mid W_0^{(N+1)} = 1) \\
&= \mathrm{Var}\left(\sum_{j=1}^{M} [\kappa(N,j) - \kappa(N,j-1)] \,\middle|\, W_0^{(N+1)} = 1] \right) \\
&= E(M \mid W_0^{(N+1)} = 1) \cdot \mathrm{Var}(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) \\
&\qquad + \mathrm{Var}(M \mid W_0^{(N+1)} = 1) \cdot \left[E(\kappa(N,1) - \kappa(N,0) \mid W_0^{(N+1)} = 1) \right]^2 \\
&\le 16 \cdot 16^{2N} + (80/3) \cdot (16^N)^2 < 16^{2(N+1)}.
\end{aligned}$$

Thus (6.38) and (6.39) (and hence also (6.40)) hold for $n = N+1$. That completes the induction step and the proof of Lemma 6.6.

Lemma 6.7. *Suppose $n \in \mathbf{N}$. Let p_n denote the probability that there exist at least two distinct integers $i, j \in \{1, 2, \dots, 6 \cdot 16^n\}$ such that $W_i^{(n)} = W_j^{(n)} = 1$. Then $p_n \ge 1/2$.*

Proof. Suppose $n \in \mathbf{N}$. Refer to (6.13)–(6.14). It suffices to prove that $P(\kappa(n,2) \le 6 \cdot 16^n) \ge 1/2$, or

$$P(\kappa(n,2) > 6 \cdot 16^n) \le 1/2. \tag{6.51}$$

Recall (from (6.5) and Remark 2.6(B)) that the sequence $W^{(n)}$ is strictly stationary. Recall from (6.13)–(6.14) that $\kappa(n,1) = T^{(n)}$ a.s., where $T^{(n)}$ is as in Lemma 6.6, and that $\{\kappa(n,0) = 0\} \doteq \{W_0^{(n)} = 1\}$. From (6.13)–(6.14) and a trivial argument, followed by Lemma 6.4(C), one has that for each positive integer j,

$$\begin{aligned}
&P(\kappa(n,1) = j) = \sum_{\ell=0}^{\infty} P(\kappa(n,0) = -\ell \text{ and } \kappa(n,1) = j) \\
&= \sum_{\ell=0}^{\infty} P(\kappa(n,0) = 0,\ \kappa(n,1) = j + \ell) \\
&= \sum_{\ell=0}^{\infty} P(\kappa(n,1) = j + \ell \mid \kappa(n,0) = 0) \cdot P(\kappa(n,0) = 0) \\
&= P(W_0^{(n)} = 1) \cdot P(\kappa(n,1) \ge j \mid \kappa(n,0) = 0) \\
&= 16^{-n} \cdot P(T^{(n)} \ge j \mid W_0^{(n)} = 1).
\end{aligned} \tag{6.52}$$

Now for (say) any positive integer–valued random variable Z, one has by a simple argument that $\sum_{j=1}^{\infty} j \cdot P(Z \geq j) \leq EZ^2$. Hence by (6.52) and Lemma 6.6,

$$\begin{aligned} E\kappa(n,1) &= \sum_{j=1}^{\infty} j \cdot 16^{-n} \cdot P(T^{(n)} \geq j \mid W_0^{(n)} = 1) \\ &\leq 16^{-n} \cdot E\left((T^{(n)})^2 \,\middle|\, W_0^{(n)} = 1\right) \leq 16^{-n} \cdot 2 \cdot 16^{2n} = 2 \cdot 16^n. \end{aligned} \tag{6.53}$$

Now by strict stationarity of the sequence $W^{(n)}$, and a standard elementary argument using Remark 2.8 (akin to certain arguments in the proof of Lemma 6.6), the random variable $\kappa(n,2) - \kappa(n,1)$ is independent of $\kappa(n,1)$ and $\mathcal{L}(\kappa(n,2) - \kappa(n,1)) = \mathcal{L}(\kappa(n,1) \mid W_0^{(n)} = 1)$. Recall again from the paragraph after (6.51) that $\kappa(n,1) = T^{(n)}$, from Lemma 6.6. One now has by Lemma 6.6 that

$$E(\kappa(n,2) - \kappa(n,1)) = E(T^{(n)} \mid W_0^{(n)} = 1) = 16^n.$$

Hence by (6.53), $E\kappa(n,2) \leq 3 \cdot 16^n$. Now (6.51) holds by Markov's inequality. Lemma 6.7 is proved.

7. Scaffolding (part 2)

Among other things, this section will include (in Construction 7.5 below) the construction of the random sequence X itself for Theorem 1.1 This section will build on Section 6. Recall from that section the given probability space $(\Omega, \mathcal{F}, P)$.

Construction 7.1. Refer to Construction 6.2, including equation (6.10). Define the $\{0,1,2,3,4,5,6\}$–valued random variables $\delta_k^{(n)}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$ as follows:

$$\delta_k^{(1)} := W_k^{(1)} \tag{7.1}$$

and for $n \geq 2$,

$$\delta_k^{(n)} := W_{k-\Psi(n-1,k,0)}^{(n)}. \tag{7.2}$$

By (7.1), (7.2), and (6.10), one has that for all $k \in \mathbf{Z}$,

$$\sigma(\delta_k^{(1)}) \subset \sigma(W^{(1)}), \quad \text{and} \quad \forall n \geq 2, \quad \sigma(\delta_k^{(n)}) \subset \sigma(W^{(n-1)}, W^{(n)}). \tag{7.3}$$

Define the $(\{0\} \cup \mathbf{N} \cup \{\infty\})$–valued random variables N_k, $k \in \mathbf{Z}$ as follows: For each $\omega \in \Omega$,

$$N_k(\omega) := \begin{cases} 0 & \text{if } \delta_k^{(1)}(\omega) = 0 \text{ (that is, } W_k^{(1)}(\omega) = 0) \\ m \in \mathbf{N} & \text{if } \delta_k^{(u)}(\omega) \neq 0 \forall u \in \{1,\dots,m\} \text{ and } \delta_k^{(m+1)}(\omega) = 0 \\ \infty & \text{if } \delta_k^{(u)}(\omega) \neq 0 \text{ for all } u \in \mathbf{N}. \end{cases} \tag{7.4}$$

By (7.4) and (7.3) (and (6.25)), for each $m \in \mathbf{N}$ and each $k \in \mathbf{Z}$,

$$\{N_k \geq m\} = \bigcap_{u=1}^{m} \{\delta_k^{(u)} \neq 0\} \subset \sigma(\overline{W}^{(m)}); \tag{7.5}$$

and for each integer $m \geq 0$,

$$\{N_k = m\} \subset \sigma(\overline{W}^{(m+1)}). \tag{7.6}$$

Also, define the integer-valued random variables $J(m,k)$, $m \in \mathbf{N}$, $k \in \mathbf{Z}$ as follows:

$$J(m,k) := \sum_{u=1}^{m} 6^{u-1}\left(\delta_k^{(u)} - 1\right). \tag{7.7}$$

By (7.3), for each $m \in \mathbf{N}$ and each $k \in \mathbf{Z}$,

$$\sigma(J(m,k)) \subset \sigma(\overline{W}^{(m)}). \tag{7.8}$$

Remark 7.2. Recall from Construction 7.1 that the random variables $\delta_k^{(n)}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$ take their values in the set $\{0,1,\dots,6\}$.

(A) Suppose $m \in \mathbf{N}$, $\omega \in \Omega$, $k \in \mathbf{Z}$, and $W_k^{(m)}(\omega) = 1$; then (i) $W_k^{(n)}(\omega) = 1$ for all $n \in \{1,\dots,m\}$ by Remark 6.3(B), (ii) for each $n \in \{1,\dots,m\}$, $\Psi(n,k,0)(\omega) = 0$ by (6.10) and Remark 2.2(B), hence (iii) $\delta_k^{(n)}(\omega) = 1$ for all $n \in \{1,\dots,m\}$ by (7.1) and (7.2), and hence (iv) $N_k(\omega) \ge m$ by (7.5).

(B) For each $m \in \mathbf{N}$, each $k \in \mathbf{Z}$, and each $\omega \in \Omega$ such that $N_k(\omega) \ge m$, one has that (i) $\delta_k^{(u)}(\omega) \in \{1,2,\dots,6\}$ for all $u \in \{1,\dots,m\}$ by (7.5), hence (ii) $\delta_k^{(u)}(\omega) - 1 \in \{0,1,\dots,5\}$ for all $u \in \{1,\dots,m\}$, and hence (iii) $J(m,k)(\omega) \in \{0,1,2,\dots,6^m - 1\}$ by (7.7) and a simple argument.

Construction 7.3.

(A) Refer to Definition 3.3. On the given probability space $(\Omega, \mathcal{F}, P)$, let $\zeta_k^{(n,\mathrm{ord})}$, $\zeta_k^{(n,\mathrm{cen})}$, $\zeta_k^{(n,\mathrm{fri})}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$ be an array of independent random variables, with this array being independent of the entire collection of random variables $\xi_k^{(n)}$ $W_k^{(n)}$, $\Psi(n,k,j)$, $\delta_k^{(n)}$, N_k, $J(n,k)$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$, $j \in \{0,1,2,\dots\}$ in Section 6 and Construction 7.1 (the redundancy here is for emphasis), such that for each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, (i) all three random vectors $\zeta_k^{(n,\mathrm{ord})}$, $\zeta_k^{(n,\mathrm{cen})}$, $\zeta_k^{(n,\mathrm{fri})}$ take their values in the set $\{-1,1\}^{\mathrm{sxtp}(n)}$ (see (2.1)), and (ii) the distribution of $\zeta_k^{(n,\mathrm{ord})}$ resp. $\zeta_k^{(n,\mathrm{cen})}$ resp. $\zeta_k^{(n,\mathrm{fri})}$ is $\nu_{\mathrm{ord}}^{(n)}$ resp. $\nu_{\mathrm{cen}}^{(n)}$ resp. $\nu_{\mathrm{fri}}^{(n)}$.

(B) For a given $n \in \mathbf{N}$, the random vector $\zeta_k^{(n,\mathrm{ord})}$ will be represented by

$$\zeta_k^{(n,\mathrm{ord})} := \left(\zeta_{k,0}^{(n,\mathrm{ord})}, \zeta_{k,1}^{(n,\mathrm{ord})}, \dots, \zeta_{k,\mathrm{sxtp}(n)-1}^{(n,\mathrm{ord})}\right),$$

that is, with the 6^n indices running through $0,1,\dots,6^n - 1$ (instead of $1,2,\dots,6^n$); and exactly the same convention will be used for the random vectors $\zeta_k^{(n,\mathrm{cen})}$ and $\zeta_k^{(n,\mathrm{fri})}$. (This fits the convention in Section 3 where, for a given $n \in \mathbf{N}$, the elements $x \in \{-1,1\}^{\mathrm{sxtp}(n)}$ were represented as $x := (x_0, x_1, \dots, x_{\mathrm{sxtp}(n)-1})$.

(C) For each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, define the $(\{-1,1\}^{\mathrm{sxtp}(n)})^3$–valued random vector

$$\zeta_k^{(n)} := \left(\zeta_k^{(n,\mathrm{ord})}, \zeta_k^{(n,\mathrm{cen})}, \zeta_k^{(n,\mathrm{fri})}\right). \tag{7.9}$$

Of course for a given fixed n, these random vectors are independent and identically distributed. For each $n \in \mathbf{N}$, define the resulting random sequence $\zeta^{(n)} := (\zeta_k^{(n)}, k \in \mathbf{Z})$.

(D) For each $n \in \mathbf{N}$ and each $k \in \mathbf{Z}$, define the $(\{-1,1\}^6)^3 \times (\{-1,1\}^{36})^3 \times \cdots \times (\{-1,1\}^{\mathrm{sxtp}(n)})^3$–valued random vector

$$\overline{\zeta}_k^{(n)} := \left(\zeta_k^{(1)}, \zeta_k^{(2)}, \dots, \zeta_k^{(n)}\right). \tag{7.10}$$

For any given fixed value of $n \in \mathbf{N}$, these random vectors $\overline{\zeta}_k^{(n)}$, $k \in \mathbf{Z}$ are independent and identically distributed. For each $n \in \mathbf{N}$, define the resulting random sequence $\overline{\zeta}^{(n)} := (\overline{\zeta}_k^{(n)}, k \in \mathbf{Z})$.

(E) For each $k \in \mathbf{Z}$, define the random item (sequence)

$$\overline{\zeta}_k^{(\infty)} := \left(\zeta_k^{(1)}, \zeta_k^{(2)}, \zeta_k^{(3)}, \dots\right). \tag{7.11}$$

These random items $\overline{\zeta}_k^{(\infty)}$, $k \in \mathbf{Z}$ are independent and identically distributed. Define the resulting random sequence $\overline{\zeta}^{(\infty)} := (\overline{\zeta}_k^{(\infty)}, k \in \mathbf{Z})$.

(F) Refer to (B) above. Purely as a formality, for each $n \in \mathbf{N}$, each $k \in \mathbf{Z}$, and each $u \in \mathbf{Z} - \{0, 1, \dots, 6^n - 1\}$, define the constant random variables $\zeta_{k,u}^{(n,\mathrm{ord})} := \zeta_{k,u}^{(n,\mathrm{cen})} := \zeta_{k,u}^{(n,\mathrm{fri})} := 1$. (This formality will ultimately turn out to be frivolous.)

Construction 7.4.

(A) Let $X^{(0)} := (X_k^{(0)}, k \in \mathbf{Z})$ be a sequence of independent, identically distributed $\{-1,1\}$–valued random variables such that (for each $k \in \mathbf{Z}$),

$$P\left(X_k^{(0)} = -1\right) = P\left(X_k^{(0)} = 1\right) = 1/2, \tag{7.12}$$

with this sequence $X^{(0)}$ being independent of the entire array of random variables $\xi_k^{(n)}$, $\zeta_k^{(n)}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}$ (and hence independent of the entire collection of random variables in Section 6 and Constructions 7.1 and 7.3).

(B) Refer to (A) above and to (6.4) and (7.11). For each $k \in \mathbf{Z}$, define the random ordered triplet

$$\eta_k := \left(X_k^{(0)}, \overline{\xi}_k^{(\infty)}, \overline{\zeta}_k^{(\infty)}\right). \tag{7.13}$$

Note that for any given $k \in \mathbf{Z}$, the three components of η_k are independent of each other. These random ordered triplets η_k, $k \in \mathbf{Z}$ are independent and identically distributed. Define the resulting random sequence $\eta := (\eta_k, k \in \mathbf{Z})$.

Construction 7.5. Now on our given probability space $(\Omega, \mathcal{F}, P)$, define the sequence $X := (X_k, k \in \mathbf{Z})$ of $\{-1,1\}$–valued random variables for Theorem 1.1 as follows: For each $k \in \mathbf{Z}$ and each $\omega \in \Omega$ (see Remark 7.2(B)),

$$X_k(\omega) := \begin{cases} X_k^{(0)}(\omega) & \text{if } N_k(\omega) = 0 \\ \zeta_{k-\Psi(\ell,k,0)(\omega), J(\ell,k)(\omega)}^{(\ell,\mathrm{cen})}(\omega) & \text{if } N_k(\omega) = \ell \in \mathbf{N} \\ 1 & \text{if } N_k(\omega) = \infty. \end{cases} \tag{7.14}$$

Lemma 7.6. *Refer to Constructions 7.4 and 7.5 and Definition 2.6(A). The ordered pair (X, η) satisfies Condition $\mathcal{M}$.*

Proof. In what follows, keep in mind the three sentences after (7.13).

For each $n \in \mathbf{N}$, the ordered pair $(W^{(n)}, \eta)$ satisfies Condition $\mathcal{M}$ by (6.3), (6.4), (6.5), and (7.13). Hence for each $n \in \mathbf{N}$ and each $j \geq 0$, the ordered pair $(\Psi(n,j), \eta)$ satisfies Condition $\mathcal{M}$ by (6.8) (see (6.10) and the phrase right before (6.8)).

Next, for each $n \geq 2$ and each $k \in \mathbf{Z}$,

$$W_{k-\Psi(n-1,k,0)}^{(n)} = \sum_{u=0}^{\infty} W_{k-u}^{(n)} I(\Psi(n-1,k,0) = u).$$

Hence by (7.1), (7.2), and both sentences in the preceding paragraph above, for each $n \in \mathbf{N}$, the ordered pair $((\delta_k^{(n)},\ k \in \mathbf{Z}), \eta)$ satisfies Condition $\mathcal{M}$.

Next, by (7.4), for each $k \in \mathbf{Z}$,

$$\begin{aligned} N_k = {} & 0 \cdot I(\delta_k^{(1)} = 0) \\ & + \sum_{m \in \mathbf{N}} m \cdot \left[\prod_{u=1}^{m} I\left(\delta_k^{(u)} \neq 0\right) \right] \cdot I\left(\delta_k^{(n+1)} = 0\right) \\ & + \infty \cdot \prod_{u \in \mathbf{N}} I\left(\delta_k^{(u)} \neq 0\right). \end{aligned}$$

Hence by the last sentence of the preceding paragraph, the ordered pair $((N_k,\ k \in \mathbf{Z}), \eta)$ satisfies Condition $\mathcal{M}$. Similarly, from (7.7), for each $m \in \mathbf{N}$, the ordered pair $((J(m,k),\ k \in \mathbf{Z}), \eta)$ satisfies condition $\mathcal{M}$.

Next, for each $\ell \in \mathbf{N}$ and each $k \in \mathbf{Z}$, by Remark 7.2(B) (see also (2.1)),

$$\begin{aligned} & I(N_k = \ell) \cdot \zeta^{(\ell,\mathrm{cen})}_{k-\Psi(\ell,k,0),J(\ell,k)} \\ = & \sum_{u=0}^{\infty} \sum_{v=0}^{\mathrm{sxtp}(\ell)-1} I(N_k = \ell) \cdot \zeta^{(\ell,\mathrm{cen})}_{k-u,v} \cdot I(\Psi(\ell,k,0) = u) \cdot I(J(\ell,k) = v). \end{aligned} \tag{7.15}$$

(If $\omega \in \Omega$ is such that $N_k(\omega) \neq \ell$, then trivially (7.15) holds for that ω with both sides being 0; for the formal definition of the left side of (7.15) for such ω, recall Construction 7.3(F) to cover the possible case $J(\ell,k)(\omega) < 0$.) By (7.15) and the observations made so far, together with equations (7.9), (7.11), and (7.13), for each $\ell \in \mathbf{N}$, the ordered pair

$$\left(\left(I(N_k = \ell) \cdot \zeta^{(\ell,\mathrm{cen})}_{k-\Psi(\ell,k,0),J(\ell,k)}, k \in \mathbf{Z}\right), \eta\right)$$

satisfies Condition $\mathcal{M}$.

Finally, for each $k \in \mathbf{Z}$, by (7.14),

$$\begin{aligned} X_k = {} & X_k^{(0)} \cdot I(N_k = 0) + 1 \cdot I(N_k = \infty) \\ & + \sum_{\ell \in \mathbf{N}} I(N_k = \ell) \cdot \zeta^{(\ell,\mathrm{cen})}_{k-\Psi(\ell,k,0),J(\ell,k)}. \end{aligned}$$

Hence by observations in the preceding two paragraphs together with (7.13), the ordered pair (X, η) satisfies Condition $\mathcal{M}$. Lemma 7.6 is proved.

Remark 7.7. Of course by Lemma 7.6 (and Definition 2.6(A)), the random sequence X is strictly stationary. Also, by (7.13), (7.12), (7.11), (7.9), and (6.4), the random variables η_k, $k \in \mathbf{Z}$ in (7.13) can be regarded as taking their values in the set $S := \{-1,1\} \times \{0,1\}^{\mathbf{N}} \times \{-1,1\}^{\mathbf{N}}$. Trivially that set is bimeasurably isomorphic to the set $\{0,1\}^{\mathbf{N}}$, and that set in turn is well known to be bimeasurably isomorphic to the open unit interval $(0,1)$ (with its Borel σ-field) and hence also to the real line $\mathbf{R}$ (with its Borel σ-field $\mathcal{R}$). Applying a particular bimeasurable isomorphism $\Theta : S \to \mathbf{R}$ to each of the random variables η_k in (7.13), one has that the random sequence η can thereby be "coded" as a sequence of independent, identically distributed *real*-valued random variables. Thus by Lemma 7.6, property (C) in Theorem 1.1 holds.

The other properties in Theorem 1.1 will be verified in sections 9 and 10, after some further preparation in Section 8. The following technical lemma will be needed in Section 9.

Lemma 7.8. *For each* $n \in \mathbf{N}$ *and each* $k \in \mathbf{Z}$, $P(N_k \geq n) = (3/8)^n$.

Proof. For each $k \in \mathbf{Z}$, by (7.4) (or (7.5)), (7.1), and Lemma 6.4(A) (and stationarity),

$$P(N_k \geq 1) = P(\delta_k^{(1)} \neq 0) = P(W_k^{(1)} \neq 0) = 3/8.$$

Now for induction, suppose that $n \in \mathbf{N}$, $k \in \mathbf{Z}$, and $P(N_k \geq n) = (3/8)^n$. If it is shown that $P(N_k \geq n+1 \mid N_k \geq n) = 3/8$, then (since $\{N_k \geq n+1\} \subset \{N_k \geq n\}$) it will follow that $P(N_k \geq n+1) = (3/8)^{n+1}$. Then Lemma 7.8 will hold by induction.

Suppose $j \in \{0, 1, 2, \ldots\}$ and that $P(\{N_k \geq n\} \cap \{\Psi(n,k,0) = j\}) > 0$. If $\omega \in \Omega_0$ (see (6.20)–(6.21)) is such that $\Psi(n,k,0)(\omega) = j$, then $W_{k-j}^{(n)}(\omega) = 1$ by (6.10) and Definition 2.2 (and Statement 6.3(i) and Definition 2.3(A)). Also, $\{N_k \geq n\} \in \sigma(\overline{W}^{(n)})$ by (7.5), and $\{\Psi(n,k,0) = j\} \in \sigma(\overline{W}^{(n)})$ by (6.10). Hence by (7.4) (see the equality in (7.5) with $m = n$ and with $m = n+1$), (6.21), (7.2), and Lemma 6.4(D),

$$\begin{aligned}
&P\left(N_k \geq n+1 \mid \{N_k \geq n\} \cap \{\Psi(n,k,0,) = j\}\right) \\
&= P\left(\delta_k^{(n+1)} \neq 0 \,\middle|\, \{N_k \geq n\} \cap \{\Psi(n,k,0) = j\} \cap \{W_{k-j}^{(n)} = 1\}\right) \\
&= P\left(W_{k-j}^{(n+1)} \neq 0 \,\middle|\, \{N_k \geq n\} \cap \{\Psi(n,k,0) = j\} \cap \{W_{k-j}^{(n)} = 1\}\right) = 3/8.
\end{aligned}$$

Hence by Remark 2.8, $P(N_k \geq n+1 \mid N_k \geq n) = 3/8$. That completes the induction argument and the proof.

8. Scaffolding (part 3)

In this section, some more foundations will be laid for the proofs, in sections 9 and 10, of the properties in Theorem 1.1 not verified in Remark 7.7.

Definition 8.1. Refer to (7.4), section 6.3(A), and Definition 2.3(A). For each $m \in \mathbf{N}$ and each $\omega \in \Omega_0$, let $\mathcal{E}_m(\omega)$ denote the family of all sets $E \subset \mathbf{Z}$ such that the following holds:

There exist integers j and ℓ such that

$$\begin{aligned}
&j < \ell, \quad W_j^{(m)}(\omega) = W_\ell^{(m)}(\omega) = 1, \quad \text{and} \\
&W_k^{(m)}(\omega) \neq 1 \quad \text{for all} \quad k \in \{j+1, j+2, \ldots, \ell-1\},
\end{aligned} \tag{8.1}$$

and

$$E = \{k \in \{j, j+1, \ldots, \ell-1\} : N_k(\omega) \geq m\}. \tag{8.2}$$

(Note that the "inner set" in (8.2) contains j but not ℓ.)

Remark 8.2.

(A) Refer to Definition 8.1. Suppose $m \in \mathbf{N}$, $\omega \in \Omega_0$, $E \in \mathcal{E}_m(\omega)$, and j and ℓ are integers such that (8.1) and (8.2) hold. Since $W_j^{(m)}(\omega) = 1$ (see (8.1)), one has that $N_j(\omega) \ge m$ by Remark 7.2(A)(iv), and hence $j \in E$ (see (8.2)). In fact (for the j in (8.1))

$$j = \min E. \tag{8.3}$$

Thus trivially the set E is nonempty and also (see (8.2)) finite.

(B) For a given $m \in \mathbf{N}$, $\omega \in \Omega_0$, and $E \in \mathcal{E}_m(\omega)$, the integers j and ℓ in (8.1) are unique, by (8.3) and (8.1) itself.

(C) If $m \in \mathbf{N}$, $\omega \in \Omega_0$, $E \in \mathcal{E}_m(\omega)$, and also $\widetilde{E} \in \mathcal{E}_m(\omega)$, then (see Section 2.1(D)) either $E < \widetilde{E}$, $E = \widetilde{E}$, or $E > \widetilde{E}$.

(D) Refer to Definition 8.1 and Section 2.1(H). Suppose $m \in \mathbf{N}$ and $\omega \in \Omega_0$. Then (recall Statement 6.3(A)(i))

$$\{k \in \mathbf{Z} : N_k(\omega) \ge m\} = \text{union } \mathcal{E}_m(\omega). \tag{8.4}$$

From this and (C) above and a simple argument (recall Statement 6.3(A)(i)), (i) one has a representation of the form $\mathcal{E}_m(\omega) := \{\dots, E_{-1}, E_0, E_1, \dots\}$ (the E_i's depend on m and ω) where (section 2.1(D)) $\dots < E_{-1} < E_0 < E_1 < \dots$, and (ii) those sets E_i, $i \in \mathbf{Z}$ form a partition of the set $\{k \in \mathbf{Z} : N_k(\omega) \ge m\}$.

(E) Suppose $m \in \mathbf{N}$, $\omega \in \Omega_0$, and $E \in \mathcal{E}_m(\omega)$. (i) If $W_{\min E}^{(m+1)}(\omega) = 0$, then $N_k(\omega) = m$ for all $k \in E$. (ii) If instead $W_{\min E}^{(m+1)}(\omega) \ne 0$, then $N_k(\omega) \ge m+1$ for all $k \in E$. (Note that by (i) and (ii) together, the respective converses of (i) and (ii) each hold.)

Proof of (E). Let the integers j and ℓ be as in (8.1)–(8.2). For each $k \in \{j, j+1, \dots, \ell-1\}$, $\Psi(m,k,0)(\omega) = k - j$ by (8.1)–(8.2), (6.10), and Definition 2.2, hence $j = k - \Psi(m,k,0)(\omega)$, and hence $\delta_k^{(m+1)}(\omega) = W_j^{(m+1)}(\omega)$ by (7.2). Of course for each $k \in E$, one has that $\delta_k^{(n)}(\omega) \ne 0$ for all $n \in \{1, \dots, m\}$ by (8.2) and (7.4). If $W_j^{(m+1)}(\omega) = 0$, then for all $k \in E$, $\delta_k^{(m+1)}(\omega) = 0$ and hence (see (7.4)) $N_k(\omega) = m$. If instead $W_j^{(m+1)}(\omega) \ne 0$, then for all $k \in E$, $\delta_k^{(m+1)}(\omega) \ne 0$ and hence (see (7.4)) $N_k(\omega) \ge m+1$. Thus (recall (8.3) once more) statement (E)(i)(ii) holds.

Definition 8.3.

(A) Refer to (6.20)–(6.21) and (7.4). For each $\omega \in \Omega_0$, let $\mathcal{D}_0(\omega)$ denote the family of all "singleton" sets $\{k\}$ (with $k \in \mathbf{Z}$) such that $N_k(\omega) = 0$.

(B) For each $m \in \mathbf{N}$ and each $\omega \in \Omega_0$, let $\mathcal{D}_m(\omega)$ denote the family of all sets $E \in \mathcal{E}_m(\omega)$ such that (see Remark 8.2(E)) $N_k(\omega) = m$ for all $k \in E$.

Remark 8.4.

(A) Refer to Definition 8.3(B), Remark 8.2(D)(E), and section 2.1(H). For each $m \in \mathbf{N}$ and each $\omega \in \Omega_0$,

$$\{k \in \mathbf{Z} : N_k(\omega) = m\} = \text{union } \mathcal{D}_m(\omega), \tag{8.5}$$

and in fact the members of $\mathcal{D}_m(\omega)$ form a partition of the set $\{k \in \mathbf{Z} : N_k(\omega) = m\}$. That sentence also holds trivially for $m = 0$ and $\omega \in \Omega_0$; see Definition 8.3(A).

(B) Refer to Remark 8.2(A), Definition 8.3(A)(B), and equations (8.4) and (8.5). (i) For a given $\omega \in \Omega_0$, none of the families $\mathcal{D}_m(\omega)$ $(m \geq 0)$ or $\mathcal{E}_m(\omega)$ $(m \geq 1)$ contains the empty set as a member. (ii) If $\omega \in \Omega_0$ and $0 \leq m < n$, then for any $D \in \mathcal{D}_m(\omega)$ and any $E \in \mathcal{E}_n(\omega)$ (in particular, any $E \in \mathcal{D}_n(\omega)$), one has that $D \cap E = \emptyset$.

(C) By (A) and (B) above and Remark 8.2(C)(D), the following holds: If $m \in \mathbf{N}$ and $\omega \in \Omega_0$, then the family

$$[\mathcal{D}_0(\omega) \cup \mathcal{D}_1(\omega) \cup \cdots \cup \mathcal{D}_{m-1}(\omega)] \cup \mathcal{E}_m(\omega)$$

gives a partition of the set $\mathbf{Z}$ itself into countably many nonempty finite sets.

Lemma 8.5. *Suppose $m \in \mathbf{N}$, $\omega \in \Omega_0$, and $E \in \mathcal{E}_m(\omega)$. Then the following statements hold:*

(A) *One has that* $\operatorname{card} E = 6^m$.

(B) *Representing the set E by $E = \{i(1), i(2), \ldots, i(6^m)\}$ where $i(1) < i(2) < i(3) < \cdots < i(6^m)$, one has (see (7.7)) that $J(m, i(v))(\omega) = v - 1$ for each $v \in \{1, 2, \ldots, 6^m\}$.*

(C) *If also $m \geq 2$, then there exist six sets $E_1, E_2, \ldots, E_6 \in \mathcal{E}_{m-1}(\omega)$ such that $E_1 < E_2 < \cdots < E_6$ (see Section 2.1(D)) and $E = E_1 \cup E_2 \cup \cdots \cup E_6$.*

Proof. We shall first prove statements (A) and (B) for the case $m = 1$.

Suppose $\omega \in \Omega_0$ and $E \in \mathcal{E}_1(\omega)$.

Let j and ℓ denote the integers such that (8.1) and (8.2) hold (with $m = 1$).

For a given $k \in \mathbf{Z}$, the following three inequalities are equivalent by (7.4) and (7.1): $N_k(\omega) \geq 1$, $\delta_k^{(1)}(\omega) \neq 0$, and $W_k^{(1)}(\omega) \neq 0$. Hence by (8.2),

$$E = \{k \in \{j, j+1, \ldots, \ell - 1\} : W_k^{(1)}(\omega) \neq 0\}. \tag{8.6}$$

Since $\omega \in \Omega_0$ (by hypothesis), the sequence $(W_k^{(1)}(\omega),\ k \in \mathbf{Z})$ of elements of $\{0, 1, \ldots, 6\}$ satisfies Condition $\mathcal{S}$ (see Statement 6.3(A)(i) and Definition 2.3(A) again). Hence by (8.1), there exist integers $i(1), i(2), \ldots, i(6)$ such that (see also (8.3)) $j = i(1) < i(2) < \cdots < i(6) < \ell$,

$$W_{i(v)}^{(1)}(\omega) = v \quad \text{for each} \quad v \in \{1, 2, \ldots, 6\}, \tag{8.7}$$

and $W_k^{(1)}(\omega) = 0$ for all other elements $k \in \{j, j+1, \ldots, \ell - 1\}$. Hence by (8.6), $E = \{i(1), i(2), \ldots, i(6)\}$. Hence $\operatorname{card} E = 6$. Also, by (7.7), (7.1), and (8.7), for each $v \in \{1, 2, \ldots, 6\}$,

$$J(1, i(v))(\omega) = \delta_{i(v)}^{(1)}(\omega) - 1 = W_{i(v)}^{(1)}(\omega) - 1 = v - 1.$$

All parts of statements (A) and (B) (of Lemma 8.5) have now been verified for the case $m = 1$.

The induction step. Now suppose $M \in \mathbf{N}$, and statements (A) and (B) in Lemma 8.5 hold for the case $m = M$. To complete the proof of Lemma 8.5 by induction, it suffices to prove that all three statements (A), (B), and (C) hold for the case $m = M + 1$.

Suppose $\omega \in \Omega_0$ and $E \in \mathcal{E}_{M+1}(\omega)$.

Referring to (8.1) and (8.2), let j and ℓ be the integers such that (for the given ω and E)

$$j < \ell, \quad W_j^{(M+1)}(\omega) = W_\ell^{(M+1)}(\omega) = 1, \quad \text{and} \tag{8.8}$$

$$W_k^{(M+1)}(\omega) \neq 1 \quad \text{for all} \quad k \in \{j+1, j+2, \dots, \ell-1\},$$

and

$$E = \{k \in \{j, j+1, \dots, \ell-1\} : N_k(\omega) \geq M+1\}. \tag{8.9}$$

By (8.8) and Remark 6.3(B), $W_j^{(M)}(\omega) = W_\ell^{(M)}(\omega) = 1$. Let $a(0)$, $a(1)$, $\dots$, $a(p)$ (where p is a positive integer) denote the integers such that

$$j = a(0) < a(1) < a(2) < \cdots < a(p) = \ell \quad \text{and} \tag{8.10}$$

$$\{k \in \{j, j+1, \dots, \ell-1\} : W_k^{(M)}(\omega) = 1\} = \{a(0), a(1), a(2), \dots, a(p-1)\}. \tag{8.11}$$

Recall that since $\omega \in \Omega_0$ (by hypothesis), the sequence $(W_k^{(M+1)}(\omega),\ k \in \mathbf{Z})$ of elements of $\{0, 1, \dots, 6\}$ satisfies Condition $\mathcal{S}$ (see Statement 6.3(A)(i) and Definition 2.3(A) again). Hence by (8.8), there exist integers $i(1), i(2), \dots, i(6)$ with $j = i(1) < i(2) < \cdots < i(6) < \ell$ such that

$$\begin{aligned} &W_{i(u)}^{(M+1)}(\omega) = u \quad \text{for} \quad u \in \{1, 2, \dots, 6\} \quad \text{and} \\ &W_k^{(M+1)}(\omega) = 0 \quad \text{for all other } k \in \{j, j+1, \dots, \ell-1\}. \end{aligned} \tag{8.12}$$

By (8.12) and Remark 6.3(B), $W_{i(u)}^{(M)}(\omega) = 1$ for each $u \in \{1, 2, \dots, 6\}$. Referring to (8.10) and (8.11), for each $u \in \{1, 2, \dots, 6\}$, let $e(u)$ denote the element of $\{0, 1, \dots, p-1\}$ such that

$$i(u) = a(e(u)). \tag{8.13}$$

Then by (8.10), (8.13), and the equality $j = i(1)$ just before (8.12), one has that $a(0) = j = i(1) = a(e(1))$; and hence (see (8.10) and the inequalities right before (8.12) again)

$$0 = e(1) < e(2) < \cdots < e(6) \leq p-1. \tag{8.14}$$

For each $u \in \{1, 2, \dots, 6\}$, referring to the inequality $a(e(u)) < a(e(u)+1)$ from (8.10) (see (8.14)), define the set

$$E_u := \{k \in \mathbf{Z} :\ a(e(u)) \leq k < a(e(u)+1) \ \text{ and } \ N_k(\omega) \geq M\}. \tag{8.15}$$

For each $u \in \{1, 2, \dots, 6\}$, from (8.10) and (8.11) (and the sentence right after (8.9)), one has that $W_{a(e(u))}^{(M)}(\omega) = W_{a(e(u)+1)}^{(M)}(\omega) = 1$ and $W_k^{(M)}(\omega) \neq 1$ for all $k \in \mathbf{Z}$ such that $a(e(u)) < k < a(e(u)+1)$. Hence by (8.15) and Definition 8.1 with $m = M$,

$$E_u \in \mathcal{E}_M(\omega) \quad \text{for each} \quad u \in \{1, 2, \dots, 6\}. \tag{8.16}$$

Hence by the induction assumption of Lemma 8.5(A)(B) for the case $m = M$,

$$\text{card } E_u = 6^M \quad \text{for each} \quad u \in \{1, 2, \dots, 6\}. \tag{8.17}$$

For each $t \in \{0, 1, \dots, p-1\}$ (see (8.10) and (8.11)), one has the following: For each $k \in \mathbf{Z}$ such that $a(t) \leq k < a(t+1)$, $\Psi(M, k, 0)(\omega) = k - a(t)$ by (8.10), (8.11), (6.10), and Definition 2.2, hence $a(t) = k - \Psi(M, k, 0)(\omega)$, and hence by (7.2),

$$\delta_k^{(M+1)}(\omega) = W_{k-\Psi(M,k,0)(\omega)}^{(M+1)}(\omega) = W_{a(t)}^{(M+1)}(\omega). \tag{8.18}$$

For each $u \in \{1, 2, \dots, 6\}$, $W_{a(e(u))}^{(M+1)}(\omega) = u$ by (8.12) and (8.13), and hence by (8.18) (and its entire sentence) one has that

$$\begin{gathered} \delta_k^{(M+1)}(\omega) = W_{a(e(u))}^{(M+1)}(\omega) = u \neq 0 \\ \text{for all } k \in \mathbf{Z} \text{ such that } a(e(u)) \leq k < a(e(u)+1). \end{gathered} \tag{8.19}$$

For each $u \in \{1, 2, \dots, 6\}$ and each $k \in E_u$, $N_k(\omega) \geq M$ by (8.15), and hence $N_k(\omega) \geq M+1$ by (8.19) and (7.4)–(7.5). Hence by (8.15), for each $u \in \{1, 2, \dots, 6\}$,

$$E_u = \{k \in \mathbf{Z} : \ a(e(u)) \leq k < a(e(u) + 1) \ \text{ and } \ N_k(\omega) \geq M + 1\}. \tag{8.20}$$

For any $t \in \{0, 1, \dots, p-1\} - \{e(1), e(2), \dots, e(6)\}$ (see (8.13) and its entire sentence), one has that $j \leq a(t) < \ell$ by (8.10), $a(t) \neq a(e(u)) = i(u)$ for each $u \in \{1, 2, \dots, 6\}$ by (8.10) and (8.13), and hence $W^{(M+1)}_{a(t)}(\omega) = 0$ by (8.12). Hence for each $t \in \{0, 1, \dots, p-1\} - \{e(1), e(2), \dots, e(6)\}$ and each $k \in \mathbf{Z}$ such that $a(t) \leq k < a(t+1)$, one has that $\delta^{(M+1)}_k(\omega) = 0$ by (8.18) and its entire sentence, and hence $N_k(\omega) \leq M$ by (7.4). Hence by (8.9), (8.10), and (8.20), the set E has no elements other than the ones in the sets E_u, $u \in \{1, 2, \dots, 6\}$. In fact by (8.9) and (8.20), one now has that

$$E = E_1 \cup E_2 \cup \cdots \cup E_6, \tag{8.21}$$

and by (8.10), (8.14), and (8.20) (see also (8.17)),

$$E_1 < E_2 < \cdots < E_6 \tag{8.22}$$

(see section 2.1(D)). Hence by (8.17),

$$\operatorname{card} E = 6^{M+1}. \tag{8.23}$$

Thus statement (A) in Lemma 8.5 holds (for the given ω and E) with $m = M+1$. By (8.16), (8.21), and (8.22), statement (C) in Lemma 8.5 holds (for the given ω and E) with $m = M+1$. To complete the induction step and the proof of Lemma 8.5, our task now is to verify statement (B) (for the given ω and E) with $m = M+1$.

Now refer to (8.16) and our induction assumption of statements (A) and (B) for the case $m = M$. For each $u \in \{1, 2, \dots, 6\}$, referring to (8.17), representing the set E_u by

$$\begin{gathered} E_u = \bigl\{\alpha(u,1), \alpha(u,2), \alpha(u,3), \dots, \alpha(u, 6^M)\bigr\} \\ \text{with } \alpha(u,1) < \alpha(u,2) < \cdots < \alpha(u, 6^M), \end{gathered} \tag{8.24}$$

one has that

$$J(M, \alpha(u,v))(\omega) = v - 1 \quad \text{for all } v \in \{1, 2, \dots, 6^M\}. \tag{8.25}$$

Referring to (8.23), represent the set E by

$$E = \bigl\{\beta(1), \beta(2), \beta(3), \dots, \beta(6^{M+1})\bigr\} \ \text{ with } \ \ \beta(1) < \beta(2) < \cdots < \beta(6^{M+1}). \tag{8.26}$$

Then for each $u \in \{1, 2, \dots, 6\}$, by (8.17), (8.21), (8.22), and (8.26), the set E_u contains precisely the elements $\beta(6^M(u-1)+v)$, $v \in \{1, 2, \dots, 6^M\}$. Hence by (8.24),

$$\forall\ u \in \{1, 2, \dots, 6\}, \quad \forall\ v \in \{1, 2, \dots, 6^M\}, \quad \beta\left(6^M(u-1)+v\right) = \alpha(u,v). \tag{8.27}$$

Now suppose $r \in \{1, 2, \dots, 6^{M+1}\}$. Let $u \in \{1, 2, \dots, 6\}$ and $v \in \{1, 2, \dots, 6^M\}$ be such that

$$r = 6^M(u-1) + v. \tag{8.28}$$

Then $\beta(r) \in E_u$ by the sentence preceding (8.27), hence $a(e(u)) \leq \beta(r) < a(e(u)+1)$ by (8.15), and hence $\delta^{(M+1)}_{\beta(r)}(\omega) = W^{(M+1)}_{a(e(u))}(\omega) = u$ by (8.19). Also, $\beta(r) =$

$\alpha(u, v)$ by (8.28) and (8.27). Hence by (7.7) (applied twice) and then (8.25) and (8.28),

$$\begin{aligned} J(M+1, \beta(r))(\omega) &= \sum_{q=1}^{M+1} 6^{q-1}\left(\delta_{\beta(r)}^{(q)}(\omega) - 1\right) \\ &= 6^M \cdot \left(\delta_{\beta(r)}^{(M+1)}(\omega) - 1\right) + \sum_{q=1}^{M} 6^{q-1}\left(\delta_{\beta(r)}^{(q)}(\omega) - 1\right) \\ &= 6^M(u-1) + \sum_{q=1}^{M} 6^{q-1}\left(\delta_{\alpha(u,v)}^{(q)}(\omega) - 1\right) \\ &= 6^M(u-1) + J(M, \alpha(u,v))(\omega) \\ &= 6^M(u-1) + v - 1 \\ &= r - 1. \end{aligned}$$

Since $r \in \{1, 2, \dots, 6^{M+1}\}$ was arbitrary, one has (see (8.26) again) that statement (B) in Lemma 8.5 holds (for the given ω and E) with $m = M + 1$. That completes the induction step and the proof of Lemma 8.5.

Definition 8.6. Recall from Definitions 8.1 and 8.3 and Lemma 8.5(A) that for each $\omega \in \Omega_0$ (see (6.20)), one has that (i) card $E = 1$ for each $E \in \mathcal{D}_0(\omega)$, and (ii) for each $m \in \mathbf{N}$ and each $E \in \mathcal{E}_m(\omega)$ (and in particular, for each $E \in \mathcal{D}_m(\omega)$), card $E = 6^m$. Recall also Remarks 8.2(D)(E) and 8.4(B)(C).

(A) For each $\ell \in \{0, 1, 2, \dots\}$ and each set $D \subset \mathbf{Z}$ such that card $D = 6^\ell$, define the set $F_D \subset \Omega_0$ (see (6.20)) as follows:

$$F_D := \{\omega \in \Omega_0 : D \in \mathcal{D}_\ell(\omega)\}. \tag{8.29}$$

(B) For each $m \in \mathbf{N}$ and each set $E \subset \mathbf{Z}$ such that card $E = 6^m$, define the set $G_E \subset \Omega_0$ as follows:

$$G_E := \{\omega \in \Omega_0 : E \in \mathcal{E}_m(\omega)\}. \tag{8.30}$$

Remark 8.7. Refer to Lemma 8.5. Suppose $m \in \mathbf{N}$, and $E \subset \mathbf{Z}$ is a set such that card $E = 6^m$.

(A) The set G_E in (8.30) is the set of all $\omega \in \Omega_0$ for which there exist integers j and ℓ such that (8.1) and (8.2) hold. It follows that G_E is an event (that is, a member of the σ-field $\mathcal{F}$ in our given probability space $(\Omega, \mathcal{F}, P)$); and further, by (7.5),

$$G_E \in \sigma(\overline{W}^{(m)}). \tag{8.31}$$

By the same argument, but with the inequality $N_k(\omega) \geq m$ in (8.2) replaced by $N_k(\omega) = m$ (see Remark 8.2(E) and Definition 8.3(B)), the set F_E in (8.29) is an event, and by (7.6),

$$F_E \in \sigma(\overline{W}^{(m+1)}). \tag{8.32}$$

(B) By Definitions 8.1, 8.3(B), and 8.6 (recall Remark 8.2(E) again),

$$F_E \subset G_E; \tag{8.33}$$

$$\forall\, \omega \in F_E,\ \forall\, k \in E, \quad N_k(\omega) = m; \quad \text{and} \tag{8.34}$$

$$\forall\, \omega \in G_E - F_E,\ \forall\, k \in E, \quad N_k(\omega) \geq m + 1. \tag{8.35}$$

Remark 8.8. Equations (8.32) and (8.34) hold with $m = 0$ for any singleton set $E = \{k\}$ where $k \in \mathbf{Z}$, by (8.29), Definition 8.3(A), and (7.6).

Definition 8.9. Refer to Constructions 7.3 and 7.4, to equations (6.10), (7.4), and (7.7), and also particularly to Remark 7.2(B).

For each $n \in \mathbf{N}$, define the sequences $Y^{(n)} := (Y_k^{(n)}, k \in \mathbf{Z})$ and $X^{(n)} := (X_k^{(n)}, k \in \mathbf{Z})$ of $\{-1,1\}$–valued random variables as follows: For each $k \in \mathbf{Z}$ and each $\omega \in \Omega$,

$$Y_k^{(n)}(\omega) := \begin{cases} X_k^{(0)}(\omega) & \text{if } N_k(\omega) = 0 \\ \zeta^{(\ell,\mathrm{cen})}_{k-\Psi(\ell,k,0)(\omega),J(\ell,k)(\omega)}(\omega) & \text{if } N_k(\omega) = \ell \in \{1,2,\dots,n-1\} \\ \zeta^{(n,\mathrm{ord})}_{k-\Psi(n,k,0)(\omega),J(n,k)(\omega)}(\omega) & \text{if } N_k(\omega) \geq n, \end{cases} \tag{8.36}$$

and

$$X_k^{(n)}(\omega) := \begin{cases} X_k^{(0)}(\omega) & \text{if } N_k(\omega) = 0 \\ \zeta^{(\ell,\mathrm{cen})}_{k-\Psi(\ell,k,0)(\omega),J(\ell,k)(\omega)}(\omega) & \text{if } N_k(\omega) = \ell \in \{1,2,\dots,n\} \\ \zeta^{(n,\mathrm{fri})}_{k-\Psi(n,k,0)(\omega),J(n,k)(\omega)}(\omega) & \text{if } N_k(\omega) \geq n+1. \end{cases} \tag{8.37}$$

Of course in the right hand side of (8.36), the "middle" case is vacuous (and should be omitted) if $n = 1$.

Remark 8.10. If $k \in \mathbf{Z}$ and $D = \{k\}$, then by Remark 8.8 (see equation (8.34)) and (8.36) and (8.37), for every $\omega \in F_D$, one has that

$$Y_k^{(n)}(\omega) = X_k^{(n)}(\omega) = X_k^{(0)}(\omega) \quad \text{for all } n \in \mathbf{N}. \tag{8.38}$$

Lemma 8.11. *Suppose $m \in \mathbf{N}$, and $E \subset \mathbf{Z}$ is a set such that*

$$\operatorname{card} E = 6^m. \tag{8.39}$$

Then in the terminology of (2.3) and (2.4), the following statements hold:

(A) *For every $\omega \in G_E$ (see (8.30)),*

$$Y_E^{(m)}(\omega) = \zeta^{(m,\mathrm{ord})}_{\min\, E}(\omega) \tag{8.40}$$

and

$$\begin{aligned} X_E^{(m)}(\omega) = &\zeta^{(m,\mathrm{cen})}_{\min\, E}(\omega) \cdot I\left(W^{(m+1)}_{\min\, E} = 0\right)(\omega) \\ &+ \zeta^{(m,\mathrm{fri})}_{\min\, E}(\omega) \cdot I\left(W^{(m+1)}_{\min\, E} \neq 0\right)(\omega). \end{aligned} \tag{8.41}$$

(B) *For every $\omega \in F_E$ (see (8.29) and (8.33)),*

$$Y_E^{(n)}(\omega) = X_E^{(n)}(\omega) = X_E^{(m)}(\omega) = \zeta^{(m,\mathrm{cen})}_{\min\, E}(\omega) \quad \textit{for all } n \geq m+1. \tag{8.42}$$

Proof. We shall prove statements (A) and (B) together.

Suppose $\omega \in G_E$.

Then by (8.39) and Definition 8.6(B), $E \in \mathcal{E}_m(\omega)$. Referring again to (8.39), represent the set E by

$$E = \{e(1), e(2), e(3), \dots, e(6^m)\} \quad \text{where} \quad e(1) < e(2) < \cdots < e(6^m). \tag{8.43}$$

Also, referring to Definition 8.1 and Remark 8.2(B), let j and ℓ be the integers such that (8.1) and (8.2) hold. We shall refer freely to (8.1) and (8.2) in the arguments below. By Remark 8.2(A), (8.2), and (8.43),

$$j = \min\, E = e(1) \quad \text{and} \quad \ell > e(6^m). \tag{8.44}$$

For each $k \in E$, by (8.1), (8.2), (6.10), and Definition 2.2, $\Psi(m,k,0)(\omega) = k - j$. Hence by (8.43),

$$\forall\ u \in \{1,2,\dots,6^m\}, \quad j = e(u) - \Psi(m, e(u), 0)(\omega). \tag{8.45}$$

Also, by (8.43) and (8.2),

$$\forall\ u \in \{1,2,\dots,6^m\}, \quad N_{e(u)}(\omega) \ge m. \tag{8.46}$$

Proof of (8.40)**.** For each $u \in \{1,2,\dots,6^m\}$, by (8.46) and (8.36), followed by (8.45), (8.43), and Lemma 8.5(B) (recall also section 7.3(B)),

$$Y_{e(u)}^{(m)}(\omega) = \zeta^{(m,\mathrm{ord})}_{e(u)-\Psi(m,e(u),0)(\omega),J(m,e(u))(\omega)}(\omega) = \zeta^{(m,\mathrm{ord})}_{j,u-1}(\omega). \tag{8.47}$$

Hence by (8.43) and (8.44) (see (2.3), (2.4), and section 7.3(B)), one obtains (8.40) via

$$Y_E^{(m)}(\omega) = \zeta_j^{(m,\mathrm{ord})}(\omega) = \zeta^{(m,\mathrm{ord})}_{\min E}(\omega). \tag{8.48}$$

Proof of (8.41)**.** If $W^{(m+1)}_{\min E}(\omega) = 0$, then $N_{e(u)}(\omega) = m$ for all $u \in \{1,2,\dots,6^m\}$ by (8.43) and Remark 8.2(E)(i), and following the argument for (8.40) (but using (8.37) instead of (8.36)), one obtains (8.47) and (8.48) with the letters "Y" and "ord" replaced by "X" and "cen," and in this case (8.41) holds. If instead $W^{(m+1)}_{\min E}(\omega) \ne 0$, then $N_{e(u)}(\omega) \ge m+1$ for all $u \in \{1,2,\dots,6^m\}$ by (8.43) and Remark 8.2(E)(ii), and with the same argument, one obtains (8.47) and (8.48) with "Y" and "ord" replaced by "X" and "fri," and in this case too equation (8.41) holds.

Proof of (8.42)**.** Suppose $\omega \in F_E$ (assumed for (8.42)). Then $E \in \mathcal{D}_m(\omega)$ by Definition 8.6(A), and hence $N_{e(u)}(\omega) = m$ for all $u \in \{1,2,\dots,6^m\}$ by (8.43) and Definition 8.3(B). Hence by (8.36) and (8.37) (applied twice), for each $n \ge m+1$ and each $u \in \{1,2,\dots,6^m\}$,

$$Y_{e(u)}^{(n)}(\omega) = X_{e(u)}^{(n)}(\omega) = \zeta^{(m,\mathrm{cen})}_{e(u)-\Psi(m,e(u),0)(\omega),J(m,e(u))(\omega)}(\omega) = X_{e(u)}^{(m)}(\omega).$$

Hence (see (8.43) again) for each $n \ge m+1$, $Y_E^{(n)}(\omega) = X_E^{(n)}(\omega) = X_E^{(m)}(\omega)$. Since $N_{e(u)}(\omega) = m$ for all $u \in \{1,2,\dots,6^m\}$ (as was noted above), $W^{(m+1)}_{\min E}(\omega) = 0$ must hold by Remark 8.2(E), and the final equality in (8.42) now follows from (8.41). That completes the proof of Lemma 8.11.

Definition 8.12. Suppose $n \in \mathbf{N}$. Suppose $\mathcal{Q} := \{Q(1), Q(2), \dots, Q(L)\}$ (where L is a positive integer) is a (finite, nonempty) collection of (pairwise) disjoint subsets of $\mathbf{Z}$ such that

$$\operatorname{card} Q(\ell) \in \{1, 6, 6^2, 6^3, \dots, 6^n\} \quad \text{for all } \ell \in \{1,2,\dots,L\}. \tag{8.49}$$

(There is no assumption of an "ordering" of these sets; elements of one set $Q(i)$ may be between elements of another set $Q(j)$. Also, it is tacitly understood that L can be 1, in which case the phrase "(pairwise) disjoint" is meaningless and should be omitted.) Define the (possibly empty) set

$$\mathcal{I} := \mathcal{I}(n, \mathcal{Q}) := \{\ell \in \{1,\dots,L\} : \operatorname{card} Q(\ell) = 6^n\}. \tag{8.50}$$

Define the event $H^{(n)}(\mathcal{Q})$ by (see (8.29)–(8.30))

$$H^{(n)}(\mathcal{Q}) := \left[\bigcap_{\ell \in \{1,\dots,L\} - \mathcal{I}} F_{Q(\ell)}\right] \bigcap \left[\bigcap_{\ell \in \mathcal{I}} G_{Q(\ell)}\right]. \tag{8.51}$$

Here (if necessary), define the "vacuous intersection" by $\bigcap_{\ell\in\emptyset}(\dots) := \Omega$ (the sample space itself).

It is easy to see that this definition of the event $H^{(n)}(\mathcal{Q})$ does not depend on the particular order in which the sets in $\mathcal{Q}$ are labeled (as $Q(\ell)$, $\ell \in \{1,\dots,L\}$).

Lemma 8.13. *Suppose $n \in \mathbf{N}$. In the context of Definition 8.12, with (8.49) satisfied and with $\mathcal{I}$ and $H^{(n)}(\mathcal{Q})$ defined as in (8.50) and (8.51), suppose*

$$P\left(H^{(n)}(\mathcal{Q})\right) > 0. \tag{8.52}$$

For each $\ell \in \{1,2,\dots,L\}$, define the integer

$$q(\ell) := \min Q(\ell). \tag{8.53}$$

Then the following statements hold:

(A) *One has that $H^{(n)}(\mathcal{Q}) \in \sigma(\overline{W}^{(n)})$. Also, for each $\omega \in H^{(n)}(\mathcal{Q})$ and each $\ell \in \mathcal{I}$ (if $\mathcal{I}$ is nonempty), $W^{(n)}_{q(\ell)}(\omega) = 1$.*

(B) *For each $\omega \in H^{(n)}(\mathcal{Q})$ and each $\ell \in \{1,2,\dots,L\}$ such that $\operatorname{card} Q(\ell) = 1$ (and hence $Q(\ell) = \{q(\ell)\}$), one has that*

$$X^{(n-1)}_{q(\ell)}(\omega) = Y^{(n)}_{q(\ell)}(\omega) = X^{(n)}_{q(\ell)}(\omega) = X^{(0)}_{q(\ell)}(\omega). \tag{8.54}$$

(C) *For each $\omega \in H^{(n)}(\mathcal{Q})$, each $m \in \{1,\dots,n-1\}$ (if $n \geq 2$), and each $\ell \in \{1,\dots,L\}$ such that $\operatorname{card} Q(\ell) = 6^m$, one has that (see (2.3) and (2.4))*

$$X^{(n-1)}_{Q(\ell)}(\omega) = Y^{(n)}_{Q(\ell)}(\omega) = X^{(n)}_{Q(\ell)}(\omega) = \zeta^{(m,\mathrm{cen})}_{q(\ell)}(\omega). \tag{8.55}$$

(D) *For any given $\omega \in H^{(n)}(\mathcal{Q})$ and any given $\ell \in \mathcal{I}$ (see (8.50)), representing the set $Q(\ell)$ by $Q(\ell) := \{z(1), z(2), \dots, z(6^n)\}$ where $z(1) < z(2) < \dots < z(6^n)$ (and hence $z(1) = q(\ell)$), and denoting $y(u) := z((u-1)\cdot 6^{n-1} + 1)$ for $u \in \{1,2,\dots,6\}$ (and hence $y(1) = z(1) = q(\ell)$), one has (see section 2.1(C)) that*

$$\begin{aligned} X^{(n-1)}_{Q(\ell)}(\omega) = \Big\langle &\zeta^{(n-1,\mathrm{fri})}_{q(\ell)}(\omega), \zeta^{(n-1,\mathrm{fri})}_{y(2)}(\omega), \zeta^{(n-1,\mathrm{fri})}_{y(3)}(\omega), \\ &\zeta^{(n-1,\mathrm{fri})}_{y(4)}(\omega), \zeta^{(n-1,\mathrm{fri})}_{y(5)}(\omega), \zeta^{(n-1,\mathrm{fri})}_{y(6)}(\omega)\Big\rangle \end{aligned} \tag{8.56}$$

(with $\zeta^{(0,\mathrm{fri})}_k(\omega) := X^{(0)}_k(\omega)$ for all $k \in \mathbf{Z}$ in the case $n = 1$),

$$Y^{(n)}_{Q(\ell)}(\omega) = \zeta^{(n,\mathrm{ord})}_{q(\ell)}(\omega) \tag{8.57}$$

and

$$\begin{aligned} X^{(n)}_{Q(\ell)}(\omega) = &\zeta^{(n,\mathrm{cen})}_{q(\ell)}(\omega)\cdot I\left(W^{(n+1)}_{q(\ell)} = 0\right)(\omega) \\ &+ \zeta^{(n,\mathrm{fri})}_{q(\ell)}(\omega)\cdot I\left(W^{(n+1)}_{q(\ell)} \neq 0\right)(\omega). \end{aligned} \tag{8.58}$$

(E) *Refer to (8.52) and (8.50). The random variables $I(W^{(n+1)}_{q(\ell)} \neq 0)$, $\ell \in \mathcal{I}$, the random variables $X^{(0)}_k$, $k \in \mathbf{Z}$, and the random variables $\zeta^{(i,\mathrm{ord})}_k$, $\zeta^{(i,\mathrm{cen})}_k$, $\zeta^{(i,\mathrm{fri})}_k$, $i \in \mathbf{N}$, $k \in \mathbf{Z}$ are conditionally independent given the event $H^{(n)}(\mathcal{Q})$. Also, for each $\ell \in \mathcal{I}$, conditional on $H^{(n)}(\mathcal{Q})$, the random variable $I(W^{(n+1)}_{q(\ell)} \neq 0)$ takes the value 0 resp. 1 with probability 5/8 resp. 3/8.*

(F) *The random vectors* $X_{Q(\ell)}^{(n-1)}$, $\ell \in \{1, \dots, L\}$ *are conditionally independent given* $H^{(n)}(Q)$.

(G) *The random vectors* $Y_{Q(\ell)}^{(n)}$, $\ell \in \{1, \dots, L\}$ *are conditionally independent given* $H^{(n)}(\mathcal{Q})$.

(H) *The random vectors* $X_{Q(\ell)}^{(n)}$, $\ell \in \{1, \dots, L\}$ *are conditionally independent given* $H^{(n)}(\mathcal{Q})$.

(I) *Refer to* (8.50)*, Section* 2.5*(A)(B), and Definition* 3.3*. For each* $\ell \in \mathcal{I}$*, one has that*

(i) $\mathcal{L}\left(X_{Q(\ell)}^{(n-1)} \,\middle|\, \mathcal{H}^{(n)}(\mathcal{Q})\right) = \left(\nu_{\text{fri}}^{(n-1)}\right)^{[6]}$ *and*

(ii) $\mathcal{L}\left(Y_{Q(\ell)}^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left(X_{Q(\ell)}^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \nu_{\text{ord}}^{(n)}$.

(J) *Recall from Section* 2.1*(H) the notation* $\text{union}(\mathcal{Q}) := Q(1) \cup Q(2) \cup \cdots \cup Q(L)$*. One has that* $\mathcal{L}\left(Y_{\text{union}(\mathcal{Q})}^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left(X_{\text{union}(\mathcal{Q})}^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right)$.

(K) *Suppose* S *is a nonempty finite subset of* $\mathbf{Z}$ *such that* $S \cap \text{union}(\mathcal{Q})$ *is nonempty and* $\text{card}(S \cap Q(\ell)) \le 5$ *for all* $\ell \in \{1, \dots, L\}$*. Then*

$$\mathcal{L}\left(X_{S \cap \text{union}(\mathcal{Q})}^{(n-1)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left(Y_{S \cap \text{union}(\mathcal{Q})}^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right).$$

(L) *For any nonempty set* $S \subset Q(1) \cup Q(2) \cup \cdots \cup Q(L)$*, one has that*

$$E\left[\left(\sum_{k \in S} X_k^{(n-1)}\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] \ge E\left[\left(\sum_{k \in S} Y_k^{(n)}\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right]. \tag{8.59}$$

(M) *Suppose* $\ell \in \mathcal{I}$ *(see* (8.50)*) and* S *is a set such that* $Q(\ell) \subset S \subset Q(1) \cup Q(2) \cup \cdots \cup Q(L)$*. Then*

$$\begin{aligned} &E\left[\left(\sum_{k \in S} X_k^{(n-1)}\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] \\ &\quad \ge E\left[\left(\sum_{k \in S} Y_k^{(n)}\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] + 720 \cdot 4^{6(n-1)}. \end{aligned} \tag{8.60}$$

Proof. Let us first prove statement (A). For each $\ell \in \{1, \dots, L\} - \mathcal{I}$, letting $m(\ell) \in \{0, 1, \dots, n-1\}$ denote the integer such that $\text{card}\, Q(\ell) = 6^{m(\ell)}$ (see (8.49) and (8.50)), one has that $F_{Q(\ell)} \in \sigma(\overline{W}^{(m(\ell)+1)}) \subset \sigma(\overline{W}^{(n)})$ by (8.32) (see section 6.3(E) again). For each $\ell \in \mathcal{I}$, $G_{Q(\ell)} \in \sigma(\overline{W}^{(n)})$ by (8.31) (and (8.50)). Hence $H^{(n)}(\mathcal{Q}) \in \sigma(\overline{W}^{(n)})$ by (8.51). Also, for each $\omega \in H^{(n)}(\mathcal{Q})$ and each $\ell \in \mathcal{I}$, one has that $\omega \in G_{Q(\ell)}$ by (8.51), hence $Q(\ell) \in \mathcal{E}_n(\omega)$ by (8.50) and Definition 8.6(B), hence $W_{q(\ell)}^{(n)}(\omega) = 1$ by (8.53), (8.3), and (8.1). Both parts of statement (A) have been proved.

Proof of (B). For each $\omega \in H^{(n)}(\mathcal{Q})$ and each $\ell \in \{1, \dots, L\}$ such that $\text{card}\, Q(\ell) = 1$, one has that $\omega \in F_{Q(\ell)}$ by (8.50) and (8.51), and hence (8.54) holds by (8.53) and (8.38) (regardless of whether $n = 1$ or $n \ge 2$).

Proof of (C). If $\omega \in H^{(n)}(\mathcal{Q})$, $m \in \{1, \dots, n-1\}$, $\ell \in \{1, \dots, L\}$, and $\text{card}\, Q(\ell) = 6^m$, then $\omega \in F_{Q(\ell)}$ by (8.50) and (8.51), and hence (8.55) holds by (8.53) and (8.42) (regardless of whether $n - 1 = m$ or $n - 1 > m$).

Proof of (D). Suppose $\omega \in H^{(n)}(\mathcal{Q})$ and $\ell \in \mathcal{I}$. Then $\omega \in G_{Q(\ell)}$ by (8.51). Referring to (8.53), one can apply (8.40) and (8.41) with $m = n$ (see (8.39) and (8.50)), and thereby one obtains (8.57) and (8.58). The remaining task is to prove (8.56).

If $n = 1$, and hence $\text{card}\, Q(\ell) = 6$, then in the terminology of statement (D), $Q(\ell) := \{z(1), z(2), \dots, z(6)\}$ where $z(1) < z(2) < \cdots < z(6)$, also $y(u) = z(u)$ for each $u \in \{1, \dots, 6\}$ (and $q(\ell) = y(1) = z(1)$), hence

$$X^{(n-1)}_{Q(\ell)}(\omega) = \left(X^{(0)}_{q(\ell)}(\omega), X^{(0)}_{y(2)}(\omega), X^{(0)}_{y(3)}(\omega), \dots, X^{(0)}_{y(6)}(\omega) \right),$$

and thus (8.56) holds (see the phrase right after (8.56)).

Now suppose instead that $n \geq 2$. Recall again from (8.51) that $\omega \in G_{Q(\ell)}$ (since $\ell \in \mathcal{I}$). By (8.30), $Q(\ell) \in \mathcal{E}_n(\omega)$. Hence by (8.2),

$$\forall\, k \in Q(\ell), \qquad N_k(\omega) \geq n. \tag{8.61}$$

Applying Lemma 8.5(C) (with $m = n$), let $E(1), E(2), \dots, E(6)$ be sets $\subset \mathbf{Z}$ such that $E(1) \cup E(2) \cup \cdots \cup E(6) = Q(\ell)$, $E(1) < E(2) < \cdots < E(6)$, and $E(u) \in \mathcal{E}_{n-1}(\omega)$ for each $u \in \{1, \dots, 6\}$. For each $u \in \{1, \dots, 6\}$, $\text{card}\, E(u) = 6^{n-1}$ by Lemma 8.5(A) (applied with $m = n - 1$). It follows that in the terminology of statement (D) (here in Lemma 8.13), for each $u \in \{1, \dots, 6\}$,

$$E(u) = \left\{ z((u-1) \cdot 6^{n-1} + 1), z((u-1) \cdot 6^{n-1} + 2), \dots, z(u \cdot 6^{n-1}) \right\},$$

and hence also $\min E(u) = y(u)$ (recall also that $y(1) = q(\ell)$). Also (see section 2.1(C))

$$X^{(n-1)}_{Q(\ell)}(\omega) = \left\langle X^{(n-1)}_{E(1)}(\omega), X^{(n-1)}_{E(2)}(\omega), \dots, X^{(n-1)}_{E(6)}(\omega) \right\rangle.$$

For each $u \in \{1, \dots, 6\}$, since $E(u) \in \mathcal{E}_{n-1}(\omega)$ (as was noted above), one has by (8.61) and Remark 8.2(E) that $W^{(n)}_{y(u)}(\omega) \neq 0$ must hold, and hence by (8.41), $X^{(n-1)}_{E(u)}(\omega) = \zeta^{(n-1,\text{fri})}_{y(u)}(\omega)$. Equation (8.56) follows. That completes the proof of statement (D).

Proof of (E). By statement (A), $H^{(n)}(\mathcal{Q}) = H^{(n)}(\mathcal{Q}) \bigcap [\bigcap_{\ell \in \mathcal{I}} \{W^{(n)}_{q(\ell)} = 1\}]$. Hence by ((8.52) and) Lemma 6.4(D), conditional on the event $H^{(n)}(\mathcal{Q})$, the $\{0,1\}$–valued random variables $I(W^{(n+1)}_{q(\ell)} \neq 0)$, $\ell \in \mathcal{I}$ are independent, with each taking the value 0 resp. 1 with probability 5/8 resp. 3.8. Also, the random sequences $\overline{\xi}^{(\infty)}$, $\overline{\zeta}^{(\infty)}$, and $X^{(0)}$ are independent by Constructions 7.3 and 7.4; and hence the random sequences $\overline{W}^{(n+1)}$, $\overline{\zeta}^{(\infty)}$, and $X^{(0)}$ are independent by (6.5) (and Definition 2.6). Also, the random variables $X^{(0)}_k$, $k \in \mathbf{Z}$ and $\zeta^{(i,\text{ord})}_k$, $\zeta^{(i,\text{cen})}_k$, and $\zeta^{(i,\text{fri})}_k$, $i \in \mathbf{N}$, $k \in \mathbf{Z}$ are all independent of each other, by Constructions 7.3(A) and 7.4(A). Putting these pieces (and the first sentence of statement (A)) together, one obtains statement (E) by an awkward but trivial calculation involving conditional probabilities.

Proof of (F), (G), and (H). We shall just give the argument for (H). The arguments for (F) and (G) are similar, using (8.56) resp. (8.57) in the places where (in the argument below for (H)) (8.58) is used. (Keep in mind that here in our context of Definition 8.12, the sets $Q(\ell)$, $\ell \in \{1, 2, \dots, L\}$ are (pairwise) disjoint (if $L \geq 2$).)

Define the random variables (vectors) Z_ℓ, $\ell \in \{1, \dots, L\}$ as follows: For $\omega \in \Omega_0$,

$$Z_\ell(\omega) := \begin{cases} X^{(0)}_{q(\ell)}(\omega) & \text{if card } Q(\ell) = 1 \\ \zeta^{(m,\text{cen})}_{q(\ell)}(\omega) & \text{if card } Q(\ell) = 6^m \text{ where } m \in \{1, \dots, n-1\} \\ [\text{RHS of } (8.58)] & \text{if } \ell \in \mathcal{I} \end{cases} \quad (8.62)$$

(Of course in the right hand side of (8.62), the "middle" part is vacuous, and should be omitted, if $n = 1$.)

By statement (E) (and the trivial fact that $I(W^{(n+1)}_{q(\ell)} = 0) = 1 - I(W^{(n+1)}_{q(\ell)} \neq 0)$ for $\ell \in \mathcal{I}$), the random variables $Z_1, Z_2, \dots, Z_L$ are conditionally independent given $H^{(n)}(\mathcal{Q})$. Also, by (8.54), (8.55), (8.58), and (8.62), $X^{(n)}_{Q(\ell)}(\omega) = Z_\ell(\omega)$ for each $\omega \in H^{(n)}(\mathcal{Q})$ and each $\ell \in \{1, \dots, L\}$. Hence (see section 2.5(A))

$$\begin{aligned} &\mathcal{L}\left(\left(X^{(n)}_{Q(1)}, X^{(n)}_{Q(2)}, \dots, X^{(n)}_{Q(L)}\right) \Big| H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left((Z_1, Z_2, \dots, Z_L) \Big| H^{(n)}(\mathcal{Q})\right) \\ &= \mathcal{L}\left(Z_1 \Big| H^{(n)}(\mathcal{Q})\right) \times \cdots \times \mathcal{L}\left(Z_L \Big| H^{(n)}(\mathcal{Q})\right) \\ &= \mathcal{L}\left(X^{(n)}_{Q(1)} \Big| H^{(n)}(\mathcal{Q})\right) \times \cdots \times \mathcal{L}\left(X^{(n)}_{Q(L)} \Big| H^{(n)}(\mathcal{Q})\right). \end{aligned}$$

Thus (H) holds.

Proof of (I). Suppose $\ell \in \mathcal{I}$. Recall the first sentence in statement (A), and recall from the proof of (E) that the random sequences $\overline{W}^{(n+1)}$, $\overline{\zeta}^{(\infty)}$, and $X^{(0)}$ are independent of each other. By Construction 7.3(A) and equation (8.56) (see also Section 2.5(A)(B)),

$$\begin{aligned} \mathcal{L}\left(X^{(n-1)}_{Q(\ell)} \Big| H^{(n)}(\mathcal{Q})\right) &= \mathcal{L}\left([\text{RHS of } (8.56)] \Big| H^{(n)}(\mathcal{Q})\right) \\ &= \mathcal{L}([\text{RHS of } (8.56)]) = \left(\nu^{(n-1)}_{\text{fri}}\right)^{[6]}. \end{aligned}$$

Thus statement (i) in (I) holds. The equality $\mathcal{L}(Y^{(n)}_{Q(\ell)} \mid H^{(n)}(\mathcal{Q})) = \nu^{(n)}_{\text{ord}}$ in statement (ii) holds by a similar argument using (8.57) instead of (8.56). To obtain the final equality in statement (ii), note that for any $a \in \{-1, 1\}^{\text{sxtp}(n)}$ (see (2.1)), by (8.58), statement (E), the first sentence of (A), Construction 7.3(A), and Remark 3.4,

$$\begin{aligned} &P\left(X^{(n)}_{Q(\ell)} = a \Big| H^{(n)}(\mathcal{Q})\right) \\ &= P\left(W^{(n+1)}_{q(\ell)} = 0 \Big| H^{(n)}(\mathcal{Q})\right) \cdot P\left(X^{(n)}_{Q(\ell)} = a \Big| \left\{W^{(n+1)}_{q(\ell)} = 0\right\} \cap H^{(n)}(\mathcal{Q})\right) \\ &\quad + P\left(W^{(n+1)}_{q(\ell)} \neq 0 \Big| H^{(n)}(\mathcal{Q})\right) \\ &\qquad \cdot P\left(X^{(n)}_{Q(\ell)} = a \Big| \left\{W^{(n+1)}_{q(\ell)} \neq 0\right\} \cap H^{(n)}(\mathcal{Q})\right) \\ &= (5/8) \cdot P\left(\zeta^{(n,\text{cen})}_{q(\ell)} = a \Big| \left\{W^{(n+1)}_{q(\ell)} = 0\right\} \cap H^{(n)}(\mathcal{Q})\right) \\ &\quad + (3/8) \cdot P\left(\zeta^{(n,\text{fri})}_{q(\ell)} = a \Big| \left\{W^{(n+1)}_{q(\ell)} \neq 0\right\} \cap H^{(n)}(\mathcal{Q})\right) \\ &= (5/8) \cdot P\left(\zeta^{(n,\text{cen})}_{q(\ell)} = a\right) + (3/8) \cdot P\left(\zeta^{(n,\text{fri})}_{q(\ell)} = a\right) \\ &= (5/8) \cdot \nu^{(n)}_{\text{cen}}(\{a\}) + (3/8) \cdot \nu^{(n)}_{\text{fri}}(\{a\}) = \nu^{(n)}_{\text{ord}}(\{a\}). \end{aligned}$$

Thus the last equality in statement (ii) in (I) holds. That completes the proof of statement (I).

Proof of (J). Recall again that the sets $Q(\ell)$, $\ell \in \{1, 2, \dots, L\}$ are (pairwise) disjoint (if $L \geq 2$). By statements (B), (C), and I(ii),

$$\mathcal{L}\left(Y^{(n)}_{Q(\ell)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left(X^{(n)}_{Q(\ell)} \,\middle|\, H^{(n)}(\mathcal{Q})\right)$$

for each $\ell \in \{1, \dots, L\}$. Hence statement (J) holds by statements (G) and (H).

Proof of (K). By statements (F) and (G), it suffices to prove that for each $\ell \in \{1, \dots, L\}$ such that $S \cap Q(\ell)$ is nonempty, one has that (see (2.4) and section 2.5(A) again)

$$\mathcal{L}\left(X^{(n-1)}_{S\cap Q(\ell)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) = \mathcal{L}\left(Y^{(n)}_{S\cap Q(\ell)} \,\middle|\, H^{(n)}(\mathcal{Q})\right).$$

For $\ell \notin \mathcal{I}$ (see (8.50)), that holds trivially by (8.54) and (8.55). For $\ell \in \mathcal{I}$, it holds by statement (I)(i)(ii), Lemma 3.7(A), and the assumption (in statement (K)) that $\operatorname{card}(S \cap Q(\ell)) \leq 5$. Thus (K) holds.

Proof of (L) and (M). First just suppose S is a nonempty subset of $Q(1) \cup Q(2) \cup \dots \cup Q(L)$. For a given vector $(k(1), k(2), \dots, k(6)) \in S^6$ (the coordinates need not be distinct), if the set $K := \{k(1), k(2), \dots, k(6)\}$ intersects at least two of the sets $Q(\ell)$ (and hence $\operatorname{card}(K \cap Q(\ell)) \leq 5$ for each $\ell \in \{1, \dots, L\}$), then by statements (F), (G), and (K),

$$E\left[\prod_{u=1}^{6} X^{(n-1)}_{k(u)} \,\middle|\, H^{(n)}(\mathcal{Q})\right] = E\left[\prod_{u=1}^{6} Y^{(n)}_{k(u)} \,\middle|\, H^{(n)}(\mathcal{Q})\right]. \tag{8.63}$$

Also, by (8.54) and (8.55), equation (8.63) holds if $k(1), \dots, k(6)$ are elements of the same set $Q(\ell)$ where $\ell \notin \mathcal{I}$. Hence the two sides of (8.63) can differ only in (some) cases where $k(1), \dots, k(6)$ are elements of the same set $Q(\ell)$ where $\ell \in \mathcal{I}$. Hence, using the notation $T \uparrow 6 := T \times T \times T \times T \times T \times T$ for a given set T, one has that

$$\begin{aligned}
&E\left[\left(\sum_{k\in S} X^{(n-1)}_k\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] - E\left[\left(\sum_{k\in S} Y^{(n)}_k\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] \\
&= \sum_{(k(1),\dots,k(6))\in S\uparrow 6} ([\text{LHS of } (8.63)] - [\text{RHS of } (8.63)]) \\
&= \sum_{\ell\in\mathcal{I}} \sum_{(k(1),\dots,k(6))\in (S\cap Q(\ell))\uparrow 6} ([\text{LHS of } (8.63)] - [\text{RHS of } (8.63)]) \\
&= \sum_{\ell\in\mathcal{I}} \Bigg[E\left[\left(\sum_{k\in S\cap Q(\ell)} X^{(n-1)}_k\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] \\
&\qquad - E\left[\left(\sum_{k\in S\cap Q(\ell)} Y^{(n)}_k\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right]\Bigg].
\end{aligned} \tag{8.64}$$

Hence by statement I(i)(ii) and Lemma 3.7(B), equation (8.59) holds. Now if also $S \supset Q(\ell)$ for some $\ell \in \mathcal{I}$, then by (8.64), statement I(i)(ii), and Lemma 3.7(B)(C), equation (8.60) holds. That completes the proofs of (L) and (M) and of Lemma 8.13.

Definition 8.14. Suppose S is a nonempty finite subset of $\mathbf{Z}$, and n is a positive integer. A "class $\mathcal{C}(n)$ covering of S" is a family $\mathcal{Q} := \{Q(1), Q(2), \dots, Q(L)\}$ of finitely many subsets of $\mathbf{Z}$ (where L is the number of sets in the family $\mathcal{Q}$) with the following four properties:

(i) For each $\ell \in \{1, \dots, L\}$, card $Q(\ell) \in \{1, 6, 6^2, \dots, 6^n\}$.
(ii) The sets $Q(1), Q(2), \dots, Q(L)$ are (pairwise) disjoint (if $L \geq 2$).
(iii) $S \subset Q(1) \cup Q(2) \cup \cdots \cup Q(L)$.
(iv) For each $\ell \in \{1, \dots, L\}$, the set $S \cap Q(\ell)$ is nonempty.

Lemma 8.15. *Suppose S is a nonempty finite subset of $\mathbf{Z}$, n is a positive integer, and $\omega \in \Omega_0$. Then there exists exactly one class $\mathcal{C}(n)$ covering $\mathcal{Q}$ of S such that (see (8.50)–(8.51)) $\omega \in H^{(n)}(\mathcal{Q})$. That covering $\mathcal{Q}$ is the family of all sets*

$$D \in [\mathcal{D}_0(\omega) \cup \mathcal{D}_1(\omega) \cup \cdots \cup \mathcal{D}_{n-1}(\omega)] \cup \mathcal{E}_n(\omega)$$

such that the set $D \cap S$ is nonempty

Proof. Suppose S, n and ω are as in the statement of Lemma 8.15. Referring to Definitions 8.1 and 8.3, define the family $\mathcal{P}_n$ of subsets of $\mathbf{Z}$ by

$$\mathcal{P}_n := [\mathcal{D}_0(\omega) \cup \mathcal{D}_1(\omega) \cup \cdots \cup \mathcal{D}_{n-1}(\omega)] \cup \mathcal{E}_n(\omega). \tag{8.65}$$

Then by Remark 8.4(C), $\mathcal{P}_n$ is a partition of the set $\mathbf{Z}$ itself into countably many nonempty finite sets.

Let $\mathcal{Q}^*$ denote the family of all sets $D \in \mathcal{P}_n$ such that the set $D \cap S$ is nonempty. Since the set S is finite and the members of $\mathcal{P}_n$ (and hence of $\mathcal{Q}^*$) are (pairwise) disjoint, the family $\mathcal{Q}^*$ has only finitely many sets. By (8.65), Lemma 8.5(A), and Definition 8.3(A)(B), the family $\mathcal{Q}^*$ satisfies property (i) in Definition 8.14. Properties (ii), (iii), and (iv) in Definition 8.14 hold for $\mathcal{Q}^*$ as trivial consequences of the definition of $\mathcal{Q}^*$ and the fact that $\mathcal{P}^n$ is a partition of $\mathbf{Z}$. Thus $\mathcal{Q}^*$ is (by Definition 8.14) a class $\mathcal{C}(n)$ covering of S.

Our next task is to show that $\omega \in H^{(n)}(\mathcal{Q}^*)$. Refer to property (i) in Definition 8.14 and refer to Lemma 8.5(A) again. For each $m \in \{0, 1, \dots, n-1\}$ and each $D \in \mathcal{Q}^*$ such that card $D = 6^m$, one must have that $D \in \mathcal{D}_m(\omega)$ by (8.65) (since $D \in \mathcal{P}_n$ but D cannot belong to $\mathcal{D}_u(\omega)$ or even $\mathcal{E}_u(\omega)$ for any $u \neq m$ by Lemma 8.5(A)), and hence $\omega \in F_D$ by (8.29). For each $D \in \mathcal{Q}^*$ such that card $D = 6^n$, one similarly has that $D \in \mathcal{E}_n(\omega)$ by (8.65) and hence $\omega \in G_D$ by (8.30). Hence by (8.50)–(8.51), $\omega \in H^{(n)}(\mathcal{Q}^*)$ (which is simply the intersection of the various events F_D and G_D mentioned above).

By the definition of $\mathcal{Q}^*$ and its properties identified above, all of Lemma 8.15 has been proved except for uniqueness.

Suppose $\mathcal{Q}$ is any class $\mathcal{C}(n)$ covering of the set S such that $\omega \in H^{(n)}(\mathcal{Q})$. To prove uniqueness in Lemma 8.15, our task is to show that $\mathcal{Q} = \mathcal{Q}^*$. Refer again to property (i) in Definition 8.14. For each $m \in \{0, 1, \dots, n-1\}$ and each $D \in \mathcal{Q}$ such that card $D = 6^m$, one has that $\omega \in F_D$ (since $H^{(n)}(\mathcal{Q}) \subset F_D$ by (8.50)–(8.51)), hence $D \in \mathcal{D}_m(\omega)$ by (8.29), hence $D \in \mathcal{P}_n$ by (8.65). For each $D \in \mathcal{Q}$ such that card $D = 6^n$, one similarly has that $\omega \in G_D$ (again by (8.50)–(8.51)), hence $D \in \mathcal{E}_n(\omega)$ by (8.30), hence $D \in \mathcal{P}_n$ by (8.65). Hence $\mathcal{Q} \subset \mathcal{P}_n$.

If $D \in \mathcal{P}_n$ and the set $D \cap S$ is empty, then (by property (iv) in Definition 8.14)) D cannot belong to $\mathcal{Q}$. If $D \in \mathcal{P}_n$ and (instead) the set $D \cap S$ is nonempty, then (by property (iii) in Definition 8.14), D must belong to $\mathcal{Q}$ (since no other set in $\mathcal{P}_n$ will contain a given $k \in D \cap S$). Hence $\mathcal{Q} = \mathcal{Q}^*$ (by the definition of $\mathcal{Q}^*$). That completes the proof of uniqueness, and of Lemma 8.15.

Lemma 8.16. *Suppose $n \in \mathbf{N}$. Then the following four statements hold:*

(A) *The random sequences $Y^{(n)}$ and $X^{(n)}$ have the same distribution (on $\{-1,1\}^{\mathbf{Z}}$).*

(B) *For any nonempty set $S \subset \mathbf{Z}$ such that $\operatorname{card} S \le 5$, the random vectors $X_S^{(n-1)}$ and $Y_S^{(n)}$ (see (2.4)) have the same distribution (on $\{-1,1\}^{\operatorname{card} S}$).*

(C) *For any nonempty finite set $S \subset \mathbf{Z}$, $E(\sum_{k\in S} X_k^{(n-1)})^6 \ge E(\sum_{k\in S} Y_k^{(n)})^6$.*

(D) *For any integer $m \ge 6 \cdot 16^n$,*

$$E\left(\sum_{k=1}^{m} X_k^{(n-1)}\right)^6 \ge E\left(\sum_{k=1}^{m} Y_k^{(n)}\right)^6 + 360 \cdot 4^{6(n-1)}.$$

Proof. Before addressing any of the four statements (A)—(D) in Lemma 8.16, let us present some arguments that will be common to the proofs of all four statements.

Let $n \in \mathbf{N}$ be arbitrary but fixed.

Suppose S is a nonempty finite subset of $\mathbf{Z}$.

Since there are only countably many finite subsets of $\mathbf{Z}$, there exist only countably many finite families of finite subsets of $\mathbf{Z}$, and hence (see Definition 8.14) only countably many class $\mathcal{C}(n)$ coverings $\mathcal{Q}$ of S. For each such $\mathcal{Q}$, $H^{(n)}(\mathcal{Q}) \subset \Omega_0$ (again see (6.20)–(6.21)) by (8.29), (8.30), and (8.50)–(8.51). By Lemma 8.15, the events $H^{(n)}(\mathcal{Q})$, for class $\mathcal{C}(n)$ coverings $\mathcal{Q}$ of S, form a countable partition of Ω_0. (Some of those events $H^{(n)}(\mathcal{Q})$ may be empty.)

In the sums below, the index $\mathcal{Q}$ ranges over all class $\mathcal{C}(n)$ coverings of S such that $P(H^{(n)}(\mathcal{Q})) > 0$. By a simple argument (see section 2.5(A) and equation (2.4)),

$$\mathcal{L}\left(Y_S^{(n)}\right) = \sum_{\mathcal{Q}} \mathcal{L}\left(Y_S^{(n)} \,\middle|\, H^{(n)}(\mathcal{Q})\right) \cdot P\left(H^{(n)}(\mathcal{Q})\right), \tag{8.66}$$

and the analogous statements holds with $Y_S^{(n)}$ replaced by $X_S^{(n-1)}$ and by $X_S^{(n)}$.

For each such covering $\mathcal{Q}$ (with $P(H^{(n)}(\mathcal{Q})) > 0$), one has that

$$\mathcal{L}(Y_S^{(n)} \mid H^{(n)}(\mathcal{Q})) = \mathcal{L}(X_S^{(n)} \mid H^{(n)}(\mathcal{Q}))$$

by Lemma 8.13(J) and property (iii) in Definition 8.14. Hence by (8.66) and its analog for $X_S^{(n)}$, one has that $\mathcal{L}(Y_S^{(n)}) = \mathcal{L}(X_S^{(n)})$. Since S was an arbitrary nonempty finite subset of $\mathbf{Z}$, statement (A) in Lemma 8.16 follows.

Next, in the case where $\operatorname{card} S \le 5$, an exactly analogous argument holds with $X_S^{(n)}$ replaced by $X_S^{(n-1)}$, using Lemma 8.13(K). Thus statement (B) in Lemma 8.16 holds.

Next (regardless of the (finite) cardinality of S),

$$\begin{aligned} &E\left(\sum_{k\in S} X_k^{(n-1)}\right)^6 - E\left(\sum_{k\in S} Y_k^{(n)}\right)^6 \\ &= \sum_{\mathcal{Q}} E\left[\left(\sum_{k\in S} X_k^{(n-1)}\right)^6 - \left(\sum_{k\in S} Y_k^{(n)}\right)^6 \,\middle|\, H^{(n)}(\mathcal{Q})\right] \cdot P(H^{(n)}(\mathcal{Q})). \end{aligned} \tag{8.67}$$

Hence by Lemma 8.13(L), statement (C) in Lemma 8.16 holds.

Proof of statement (D). Suppose (for our given $n \in \mathbf{N}$) that $m \ge 6 \cdot 16^n$. Now let $S := \{1, 2, \ldots, m\}$.

Let B denote the set (event) of all $\omega \in \Omega_0$ such that there exist distinct integers $i, j \in \{1, 2, \dots, 6 \cdot 16^n\}$ such that $W_i^{(n)}(\omega) = W_j^{(n)}(\omega) = 1$.

Suppose $\omega \in B$. Then by Definition 8.1 and a trivial argument, there exists a set $E \in \mathcal{E}_n(\omega)$ such that $E \subset \{1, 2, \dots, 6 \cdot 16^n\}$ (and hence $E \subset \{1, 2, \dots, m\}$). Now card $E = 6^n$ by Lemma 8.5(A). Also, by Lemma 8.15, E is a member of the unique class $\mathcal{C}(n)$ covering $\mathcal{Q}$ of $\{1, 2, \dots, m\}$ such that $\omega \in H^{(n)}(\mathcal{Q})$.

Hence $B \subset \bigcup_{\mathcal{Q} \in \mathcal{J}} H^{(n)}(\mathcal{Q})$ where $\mathcal{J}$ denotes the family of all class $\mathcal{C}(n)$ coverings $\mathcal{Q}$ of $\{1, 2, \dots, m\}$ such that $\mathcal{Q}$ contains at least one member E such that $E \subset \{1, 2, \dots, m\}$ and card $E = 6^n$. Let $\mathcal{J}+$ denote the family of all $\mathcal{Q} \in \mathcal{J}$ such that $P(H^{(n)}(\mathcal{Q})) > 0$. Now by Lemma 6.7, $P(B) \ge 1/2$. Hence $\sum_{\mathcal{Q} \in \mathcal{J}+} P(H^{(n)}(\mathcal{Q})) \ge 1/2$.

Now by (8.67), together with Lemma 8.13(M) (for $\mathcal{Q} \in \mathcal{J}+$) and Lemma 8.13(L) (for the other $\mathcal{Q}$'s such that $P(H^{(n)}(\mathcal{Q})) > 0$),

$$[\text{LHS of } (8.67)] \ge \sum_{\mathcal{Q} \in \mathcal{J}+} 720 \cdot 4^{6(n-1)} \cdot P(H^{(n)}(\mathcal{Q})) \ge 360 \cdot 4^{6(n-1)}.$$

Thus statement (D) in Lemma 8.16 holds. That completes the proof.

Lemma 8.17.

(A) *For every $n \in \mathbf{N}$ and every nonempty set $S \subset \mathbf{Z}$ with* card $S \le 5$, *one has that (see* (2.4) *and section* 2.5(A))

$$\mathcal{L}(X_S^{(n)}) = \mathcal{L}(X_S^{(0)}).$$

(B) *If m and N are positive integers such that $m \ge 6 \cdot 16^N$, then for every $n \ge N$,*

$$E\left(\sum_{k=1}^{m} X_k^{(n)}\right)^6 \le 15m^3 - 360 \cdot 4^{6(N-1)}.$$

Proof. Statement (A) follows from Lemma 8.16(A)(B) and induction.

Proof of (B). Suppose m and N are as in statement (B). Refer to Construction 7.4(A). Obviously $E(X_0^{(0)})^\ell = 1$ resp. 0 if ℓ is an even resp. odd integer. Hence by a well known, elementary calculation (the "6th moment" analog of the argument, involving 4th moments, in Billingsley, 1995, proof of Theorem 6.1), one that $E(\sum_{k=1}^m X_k^{(0)})^6 \le 15m^3$. Also, by Lemma 8.16 (A)(C), the sequence of numbers $E(\sum_{k=1}^m X_k^{(n)})^6$, $n \in \{0, 1, 2, \dots\}$ is nonincreasing. Also, by Lemma 8.16(A)(D),

$$E\left(\sum_{k=1}^{m} X_k^{(N)}\right)^6 \le E\left(\sum_{k=1}^{m} X_k^{(N-1)}\right)^6 - 360 \cdot 4^{6(N-1)}.$$

Combining these three preceding sentences, one obtains statement (B) in Lemma 8.17.

9. Proof of properties (A), (B), (E), and (F) in Theorem 1.1

This section contains arguments somewhat reminiscent of ones in Bradley and Pruss (2009); but because of some nontrivial differences, the arguments here will be given in full.

By Lemma 7.6 and Remark 7.7, the random sequence $X := (X_k,\, k \in \mathbf{Z})$ defined in (7.14) satisfies strict stationarity and Property (C) in Theorem 1.1 Here in Section 9, it will be shown that the sequence X satisfies properties (A), (B), (E), and (F) in Theorem 1.1. Finally in Section 10, property (D) (the triviality of the double tail σ-field) will be verified, and the proof of Theorem 1.1 will then be complete.

We start with some preliminary arguments. By (7.14), (8.36), (8.37), and Constructions 7.3 and 7.4, the random variables X_k, $X_k^{(0)}$, $X_k^{(n)}$, $Y_k^{(n)}$, $k \in \mathbf{Z}$, $n \in \mathbf{N}$ all take their values in the set $\{-1, 1\}$.

For each $k \in \mathbf{Z}$, $P(N_k = \infty) = 0$ by Lemma 7.8. For each $k \in \mathbf{Z}$ and each $\omega \in \Omega$ such that $N_k(\omega) < \infty$, one has by (7.14) and (8.37) that $X_k(\omega) = X_k^{(n)}(\omega)$ for all $n \geq N_k(\omega)$. Thus trivially,

$$\forall\, k \in \mathbf{Z}, \quad X_k^{(n)} \to X_k \ \text{ a.s. as } n \to \infty. \tag{9.1}$$

Now let us prove properties (A), (B), (E), and (F) in Theorem 1.1.

Proof of (A) and (B). Suppose S is any nonempty subset of $\mathbf{Z}$ such that $\operatorname{card} S \leq 5$. By (9.1) (see also (2.4)), $X_S^{(n)} \to X_S$ a.s. as $n \to \infty$. As an elementary consequence, $X_S^{(n)} \Rightarrow X_S$ (convergence in distribution) as $n \to \infty$. Hence by Lemma 8.17(A) (see also Section 2.5(A)), $\mathcal{L}(X_S) = \mathcal{L}(X_S^{(0)})$.

That last equality has two consequences: First, taking $S = \{k\}$ for an arbitrary $k \in \mathbf{Z}$ and applying (7.12), one obtains property (A) in Theorem 1.1. Second, taking arbitrary subsets $S \subset \mathbf{Z}$ with $\operatorname{card} S = 5$, and applying the independence of the $X_k^{(0)}$'s in Construction 7.4(A), one obtains property (B) in Theorem 1.1.

Proof of property (E). We shall now use the notations in (1.1). Let M be an arbitrary but fixed integer such that $M \geq 96$. To prove property (E), it suffices to show that

$$E\left[M^{-1/2} S(X, M)\right]^6 \leq 15 - 16^{-6}. \tag{9.2}$$

Let N be the positive integer such that

$$6 \cdot 16^N \leq M < 6 \cdot 16^{N+1}. \tag{9.3}$$

By (9.3) and Lemma 8.17(B),

$$\forall\, n \geq N, \quad E\left[S(X^{(n)}, M)\right]^6 \leq 15M^3 - 360 \cdot 4^{6(N-1)}. \tag{9.4}$$

Now for each $n \in \mathbf{N}$, $[S(X^{(n)}, M)]^6 \leq M^6$ (since each $X_k^{(n)}$ takes its values in $\{-1, 1\}$). Also, as a trivial consequence of (9.1), $[S(X^{(n)}, M)]^6 \to [S(X, M)]^6$ a.s. as $n \to \infty$. Hence by dominated convergence, $E[S(X^{(n)}, M)]^6 \to E[S(X, M)]^6$ as $n \to \infty$. Hence by (9.4), $E[S(X, M)]^6 \leq$ [RHS of (9.4)]. Dividing both sides of this by M^3 and then applying the second inequality in (9.3) and simple arithmetic, one obtains

$$\begin{aligned}
\text{[LHS of (9.2)]} &\leq 15 - 360 \cdot 4^{6(N-1)}/M^3 \\
&\leq 15 - [360 \cdot 16^{3(N-1)}]/[216 \cdot 16^{3(N+1)}] \\
&= 15 - (360/216) \cdot 16^{-6}.
\end{aligned}$$

Thus (9.2) holds, and property (E) is proved.

Proof of property (F). This is analogous to a corresponding argument in Bradley and Pruss (2009). However, it is short and is also at the core of this construction. It will be given in full here.

By property (A) (proved above), $EX_0 = 0$ and $EX_0^2 = EX_0^4 = 1$. Hence by strict stationarity, property (B), and the well known, elementary calculation in Billingsley (1995, proof of Theorem 6.1) (which requires only 4–tuplewise independence), one has that for each $n \in \mathbf{N}$ (again see (1.1)),

$$E\left(n^{-1/2}S_n\right) = 0, \tag{9.5}$$

$$E\left(n^{-1/2}S_n\right)^2 = 1, \quad \text{and} \tag{9.6}$$

$$E\left(n^{-1/2}S_n\right)^4 \le 3. \tag{9.7}$$

By (say) (9.6) and Chebyshev's inequality, the family of distributions of the normalized partial sums $(n^{-1/2}S_n,\, n \in \mathbf{N})$ is tight.

Now for the proof of property (F), suppose Q is an infinite subset of $\mathbf{N}$. Because of tightness, there exists an infinite set $T \subset Q$ and a probability measure μ on $(\mathbf{R}, \mathcal{R})$ (both T and μ henceforth fixed) such that

$$n^{-1/2}S_n \Rightarrow \mu \quad \text{as } n \to \infty, n \in T. \tag{9.8}$$

To complete the proof of property (F), our task is to show that μ is neither degenerate nor normal.

Because of (9.7), (9.8), and Billingsley (1995, p. 338, the Corollary), one has by (9.5) and (9.6) that

$$\int_{x\in\mathbf{R}} x\mu(dx) = 0 \quad \text{and} \quad \int_{x\in\mathbf{R}} x^2\mu(dx) = 1. \tag{9.9}$$

Hence the probability measure μ has positive variance and is therefore nondegenerate. Our task now is to prove that μ is not normal.

If μ were normal, then by (9.9) it would have to be the $N(0,1)$ distribution, and hence by a well known calculation (e.g. Billingsley, 1995, p. 275, Example 21.1) one would have $\int_{x\in\mathbf{R}} x^6\mu(dx) = 15$. By (9.8) and Billingsley (1995, p. 334, Corollary 1, and p. 338, Theorem 25.11), one would then have

$$\liminf_{n\to\infty} E\left(n^{-1/2}S_n\right)^6 \ge 15.$$

But that contradicts property (E) (proved above). Hence μ cannot be normal. That completes the proof of property (F).

10. Proof of property (D) in Theorem 1.1

This is the final piece in the proof of Theorem 1.1. This argument will be divided into thirteen "steps" (including some "lemmas"), numbered 10.1, 10.2, etc.

Step 10.1. Let F' be an arbitrary but fixed event such that

$$F' \in \mathcal{T}_{\text{double}}(X). \tag{10.1}$$

To prove property (D), i.e. to show that $\mathcal{T}_{\text{double}}(X)$ is trivial, our task is to show that $P(F') = 0$ or 1.

Suppose instead that

$$0 < P(F') < 1. \tag{10.2}$$

Our task is to produce a contradiction. That task will consume the rest of Section 10 here.

Step 10.2. By the assumption of (10.2), one has that $[P(F')]^2 < P(F')$. Let $\varepsilon \in (0,1)$ be fixed such that

$$[P(F')+\varepsilon] \cdot [P(F')+3\varepsilon] < P(F') - 4\varepsilon. \tag{10.3}$$

By (10.1), (7.13) (and its entire paragraph), (7.14), Lemma 7.6, and Definition 2.6, one trivially has that $F' \in \sigma(X) \dot{\subset} \sigma(\eta)$. (One can do better, but this will suffice.) Hence from Constructions 6.1, 7.3, and 7.4 and a standard measure–theoretic argument (again see (7.13)), there exists a positive integer N' and an event

$$\begin{aligned} F'' \in \sigma &\left(X_k^{(0)},\, -N' \le k \le N'\right) \\ &\vee \sigma\left(\xi_k^{(n)},\ 1 \le n \le N',\ -N' \le k \le N'\right) \\ &\vee \sigma\left(\zeta_k^{(n)},\ 1 \le n \le N',\ -N' \le k \le N'\right) \end{aligned} \tag{10.4}$$

(see (7.9)) such that

$$P\left(F'' \triangle F'\right) \le \varepsilon \tag{10.5}$$

(where Δ denotes symmetric difference).

By (10.5) and a trivial argument,

$$P(F'') \le P(F') + \varepsilon. \tag{10.6}$$

Increasing N' if necessary, assume further that

$$N' \ge 1 + 6 \cdot (8/5)^{30} \cdot (8/3)^6. \tag{10.7}$$

(The positive integer N' and the event F'', satisfying (10.4)–(10.7), are now taken as “fixed.”)

Step 10.3. Refer to section 6.3(A). Random variables defined in this step will be defined on Ω_0 and left undefined on Ω_0^c.

For each $n \in \mathbf{N}$, we shall define a random positive integer L_n and a sequence $(H(n,1), H(n,2), H(n,3), \dots)$ of random positive integers such that for each $\omega \in \Omega_0$,

$$L_n(\omega) < H(n,1)(\omega) < H(n,2)(\omega) < H(n,3)(\omega) < \dots \tag{10.8}$$

and

$$\begin{aligned} &\left\{k \in \mathbf{Z} : k > L_n(\omega) \text{ and } W_k^{(n)}(\omega) = 1\right\} \\ &\qquad = \{H(n,1)(\omega), H(n,2)(\omega), H(n,3)(\omega), \dots\}, \end{aligned} \tag{10.9}$$

and also (see section 6.1(C))

$$\sigma(L_n, H(n,1), H(N,2), H(n,3), \dots) \dot{\subset} \sigma\left(\overline{\xi}^{(n)}\right). \tag{10.10}$$

This will be done in such a way that

$$EL_1 \le N' + 6 \cdot (8/5)^{30} \cdot (8/3)^6. \tag{10.11}$$

Also, for each $n \ge 2$, we shall define a random positive integer Φ_n, satisfying

$$\sigma(\Phi_n) \dot{\subset} \sigma\left(\overline{\xi}^{(n)}\right) \quad \text{and} \quad E\Phi_n \le 6 \cdot (8/5)^{30} \cdot (8/3)^6. \tag{10.12}$$

The definition will be recursive, and is as follows:

To start with $n = 1$, refer to Statement 6.3(A)(ii) (and Section 2.4(A)(B)(C)(D)). Define the positive integer-valued random variable L_1 as follows: For every $\omega \in \Omega_0$,

$$L_1(\omega) := \min\Big\{k \in \mathbf{Z} : k \geq N' + 6 \text{ and } \big[\xi_{k-5}^{(1)t}(\omega) \mid \xi_{k-4}^{(1)t}(\omega) \mid \cdots \mid \xi_k^{(1)t}(\omega)\big] = I_6\Big\} \tag{10.13}$$

(where as usual, the superscript t denotes transpose). By Lemma 5.2(C) (see Section 6.1(B)), equation (10.11) holds. Now define the random positive integers $H(1,i)$, $i \in \mathbf{N}$ uniquely by (10.8) and (10.9) (for $n = 1$) for $\omega \in \Omega_0$. Trivially by (10.13), (10.8)–(10.9), and (6.5) (and Remark 2.6(B)), equation (10.10) holds for $n = 1$.

Now suppose $n \geq 2$, and the positive integer–valued random variables L_{n-1} and $H(n-1,i)$, $i \in \mathbf{N}$ are already defined, satisfying (10.8), (10.9), and (10.10) with n replaced by $n - 1$.

Referring to (10.8) – (10.9) and Statement 6.3(A)(iii) (with n replaced by $n-1$), define the positive integer-valued random variable Φ_n as follows: For each $\omega \in \Omega_0$,

$$\Phi_n(\omega) := \min\Big\{k \geq 6 : \Big[\xi_{H(n-1,k-5)(\omega)}^{(n)t}(\omega) \mid \xi_{H(n-1,k-4)(\omega)}^{(n)t}(\omega) \mid \cdots \mid \xi_{H(n-1,k)(\omega)}^{(n)t}(\omega)\Big] = I_6\Big\}. \tag{10.14}$$

Next, define the positive integer-valued random variable L_n as follows: For all $\omega \in \Omega_0$,

$$L_n(\omega) := H(n-1, \Phi_n(\omega))(\omega). \tag{10.15}$$

Now define the random positive integers $H(n,i)$, $i \in \mathbf{N}$ (uniquely) by (10.8) and (10.9) (for the given n) for $\omega \in \Omega_0$.

By (10.10) (with n replaced by $n-1$) and (10.14) (for the given n), the first part $(\sigma(\Phi_n) \dot\subset \sigma(\overline{\xi}^{(n)}))$ of (10.12) holds; hence by (10.10) (with n replaced by $n - 1$) and (10.15) (for the given n), $\sigma(L_n) \dot\subset \sigma(\overline{\xi}^{(n)})$; and hence (10.10) holds for the given n by (10.8)–(10.9) and (6.5). To complete this recursive definition, all that remains is to verify the second part $(E\Phi_n \leq \dots)$ in (10.12).

For that purpose, we shall apply Lemma 5.2(D), with the random integers $\kappa(j)$, $j \in \mathbf{Z}$ there defined by $\kappa(j) := H(n-1, j)$ for $j \geq 1$ and (just as a frivolous formality) $\kappa(j) := j$ (a constant random variable) for $j \leq 0$ (see (10.8)). Then $\sigma(\kappa(j))$, $j \in \mathbf{Z}) \dot\subset \sigma(\overline{\xi}^{(n-1)})$ by (10.10) with n replaced by $n-1$. Since the sequences $\xi^{(n)}$ and $\overline{\xi}^{(n-1)}$ are independent (and $\xi^{(n)}$ has the appropriate distribution), all conclusions of Lemma 5.2(D) apply to the sequence $(\xi_{\kappa(j)}^{(n)}, j \in \mathbf{Z})$. Applying Lemma 5.2(D)(iv) to that sequence, one obtains that the random variable Φ_n (see (10.14)) satisfies the second part of (10.12). That completes this recursive definition.

By (10.13), (10.8), (10.14) (which gives $\Phi_n(\omega) \geq 6$ for $n \geq 2$ and $\omega \in \Omega_0$), (10.15), and induction, one has that for all $\omega \in \Omega_0$,

$$N' + 6 \leq L_1(\omega) < H(1,1)(\omega) < L_2(\omega) < H(2,1)(\omega) < L_3(\omega) < H(3,1)(\omega) < \dots . \tag{10.16}$$

Step 10.4. The "universe" of "scaffolding" in Sections 6, 7, and 8 was based on the following array of independent random variables from Constructions 6.1, 7.3 (recall (7.9)), and 7.4:

$$X_k^{(0)},\ k \in \mathbf{Z};\ \text{and } \xi_k^{(n)}, \zeta_k^{(n,\mathrm{ord})}, \zeta_k^{(n,\mathrm{cen})}, \zeta_k^{(n,\mathrm{fri})},\ (n,k) \in \mathbf{N} \times \mathbf{Z}. \tag{10.17}$$

Here in Step 10.4, we shall construct an "alternate universe" based on an array

$$X_k^{*(0)},\ k \in \mathbf{Z}; \quad \text{and} \quad \xi_k^{*(n)}, \zeta_k^{*(n,\mathrm{ord})}, \zeta_k^{*(n,\mathrm{cen})}, \zeta_k^{*(n,\mathrm{fri})},\ (n,k) \in \mathbf{N} \times \mathbf{Z}. \tag{10.18}$$

To create this "alternate universe," we shall keep the "original" random variables in the array (10.17), except that the random variables in (10.17) that are mentioned in the right hand side of (10.4) will be replaced by "independent copies."

(A) Refer to the positive integer N' in (10.4) and (10.7). Let

$$\begin{aligned} X_k^{*(0)},\ k \in \{-N', -N'+1, -N'+2, \dots, N'\} \text{ and} \\ \xi_k^{*(n)}, \zeta_k^{*(n,\mathrm{ord})}, \zeta_k^{*(n,\mathrm{cen})}, \zeta_k^{*(n,\mathrm{fri})}, \\ n \in \{1, 2, \dots, N'\}, \quad k \in \{-N', -N'+1, \dots, N'\} \end{aligned} \tag{10.19}$$

be an array of independent random variables, with this array being independent of the entire array (10.17) (and hence independent of the entire collection of random variables studied in sections 6, 7, 8, and 9), such that for each $k \in \{-N', -N'+1, \dots, N'\}$ and $n \in \{1, 2, \dots, N'\}$, the random variable $X_k^{*(0)}$ resp. $\xi_k^{*(n)}$ resp. $\zeta_k^{*(n,\mathrm{ord})}$ resp. $\zeta_k^{*(n,\mathrm{cen})}$ resp. $\zeta_k^{*(n,\mathrm{fri})}$ takes its values in the space $\{-1,1\}$ resp. $\{0,1\}^6$ resp. $\{-1,1\}^{\mathrm{sxtp}(n)}$ (see (2.1) again) resp. $\{-1,1\}^{\mathrm{sxtp}(n)}$ resp. $\{-1,1\}^{\mathrm{sxtp}(n)}$ and has the same distribution as the random variable $X_k^{(0)}$ resp. $\xi_k^{(n)}$ resp. $\zeta_k^{(n,\mathrm{ord})}$ resp. $\zeta_k^{(n,\mathrm{cen})}$ resp. $\zeta_k^{(n,\mathrm{fri})}$.

(B) For each $k \in \mathbf{Z} - \{-N', -N'+1, \dots, N'\}$, define the $\{-1,1\}$–valued random variable $X_k^{*(0)}$ by $X_k^{*(0)} := X_k^{(0)}$ (that is, for all $\omega \in \Omega$, $X_k^{*(0)}(\omega) := X_k^{(0)}(\omega)$).

(C) For each $(n,k) \in (\mathbf{N} \times \mathbf{Z}) - (\{1, 2, \dots, N'\} \times \{-N', -N'+1, \dots, N'\})$, define the $\{0,1\}^6$–valued random variable $\xi_k^{*(n)}$ by $\xi_k^{*(n)} := \xi_k^{(n)}$, and define the $\{-1,1\}^{\mathrm{sxtp}(n)}$–valued random variables (see (2.1)) $\zeta_k^{*(n,\mathrm{ord})}$, $\zeta_k^{*(n,\mathrm{cen})}$, and $\zeta_k^{*(n,\mathrm{fri})}$ by $\zeta_k^{*(n,\mathrm{ord})} := \zeta_k^{(n,\mathrm{ord})}$, $\zeta_k^{*(n,\mathrm{cen})} := \zeta_k^{(n,\mathrm{cen})}$, and $\zeta_k^{*(n,\mathrm{fri})} := \zeta_k^{(n,\mathrm{fri})}$. That completes the definition of the array (10.18).

(D) Recall from Constructions 6.1, 7.3, and 7.4 that the random variables in the array (10.17) are independent of each other. It follows from the conditions in (A), (B), and (C) above that the random variables in the array (10.18) are independent of each other, and that in fact the two arrays (10.17) and (10.18) have the same distribution (on the appropriate space).

(E) Further, the array (10.18) is independent of the entire σ–field in the right hand side of (10.4). Hence by (10.4) itself,

$$\text{the event } F'' \text{ is independent of the array (10.18).} \tag{10.20}$$

(F) Starting with (A), (B), (C), and (D) above, we construct analogs of all of the random variables constructed in Sections 6, 7, and 8, using the array (10.18) in place of the array (10.17). The notations will be the same except that (as in (10.18)) an asterisk will be inserted after each "main symbol." The general pattern is that for a given random variable Z of the form $Z := f(\text{the array (10.17)})$, where f is an appropriate measurable function, the analogous random variable Z^* will be given by $Z^* = f(\text{the array (10.18)})$

(with the same function f). Thus the analog of (6.3) is

$$\overline{\xi}_k^{*(n)} := \left(\xi_k^{*(1)}, \xi_k^{*(2)}, \dots, \xi_k^{*(n)}\right),$$

the analog of (6.9) is

$$W_k^{*(1)} := g_{\text{spaced}}\left(\xi_k^{*(1)}, \xi_{k-1}^{*(1)}, \xi_{k-2}^{*(1)}, \dots\right),$$

the analog of (6.10) is

$$\Psi^*(n,k,j) := \psi_j\left(W_k^{*(n)}, W_{k-1}^{*(n)}, W_{k-2}^{*(n)}, \dots\right),$$

and so on. Along the way, corresponding analogs of the events $\Omega_{\text{good}}^{(n)}$ and Ω_0 in Section 6.2 and equation (6.20) are defined and denoted $\Omega_{\text{good}}^{*(n)}$ and Ω_0^*.

Of course because of (B) and (C) above, some of the new ("asterisk") random variables defined in this way will be exactly the same as the original counterparts in Sections 6, 7, and 8.

(G) Now define the event

$$\Omega' := \Omega_0 \cap \Omega_0^*. \tag{10.21}$$

By (6.21) and its "asterisk" counterpart ($P(\Omega_0^*) = 1$), one has that

$$P(\Omega') = 1. \tag{10.22}$$

Also (for example), for every $\omega \in \Omega'$, Statements 6.3(A)(i)(ii)(iii) and their "asterisk" counterparts all hold.

(H) The random variables formulated in Step 10.3 (L_n and $H(n,i)$ for $n, i \in \mathbf{N}$, and Φ_n for $n \geq 2$), are defined (and positive integer-valued) at every $\omega \in \Omega'$, by (10.21) and the second sentence in Step 10.3 itself. For those random variables, we will not need to refer to, and will therefore not formally define, "asterisk" counterparts.

Step 10.5.

(A) Refer to Step 10.4(B)(C). The portion of the array (10.18) involving indices $k \leq -N' - 1$ (and any $n \in \mathbf{N}$) coincides with the corresponding portion of the array (10.17).

(B) In particular, for any $k \leq -N' - 1$, $\xi_k^{*(1)} = \xi_k^{(1)}$ (that is, $\xi_k^{*(1)}(\omega) = \xi_k^{(1)}(\omega)$ for all $\omega \in \Omega$). Hence $W_k^{*(1)} = W_k^{(1)}$ for every $k \leq -N' - 1$, since by (6.9) (in both arrays (10.17) and (10.18)), for any $k \leq -N' - 1$ and any $\omega \in \Omega$,

$$\begin{aligned} W_k^{*(1)}(\omega) &= g_{\text{spaced}}\left(\xi_k^{*(1)}(\omega), \xi_{k-1}^{*(1)}(\omega), \xi_{k-2}^{*(1)}(\omega), \dots\right) \\ &= g_{\text{spaced}}\left(\xi_k^{(1)}(\omega), \xi_{k-1}^{(1)}(\omega), \xi_{k-2}^{(1)}(\omega), \dots\right) = W_k^{(1)}(\omega). \end{aligned} \tag{10.23}$$

By the same type of argument, using (6.10) and (6.11) (in both arrays (10.17) and (10.18)) together with induction on n, one can show the following two facts together: that $\Psi^*(n,k,j) = \Psi(n,k,j)$ for each $n \in \mathbf{N}$, each $k \leq -N' - 1$, and each $j \geq 0$, and that

$$\forall\, n \in \mathbf{N},\ \forall\, k \leq -N' - 1, \quad W_k^{*(n)} = W_k^{(n)}. \tag{10.24}$$

(C) Continuing to apply (A) above in the same way as in (B), one obtains for any $k \leq -N' - 1$ and $n \in \mathbf{N}$ the following equalities of random variables: $\overline{\xi}_k^{*(n)} = \overline{\xi}_k^{(n)}$ (see (6.3)), $\overline{W}_k^{*(n)} = \overline{W}_k^{(n)}$ (see (6.25)), $\delta_k^{*(n)} = \delta_k^{(n)}$ (see (7.1) and (7.2)), $N_k^* = N_k$ (see (7.4)), $J^*(n,k) = J(n,k)$ (see (7.7)), $\zeta_k^{*(n)} = \zeta_k^{(n)}$ (see (7.9)), $\overline{\zeta}_k^{*(n)} = \overline{\zeta}_k^{(n)}$ (see (7.10)), and finally, from (7.14),

$$\forall\, k \leq -N' - 1, \qquad X_k^* = X_k. \tag{10.25}$$

Lemma 10.6. *Refer to Step 10.3 (including equation (10.16)). Refer also to (10.21)–(10.22) and Step 10.4(H). Suppose $n \in \mathbf{N}$. Then for every $\omega \in \Omega'$, the following statements hold:*

$$\forall\ k \geq L_n(\omega) + 1, \quad W_k^{*(n)}(\omega) = W_k^{(n)}(\omega); \tag{10.26}$$

$$\forall\ k \geq H(n,1)(\omega), \quad \delta_k^{*(n)}(\omega) = \delta_k^{(n)}(\omega); \tag{10.27}$$

$$\forall\ k \geq H(n,1)(\omega), \quad \Psi^*(n,k,0)(\omega) = \Psi(n,k,0)(\omega);\ \textit{and} \tag{10.28}$$

$$\forall\ k \geq H(n,1)(\omega), \quad k - \Psi^*(n,k,0)(\omega) = k - \Psi(n,k,0)(\omega) \geq H(n,1)(\omega). \tag{10.29}$$

Proof. Throughout this proof, let $\omega \in \Omega'$ be arbitrary but fixed. (Again recall Step 10.4(G)(H).)

For each $n \in \mathbf{N}$, let the function $\varphi_n : (\{0,1\}^6)^n \to \{0,1,2,3,4,5,6\}$ be as in Lemma 4.7.

We shall first verify (10.26) and (10.27) for $n = 1$. For each $i \geq L_1(\omega) - 5$, one has that $\xi_i^{*(1)}(\omega) = \xi_i^{(1)}(\omega)$ by (10.16) and Step 10.4(C). Hence for each integer $k \geq L_1(\omega) + 1$, by the "asterisk" counterpart of (6.9), then two applications of (10.13) and Lemma 4.7 together, followed by (6.9) itself, with $L_1(\omega)$ written here as $L(1)(\omega)$,

$$\begin{aligned} W_k^{*(1)}(\omega) &= g_{\text{spaced}}\left(\xi_k^{*(1)}(\omega), \xi_{k-1}^{*(1)}(\omega), \xi_{k-2}^{*(1)}(\omega), \ldots\right) \\ &= \varphi_{k-L(1)(\omega)}\left(\xi_{L(1)(\omega)+1}^{*(1)}(\omega), \xi_{L(1)(\omega)+2}^{*(1)}(\omega), \ldots, \xi_k^{*(1)}(\omega)\right) \\ &= \varphi_{k-L(1)(\omega)}\left(\xi_{L(1)(\omega)+1}^{(1)}(\omega), \xi_{L(1)(\omega)+2}^{(1)}(\omega), \ldots, \xi_k^{(1)}(\omega)\right) \\ &= g_{\text{spaced}}\left(\xi_k^{(1)}(\omega), \xi_{k-1}^{(1)}(\omega), \xi_{k-2}^{(1)}(\omega), \ldots\right) \\ &= W_k^{(1)}(\omega). \end{aligned}$$

Thus (10.26) holds for $n = 1$. By (10.26) (for $n = 1$) and (7.1) (and its "asterisk" counterpart), $\delta_k^{*(1)}(\omega) = \delta_k^{(1)}(\omega)$ for all $k \geq L_1(\omega) + 1$ (and in particular for all $k \geq H(1,1)(\omega)$ by (10.16)). Thus (10.27) holds for $n = 1$.

Now we use induction. Suppose $N \in \mathbf{N}$, and suppose (for the given $\omega \in \Omega'$) that (10.26) and (10.27) hold for $n = N$. For the induction step, we shall show (for the given $\omega \in \Omega'$) that (10.28) and (10.29) hold for $n = N$, and then show that (10.26) and (10.27) hold for $n = N + 1$. This actually suffices to prove Lemma 10.6 by induction.

Verification of (10.28)–(10.29) *for $n = N$.* Suppose $k \geq H(N,1)(\omega)$. Referring to (10.8) and (10.9), let m denote the positive integer such that $H(N,m)(\omega) \leq k < H(N,m+1)(\omega)$. By the inductive assumption of (10.26) for $n = N$, equation (10.9) holds for $n = N$ with $W_k^{(N)}(\omega)$ replaced by $W_k^{*(N)}(\omega)$. Hence by (6.10) and its "asterisk" analog (and Definition 2.2(A)), together with (10.8) and (10.9),

$$\begin{aligned}\Psi^*(N,k,0)(\omega) &= \psi_0\left(W_k^{*(N)}(\omega), W_{k-1}^{*(N)}(\omega), W_{k-2}^{*(N)}(\omega), \dots\right)\\ &= k - H(N,m)(\omega)\\ &= \psi_0\left(W_k^{(N)}(\omega), W_{k-1}^{(N)}(\omega), W_{k-2}^{(N)}(\omega), \dots\right)\\ &= \Psi(N,k,0)(\omega),\end{aligned}$$

and hence also (see (10.8) again)

$$k - \Psi^*(N,k,0)(\omega) = k - \Psi(N,k,0)(\omega) = H(N,m)(\omega) \ge H(N,1)(\omega).$$

Thus (10.28) and (10.29) hold for $n = N$.

Verification of (10.26) *for* $n = N+1$. By (10.15) (see (10.14)) $L_{N+1}(\omega) = H(N, \Phi_{N+1}(\omega))(\omega)$. If $k \ge L_{N+1}(\omega)+1$ but k is not one of the integers $H(N,m)(\omega)$, $m \ge \Phi_{N+1}(\omega)+1$, then $W_k^{(N)}(\omega) \ne 1$ by ((10.8) and) (10.9) for $n = N$, hence $W_k^{*(N)} = W_k^{(N)} \ne 1$ by ((10.16) and) the inductive assumption of (10.26) for $n = N$, and hence $W_k^{*(N+1)}(\omega) = W_k^{(N+1)}(\omega) = 0$ by both Remark 6.3(B) and its " asterisk" counterpart. Hence (see (10.8) again) to complete the proof of (10.26) for $n = N+1$, what remains is to show that for each integer $m \ge \Phi_{N+1}(\omega)+1$, one has that $W_{H(N,m)(\omega)}^{*(N+1)}(\omega) = W_{H(N,m)(\omega)}^{(N+1)}(\omega)$.

Suppose $m \ge \Phi_{N+1}(\omega)+1$. To obtain the desired equality, using the function φ_u in Lemma 4.7 for the positive integer $u := m - \Phi_{N+1}(\omega)$, using the convention in Notation 2.1(G), and writing the positive integer $\Phi_{N+1}(\omega)$ as $\Phi(N+1)(\omega)$, we shall show that

$$\begin{aligned}&W_{H(N,m)(\omega)}^{*(N+1)}(\omega)\\ &= g_{\text{spaced}}\left(\xi_j^{*(N+1)},\ j \in \{i \le H(N,m)(\omega) : W_i^{*(N)}(\omega) = 1\}\right)\\ &= \varphi_u\left(\xi_{H(N,\Phi(N+1)(\omega)+1)(\omega)}^{*(N+1)}(\omega), \xi_{H(N,\Phi(N+1)(\omega)+2)(\omega)}^{*(N+1)}(\omega), \dots, \xi_{H(N,m)(\omega)}^{*(N+1)}(\omega)\right)\\ &= \varphi_u\left(\xi_{H(N,\Phi(N+1)(\omega)+1)(\omega)}^{(N+1)}(\omega), \xi_{H(N,\Phi(N+1)(\omega)+2)(\omega)}^{(N+1)}(\omega), \dots, \xi_{H(N,m)(\omega)}^{(N+1)}(\omega)\right)\\ &= g_{\text{spaced}}\left(\xi_j^{(N+1)},\ j \in \{i \le H(N,m)(\omega) : W_i^{(N)}(\omega) = 1\}\right)\\ &= W_{H(N,m)(\omega)}^{(N+1)}(\omega).\end{aligned} \tag{10.30}$$

To verify this, first note that by (10.16) and Step 10.4(C), the equality $\xi_k^{*(N+1)}(\omega) = \xi_k^{(N+1)}(\omega)$ holds for each $k \ge H(N,1)(\omega)$, *and in particular it holds for each k of the form $k = H(N,\ell)(\omega)$ for $\ell \ge \Phi_{N+1}(\omega) - 5$* by (10.8) (with $n = N$) and (10.14) (with $n = N+1$). That trivially yields the third equality in (10.30). Next, by (10.8)–(10.9) and the inductive assumption of (10.26) for $n = N$, $W_k^{*(N)}(\omega) = W_k^{(N)}(\omega) = 1$ for $k \in \{H(N,1)(\omega), H(N,2)(\omega),\ H(N,3)(\omega),\ \dots\}$, and $W_k^{*(N)}(\omega) = W_k^{(N)}(\omega) \ne 1$ for all other $k \ge H(N,1)(\omega)$. This fact has a couple of consequences: First (just consider that fact for $k = H(N,m)(\omega)$), by (6.18) and (6.13)–(6.14) and their "asterisk" counterparts (for $n = N$), the first and fifth (i.e. last) equalities in (10.30) hold. Then, by (10.14) with $n = N+1$ (combined with the observation in italics in the first sentence after (10.30)), one has from Lemma 4.7 that the second and fourth equalities in (10.30) also hold. Thus (10.30) holds. That completes the proof of (10.26) for $n = N + 1$.

Verification of (10.27) *for* $n = N + 1$. Now suppose $k \geq H(N+1,1)(\omega)$. Referring to (10.8) for $n = N + 1$, let $m \in \mathbf{N}$ be such that $H(N+1,m)(\omega) \leq k < H(N+1,m+1)(\omega)$. Now $W^{*(N+1)}_{H(N+1,m)(\omega)}(\omega) = W^{(N+1)}_{H(N+1,m)(\omega)}(\omega) = 1$ by (10.9) (for $n = N + 1$) and (10.26) for $n = N + 1$ (just verified above), and hence $W^{*(N)}_{H(N+1,m)(\omega)}(\omega) = 1$ and $W^{(N)}_{H(N+1,m)(\omega)}(\omega) = 1$ by Remark 6.3(B) (and its "asterisk" counterpart). Hence $\Psi(N,k,0)(\omega) \leq k - H(N+1,m)(\omega)$ by (6.10) (and Definition 2.2(A)), hence $k - \Psi(N,k,0)(\omega) \geq H(N+1,m)(\omega) \geq L_{N+1}(\omega) + 1$ (see (10.9) with $n = N + 1$), hence

$$W^{*(N+1)}_{k-\Psi(N,k,0)(\omega)}(\omega) = W^{(N+1)}_{k-\Psi(N,k,0)(\omega)}(\omega) \tag{10.31}$$

by (10.26) for $n = N+1$ (verified above). Also (since $k \geq H(N+1,m)(\omega) \geq H(N+1,1)(\omega) > H(N,1)(\omega)$ by (10.16)) $\Psi^*(N,k,0)(\omega) = \Psi(N,k,0)(\omega)$ by (10.28) for $n = N$ (verified above). Substituting that into the left hand side of (10.31) and then applying (7.2) and its "asterisk" counterpart, one obtains $\delta^{*(N+1)}_k(\omega) = \delta^{(N+1)}_k(\omega)$. Thus (10.27) has been verified for $n = N + 1$. That completes the induction step and the proof of Lemma 10.6.

Step 10.7. For each $n \in \mathbf{N}$, each $\omega \in \Omega'$ (see Step 10.4(G)), and each set $S \subset \mathbf{R}$, define the sets

$$G^{(n)}(S)(\omega) := \left\{k \in \mathbf{Z} : k \in S \ \text{ and } \ W^{(n)}_k(\omega) = 1\right\} \tag{10.32}$$

and

$$G^{*(n)}(S)(\omega) := \left\{k \in \mathbf{Z} : k \in S \ \text{ and } \ W^{*(n)}_k(\omega) = 1\right\}. \tag{10.33}$$

For each $n \in \mathbf{N}$, define the random sets Γ_n and Γ^*_n as follows: For each $\omega \in \Omega'$ (see Step 10.4(G) again),

$$\begin{aligned} \Gamma_n(\omega) :=& G^{(n)}([-N', L_n(\omega)])(\omega) \\ =& \left\{k \in \mathbf{Z} : -N' \leq k \leq L_n(\omega) \text{ and } W^{(n)}_k(\omega) = 1\right\} \end{aligned} \tag{10.34}$$

and

$$\begin{aligned} \Gamma^*_n(\omega) :=& G^{*(n)}([-N', L_n(\omega)])(\omega) \\ =& \left\{k \in \mathbf{Z} : -N' \leq k \leq L_n(\omega) \text{ and } W^{*(n)}_k(\omega) = 1\right\}. \end{aligned} \tag{10.35}$$

For each $n \in \mathbf{N}$, by (10.34), (10.35), (10.10), and (6.5) and its "asterisk" counterpart, one has that

$$\sigma\left(\Gamma_n, \Gamma^*_n\right) \ \dot{\subset}\ \sigma\left(\overline{\xi}^{(n)}, \overline{\xi}^{*(n)}\right). \tag{10.36}$$

Also, for $n \in \mathbf{N}$ and $\omega \in \Omega'$, by (6.13) and (6.14), one has (see (10.32)) the following reformulation of (6.18) in the convention of Notation 2.1(G):

$$\begin{aligned} &\text{If } k \in \mathbf{Z} \text{ is such that } W^{(n)}_k(\omega) = 1, \\ &\text{then } W^{(n+1)}_k(\omega) = g_{\text{spaced}}\left(\xi^{(n+1)}_j(\omega),\ j \in G^{(n)}((-\infty,k])(\omega)\right). \end{aligned} \tag{10.37}$$

Analogously (see (10.33)), for $n \in \mathbf{N}$ and $\omega \in \Omega'$,

$$\begin{gathered}\text{If } k \in \mathbf{Z} \text{ is such that } W_k^{*(n)}(\omega) = 1, \\ \text{then } W_k^{*(n+1)}(\omega) = g_{\text{spaced}}\left(\xi_j^{*(n+1)}(\omega),\, j \in G^{*(n)}((-\infty, k])(\omega)\right).\end{gathered} \tag{10.38}$$

Lemma 10.8. *For each* $n \in \mathbf{N}$*,*

$$E(\operatorname{card}\Gamma_n) \le 3N' \quad \text{and} \tag{10.39}$$

$$E(\operatorname{card}\Gamma_n^*) \le 3N'. \tag{10.40}$$

Proof. It suffices to give the argument for (10.39). The argument for (10.40) is exactly analogous (using also Lemma 10.6).

First, for each $\omega \in \Omega'$, by (10.34), $\Gamma_1(\omega) \subset \{-N', -N'+1, \dots, L_1(\omega)\}$ and hence $\operatorname{card}\Gamma_1(\omega) \le 1 + N' + L_1(\omega)$. Hence by (10.11) and (10.7),

$$\begin{aligned} E(\operatorname{card}\Gamma_1) &\le 1 + N' + EL_1 \\ &\le 1 + N' + N' + 6 \cdot (8/5)^{30} \cdot (8/3)^6 \\ &\le 3N'. \end{aligned} \tag{10.41}$$

Next, for each $n \ge 2$ and each $\omega \in \Omega'$, by (10.34), (10.32), Remark 6.3(C), (10.16), (10.8), (10.9), and (10.15),

$$\begin{aligned} \operatorname{card}\Gamma_n(\omega) &\le 1 + (1/6) \cdot \operatorname{card} G^{(n-1)}([-N', L_n(\omega)])(\omega) \\ &= 1 + (1/6) \cdot \operatorname{card} G^{(n-1)}([-N', L_{n-1}(\omega)])(\omega) \\ &\qquad + (1/6) \cdot \operatorname{card} G^{(n-1)}((L_{n-1}(\omega), L_n(\omega)])(\omega) \\ &= 1 + (1/6) \cdot \operatorname{card}\Gamma_{n-1}(\omega) \\ &\qquad + (1/6) \cdot \operatorname{card}\big\{H(n-1,1)(\omega), H(n-1,2)(\omega), \\ &\qquad\qquad\qquad \dots, H(n-1, \Phi_n(\omega))(\omega)\big\} \\ &= 1 + (1/6) \cdot \operatorname{card}\Gamma_{n-1}(\omega) + (1/6) \cdot \Phi_n(\omega). \end{aligned}$$

Hence for each $n \ge 2$, by (10.12) and (10.7),

$$\begin{aligned} E(\operatorname{card}\Gamma_n) &\le 1 + (1/6) \cdot E(\operatorname{card}\Gamma_{n-1}) + (1/6) \cdot E\Phi_n \\ &\le 1 + (1/6) \cdot E(\operatorname{card}\Gamma_{n-1}) + N'. \end{aligned}$$

Hence for any $n \in \mathbf{N}$ such that $E(\operatorname{card}\Gamma_{n-1}) \le 3N'$, one trivially has that $E(\operatorname{card}\Gamma_n) \le 3N'$. Hence by (10.41) and induction, (10.39) holds for all $n \in \mathbf{N}$. That completes the proof of Lemma 10.8.

Step 10.9. Recall the number $\varepsilon \in (0,1)$ fixed in the sentence containing (10.3). Let M be a positive integer sufficiently large that

$$6N'/M \le \varepsilon. \tag{10.42}$$

By Lemma 10.8, $E(\operatorname{card}(\Gamma_n \cup \Gamma_n^*)) \le 6N'$ for each positive integer n. Hence by Fatou's Lemma,

$$E\left(\liminf_{n\to\infty} \operatorname{card}(\Gamma_n \cup \Gamma_n^*)\right) \le 6N'.$$

Hence by Markov's inequality and (10.42),

$$P\left(\liminf_{n\to\infty} \operatorname{card}(\Gamma_n \cup \Gamma_n^*) \ge M\right) \le 6N'/M \le \varepsilon. \tag{10.43}$$

Next, define the number $\theta \in (0,1)$ by

$$\theta := (5/8)^{6M}. \tag{10.44}$$

Also, for each $n \geq N'$, define the event

$$B_n := \{\text{card}(\Gamma_n \cup \Gamma_n^*) \leq M\}; \tag{10.45}$$

and for each $n \geq N'+1$, define the event

$$D_n := \left\{\omega \in \Omega' : \xi_k^{(n)}(\omega) = (0,0,0,0,0,0) \quad \text{for all } k \in \Gamma_{n-1}(\omega) \cup \Gamma_{n-1}^*(\omega)\right\}. \tag{10.46}$$

(That event includes "by default" such sample points for which the specified set of k's is empty.) By (10.36), (10.45), and (10.46),

$$\forall\, n \geq N', \quad B_n \,\dot{\in}\, \sigma\left(\overline{\xi}^{(n)}, \overline{\xi}^{*(n)}\right); \quad \text{and} \tag{10.47}$$

$$\forall\, n \geq N'+1, \quad D_n \,\dot{\in}\, \sigma\left(\overline{\xi}^{(n)}, \overline{\xi}^{*(n)}\right). \tag{10.48}$$

Lemma 10.10. *Refer to (10.44), (10.45), and (10.46). Suppose $n \geq N'$, $A \in \sigma(\overline{\xi}^{(n)}, \overline{\xi}^{*(n)})$, and $P(A \cap B_n) > 0$. Then $P(D_{n+1} \mid A \cap B_n) \geq \theta$.*

Proof. By (10.45), the event B_n can be partitioned into an at most countable collection of events of the form $\{\Gamma_n \cup \Gamma_n^* = S\}$ for sets $S \subset \mathbf{Z}$ such that $\text{card}\, S \leq M$ (including the empty set). Accordingly, the event $A \cap B_n$ is partitioned into events $A \cap \{\Gamma_n \cup \Gamma_n^* = S\}$ for such S.

For any such S such that $P(A \cap \{\Gamma_n \cup \Gamma_n^* = S\}) > 0$, one has by (10.36) (and the hypothesis of Lemma 10.10 here) that $A \cap \{\Gamma_n \cup \Gamma_n^* = S\} \dot{\in} \sigma(\overline{\xi}^{n}, \overline{\xi}^{*(n)})$ and is therefore independent of the sequence $\xi^{(n+1)}$ (recall again Construction 6.1(A)(C)), and hence by (10.46), Construction 6.1(A) (including (6.1) there), and (10.44),

$$\begin{aligned}
&P(D_{n+1} \mid A \cap \{\Gamma_n \cup \Gamma_n^* = S\}) \\
&= P\left(\xi_k^{(n+1)} = (0,0,0,0,0,0)\ \forall\, k \in S \mid A \cap \{\Gamma_n \cup \Gamma_n^* = S\}\right) \\
&= P\left(\xi_k^{(n+1)} = (0,0,0,0,0,0)\ \forall\, k \in S\right) \\
&= ((5/8)^6)^{\text{card}\, S} \\
&\geq (5/8)^{6M} = \theta.
\end{aligned}$$

Lemma 10.10 now follows from Remark 2.8.

Lemma 10.11. *$P\left(\bigcup_{n=N'+1}^{\infty} D_n\right) \geq 1 - 2\varepsilon$.*

Here again of course the number $\varepsilon \in (0,1)$ is as fixed in the sentence containing (10.3). One can do better than Lemma 10.11, but that will not be necessary.

Proof. Suppose Lemma 10.11 is false. That is, suppose instead that

$$s := P\Big(\bigcap_{n=N'+1}^{\infty} D_n^c\Big) > 2\varepsilon. \tag{10.49}$$

We shall aim for a contradiction.

Referring to (10.44) and (10.49), let Θ be an integer such that

$$\Theta \geq N'+1 \quad \text{and} \quad P\Big(\bigcap_{n=N'+1}^{\Theta} D_n^c\Big) \leq s + \varepsilon\theta/2. \tag{10.50}$$

In the arguments below, we shall use more compact notations for the events in (10.49) and (10.50):

$$E := \bigcap_{n=N'+1}^{\infty} D_n^c \quad \text{and} \quad E' := \bigcap_{n=N'+1}^{\Theta} D_n^c. \tag{10.51}$$

Referring to (10.50), (10.45) and (10.46), define the events $A_j,\ j \in \{\Theta+1, \Theta+2, \Theta+3, \dots\} \cup \{\infty\}$ as follows:

$$\begin{aligned} &A_{\Theta+1} := B_{\Theta+1}; \\ &\forall j \in \{\Theta+2, \Theta+3, \Theta+4, \dots\}, \quad A_j := B_j \bigcap \Big(\bigcap_{i=\Theta+1}^{j-1} B_i^c \Big); \quad \text{and} \\ &A_\infty := \Big(\bigcap_{i=\Theta+1}^{\infty} B_i^c \Big). \end{aligned} \tag{10.52}$$

These events $A_{\Theta+1}, A_{\Theta+2}, A_{\Theta+3}, \dots$ and A_∞ form a partition of Ω. Also, by (10.52) and (10.45), $A_\infty \,\dot\subset\, \{\liminf_{n\to\infty} \text{card}(\Gamma_n \cup \Gamma_n^*) \geq M+1\}$, and hence by (10.43), $P(A_\infty) \leq \varepsilon$. Hence $P(E' \cap A_\infty) \leq \varepsilon$. Now $P(E') \geq P(E) > 2\varepsilon$ by (10.51) and (10.49). Hence

$$P(E' \cap A_\infty^c) > \varepsilon. \tag{10.53}$$

Now suppose $j \in \{\Theta+1, \Theta+2, \Theta+3, \dots\}$ is such that $P(E' \cap A_j) > 0$. (Then $j \geq N'+2$ by (10.50).) By (10.47), (10.48), (10.51), and (10.52) (and Constructions 6.1(A)(C)), $E' \cap A_j \cap B_j = E' \cap A_j \,\dot\in\, \sigma(\bar\xi^{(j)}, \bar\xi^{*(j)})$. Hence by Lemma 10.10, $P(D_{j+1} | E' \cap A_j) \geq \theta$. Of course by (10.51) the events D_{j+1} and E are disjoint. It follows that $P(E | E' \cap A_j) \leq 1 - \theta$.

Now by the sentence right after (10.52), the events $E' \cap A_j,\ j \in \{\Theta+1, \Theta+2, \Theta+3, \dots\}$ form a partition of the event $E' \cap A_\infty^c$. Hence by the calculations in the preceding paragraph, together with Remark 2.8, one has that $P(E | E' \cap A_\infty^c) \leq 1-\theta$. Since $E \subset E'$ (recall (10.51)), this can be rewritten as $P(E \cap A_\infty^c) \leq (1-\theta) P(E' \cap A_\infty^c)$. Combining that with (10.53), one now has

$$P(E' \cap A_\infty^c) - P(E \cap A_\infty^c) \geq \theta \cdot P(E' \cap A_\infty^c) \geq \theta\varepsilon. \tag{10.54}$$

Since (again) $E \subset E'$, one also has that $P(E' \cap A_\infty) - P(E \cap A_\infty) \geq 0$. Combining that with (10.54), one obtains

$$P(E') - P(E) \geq \theta\varepsilon. \tag{10.55}$$

However, by (10.49). (10.50). and (10.51), $P(E') - P(E) \leq s + \varepsilon\theta/2 - s = \varepsilon\theta/2$, which contradicts (10.55). Hence Lemma 10.11 must hold after all.

Lemma 10.12. *For every integer $N \geq N'$ and every $\omega \in D_{N+1}$, there exists a positive integer $K = K(\omega)$ such that $X_k^*(\omega) = X_k(\omega)$ for all $k \geq K$.*

Proof. Throughout this proof, let N and ω be fixed such that

$$N \geq N' \quad \text{and} \quad \omega \in D_{N+1}. \tag{10.56}$$

Of course $\omega \in \Omega'$ by (10.46); and all properties of ω mentioned in the last sentence of Step 10.4(G) will be tacitly taken for granted here.

Suppose $k \in \{-N', -N'+1, \dots, L_N(\omega)\}$ is such that $W_k^{*(N)}(\omega) = 1$. Then $k \in \Gamma_N^*(\omega)$ by (10.35), hence $\xi_k^{(N+1)}(\omega) = (0,0,0,0,0,0)$ by (10.56) and (10.46), and hence also $\xi_k^{*(N+1)}(\omega) = (0,0,0,0,0,0)$ by (10.56) and Step 10.4(C). Also (trivially) by (10.33), $k = \max G^{*(N)}((-\infty, k])(\omega)$, and hence (in the convention of Notation 2.1(G))

$$W_k^{*(N+1)}(\omega) = g_{\text{spaced}}\left(\xi_j^{*(N+1)}(\omega),\, j \in G^{*(N)}((-\infty, k])(\omega)\right) = 0$$

by (10.38) and Lemma 4.8 (equation (4.26)).

We have shown that (under the hypothesis of Lemma 10.12) $W_k^{*(N+1)}(\omega) = 0$ for every $k \in \{-N', -N'+1, \ldots, L_N(\omega)\}$ such that $W_k^{*(N)}(\omega) = 1$. For all other values of $k \in \{-N', -N'+1, \ldots, L_N(\omega)\}$, $W_k^{*(N+1)}(\omega) = 0$ by (the "asterisk" analog of) Remark 6.3(B). Hence $W_k^{*(N+1)}(\omega) = 0$ for all $k \in \mathbf{Z}$ such that $-N' \le k \le L_N(\omega)$. By an exactly analogous argument (using (10.34) and (10.32) instead of (10.35) and (10.33)), $W_k^{(N+1)}(\omega) = 0$ for all such k. That is,

$$\forall\, k \in \{-N', -N'+1, \ldots, L_N(\omega)\}, \quad W_k^{*(N+1)}(\omega) = W_k^{(N+1)}(\omega) = 0. \tag{10.57}$$

Next, suppose $u \in \mathbf{N}$. Then $W^{*(N)}_{H(N,u)(\omega)}(\omega) = W^{(N)}_{H(N,u)(\omega)}(\omega) = 1$ by Lemma 10.6 (equation (10.26)), (10.8), and (10.9); and hence by (10.38),

$$W^{*(N+1)}_{H(N,u)(\omega)}(\omega) = g_{\text{spaced}}\left(\xi_j^{*(N+1)}(\omega),\, j \in G^{*(N)}((-\infty, H(N,u)(\omega)])(\omega)\right). \tag{10.58}$$

Again, for every $k \in G^{*(N)}((-\infty, H(N,u)(\omega)])(\omega)$ such that $-N' \le k \le L_N(\omega)$, one has that $\xi_k^{*(N+1)}(\omega) = (0,0,0,0,0,0)$, by (10.33) and the arguments in the paragraph after that of (10.56). Referring to (10.8)–(10.9) and applying Lemma 4.8 (equation (4.27)), one can delete those particular elements k from the set $G^{*(N)}((-\infty, H(N,u)(\omega)])(\omega)$ without changing the right hand side of (10.58). That is, one obtains (see (10.33) and (10.8)–(10.9) again)

$$\begin{aligned} W^{*(N+1)}_{H(N,u)}(\omega) &= g_{\text{spaced}}\Big(\xi_j^{*(N+1)}(\omega),\, j \in G^{*(N)}((-\infty, -N'-1])(\omega) \\ &\qquad\qquad \cup\, G^{*(N)}([L_N(\omega)+1, H(N,u)(\omega)])(\omega)\Big) \\ &= g_{\text{spaced}}\Big(\xi_j^{*(N+1)}(\omega),\, j \in G^{*(N)}((-\infty, -N'-1])(\omega) \\ &\qquad\qquad \cup\, \{H(N,1)(\omega), H(N,2)(\omega), \ldots, H(N,u)(\omega)\}\Big) \end{aligned} \tag{10.59}$$

Now by (10.56) and Step 10.4(C), $\xi_j^{*(N+1)}(\omega) = \xi_j^{(N+1)}(\omega)$ for every $j \in \mathbf{Z}$. Also,

$$G^{*(N)}((-\infty, -N'-1])(\omega) = G^{(N)}((-\infty, -N'-1])(\omega)$$

by (10.32), (10.33), and (10.24). Hence by (10.59),

$$\begin{aligned} W^{*(N+1)}_{H(N,u)(\omega)}(\omega) = g_{\text{spaced}}\Big(&\xi_j^{(N+1)}(\omega),\, j \in G^{(N)}((-\infty, -N'-1])(\omega) \\ &\cup \{H(N,1)(\omega), H(N,2)(\omega), \ldots, H(N,u)(\omega)\}\Big). \end{aligned} \tag{10.60}$$

By an exactly analogous argument, one obtains

$$W^{(N+1)}_{H(N,u)(\omega)}(\omega) = [\text{RHS of } (10.60)],$$

and hence by (10.60) itself,

$$W^{*(N+1)}_{H(N,u)(\omega)}(\omega) = W^{(N+1)}_{H(N,u)(\omega)}(\omega). \tag{10.61}$$

Equation (10.61) was obtained for arbitrary $u \in \mathbf{N}$. For all other $k \ge L_N(\omega)+1$ besides $\{H(N,1)(\omega), H(N,2)(\omega), \dots, \}$, one has that $W_k^{*(N)}(\omega) = W_k^{(N)}(\omega) \ne 1$ by Lemma 10.6 (equation (10.26)) and (10.9), and hence $W_k^{*(N+1)}(\omega) = W_k^{(N+1)}(\omega) = 0$ by Remark 6.3(B) and its "asterisk" counterpart. Combining that with (10.61), one now has that

$$\forall\, k \ge L_N(\omega)+1, \quad W_k^{*(N+1)}(\omega) = W_k^{(N+1)}(\omega). \tag{10.62}$$

Now by (10.57), (10.62), and (10.24), one now has that

$$\forall\, k \in \mathbf{Z}, \quad W_k^{*(N+1)}(\omega) = W_k^{(N+1)}(\omega). \tag{10.63}$$

Now recall from Step 10.4(C) that $\xi_k^{*(n)}(\omega) = \xi_k^{(n)}(\omega)$ for all $n \ge N'+1$ and all $k \in \mathbf{Z}$. Recall (10.56) again. Starting with (10.63) and using (6.10) and (6.11) (and their "asterisk" counterparts) and induction on n, one has that

$$\forall\, n \ge N+1,\ \forall\, k \in \mathbf{Z}, \quad W_k^{*(n)}(\omega) = W_k^{(n)}(\omega);\ \text{and} \tag{10.64}$$

$$\forall\, n \ge N+1, \forall\, k \in \mathbf{Z}, \forall\, j \ge 0,\ \Psi^*(n,k,j)(\omega) = \Psi(n,k,j)(\omega). \tag{10.65}$$

Now recall (10.16). Our candidate for the integer $K = K(\omega)$ in Lemma 10.12 will be $H(N,1)(\omega)$.

By (10.16) and Step 10.4(B), one has that

$$\forall\, k \ge H(N,1)(\omega), \quad X_k^{*(0)}(\omega) = X_k^{(0)}(\omega). \tag{10.66}$$

Now keep in mind that $H(N,1)(\omega) \ge H(n,1)(\omega)$ for all $n \in \{1,\dots,N\}$ by (10.16). By (6.24) and Lemma 10.6 (equation (10.29), applied twice), one has that

$$\begin{aligned} &\forall\, n \in \{1,\dots,N\}, \qquad \forall\, k \ge H(N,1)(\omega), \\ &k - \Psi^*(n,k,0)(\omega) = k - \Psi(n,k,0)(\omega) \ge k - \Psi(N,k,0)(\omega) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ge H(N,1)(\omega). \end{aligned} \tag{10.67}$$

By Lemma 10.6 (equation (10.27)),

$$\forall\, n \in \{1,\dots,N\},\ \forall\, k \ge H(N,1)(\omega), \quad \delta_k^{*(n)}(\omega) = \delta_k^{(n)}(\omega). \tag{10.68}$$

Also, by (7.2) and its "asterisk" counterpart, equation (10.67) for $n = N$, and equation (10.63),

$$\forall\, k \ge H(N,1)(\omega), \quad \delta_k^{*(N+1)}(\omega) = \delta_k^{(N+1)}(\omega). \tag{10.69}$$

Next, recall from Step 10.4(C) (and (7.9) and its "asterisk" counterpart) that $\zeta_k^{*(n)}(\omega) = \zeta_k^{(n)}(\omega)$ for all $n \in \mathbf{N}$ and all $k \ge N'+1$. Since $H(N,1)(\omega) \ge N'+1$ by (10.16), one has by (10.67) that

$$\begin{aligned} &\forall\, n \in \{1,\dots,N\}, \quad \forall\, k \ge H(N,1)(\omega), \\ &\zeta_{k-\Psi^*(n,k,0)(\omega)}^{*(n)}(\omega) = \zeta_{k-\Psi(n,k,0)(\omega)}^{*(n)}(\omega) = \zeta_{k-\Psi(n,k,0)(\omega)}^{(n)}(\omega). \end{aligned} \tag{10.70}$$

Next by Step 10.4(C) and (10.56), one has that for every $n \ge N+1$ (in fact every $n \ge N'+1$) and every $k \in \mathbf{Z}$, $\zeta_k^{*(n)}(\omega) = \zeta_k^{(n)}(\omega)$. Hence by (10.65), for every $n \ge N+1$ and every $k \in \mathbf{Z}$, $\zeta_{k-\Psi^*(n,k,0)(\omega)}^{*(n)}(\omega) = \zeta_{k-\Psi(n,k,0)(\omega)}^{(n)}(\omega)$. Also, by (10.64), (10.65), and (7.2) and its "asterisk" counterpart, for every $n \ge N+2$

and every $k \in \mathbf{Z}$, $\delta_k^{*(n)}(\omega) = \delta_k^{(n)}(\omega)$. Combining these facts with (10.68), (10.69), and (10.70), one now has that

$$\forall\, n \in \mathbf{N},\ \forall\, k \geq H(N,1)(\omega), \quad \delta_k^{*(n)}(\omega) = \delta_k^{(n)}(\omega); \quad \text{and} \tag{10.71}$$

$$\forall\, n \in \mathbf{N}, \forall\, k \geq H(N,1)(\omega), \quad \zeta^{*(n)}_{k-\Psi^*(n,k,0)(\omega)}(\omega) = \zeta^{(n)}_{k-\Psi(n,k,0)(\omega)}(\omega). \tag{10.72}$$

By (7.7) and (7.4) and their "asterisk" counterparts, together with (10.71), one obtains

$$\forall\, n \in \mathbf{N}, \forall\, k \geq H(N,1)(\omega), \quad J^*(n,k)(\omega) = J(n,k)(\omega); \quad \text{and} \tag{10.73}$$

$$\forall\, k \geq H(N,1)(\omega), \quad N_k^*(\omega) = N_k(\omega). \tag{10.74}$$

Now by (7.14) (and its "asterisk" counterpart), (10.66), (10.72) (recall (7.9) and its "asterisk" counterpart), (10.73), and (10.74), one has that $X_k^*(\omega) = X_k(\omega)$ for all $k \geq H(N,1)(\omega)$. Thus Lemma 10.12 holds with $K = K(\omega) = H(N,1)(\omega)$.

Step 10.13. By Lemma 10.12 and Lemma 10.11,

$$P\left(\bigcup_{J \in \mathbf{N}} \{X_k^* = X_k\ \forall\, k \geq J\}\right) \geq P\left(\bigcup_{n=N'+1}^{\infty} D_n\right) \geq 1 - 2\varepsilon. \tag{10.75}$$

Also (trivially) for each $J \in \mathbf{N}$,

$$\big\{X_k^* = X_k\ \forall\, k \geq J\big\} \subset \big\{X_k^* = X_k\ \forall\, k \geq J+1\big\}.$$

Hence by (10.75), $\lim_{J\to\infty} P(X_k^* = X_k\ \forall\, k \geq J) \geq 1 - 2\varepsilon$. Accordingly, let J' be a positive integer such that

$$J' \geq N' + 1 \quad \text{and} \tag{10.76}$$

$$P\left(X_k^* = X_k\ \forall\, k \geq J'\right) \geq 1 - 3\varepsilon. \tag{10.77}$$

Now recall (10.1). One has that $F' \in \sigma(X_k, |k| \geq J')$. Thus (see e.g. Billingsley, 1995, Theorem 20.1(i), trivially extended, and recall property (A) in Theorem 1.1) there exists a Borel set $B' \subset \{-1,1\}^{\mathbf{N}}$ such that

$$F' = \{(X_{J'}, X_{-J'}, X_{J'+1}, X_{-J'-1}, X_{J'+2}, X_{-J'-2}, \dots) \in B'\} \tag{10.78}$$

Define the ("asterisk counterpart") event

$$F''' := \big\{(X^*_{J'}, X^*_{-J'}, X^*_{J'+1}, X^*_{-J'-1}, X^*_{J'+2}, X^*_{-J'-2}, \dots) \in B'\big\} \tag{10.79}$$

By (10.76), (10.77), and (10.25),

$$P\left(X_k^* = X_k \text{ for all } k \in \mathbf{Z} \text{ such that } |k| \geq J'\right) \geq 1 - 3\varepsilon.$$

Hence by (10.78), (10.79), and a simple argument,

$$P(F' \triangle F''') \leq 3\varepsilon \tag{10.80}$$

(where $\triangle$ denotes symmetric difference). Hence by a simple standard argument,

$$P(F''') \leq P(F') + 3\varepsilon. \tag{10.81}$$

Refer to Construction 7.4(B). By Lemma 7.6 (and Remark 2.6(B)), one has that $\sigma(X)\dot{\subset}\sigma(\eta)$, the σ-field generated by the array (10.17). The "asterisk" counterpart is $\sigma(X^*)\dot{\subset}\sigma(\eta^*)$, the σ-field generated by the array (10.18). Hence by (10.20) and (10.79),

$$\text{the events } F'' \text{ and } F''' \text{ are independent.} \tag{10.82}$$

Now by (10.5), (10.80), and a standard elementary argument (see e.g.Bradley, 2007a, Vol. 1, Appendix, Section A053(V)),

$$\begin{aligned} P((F'' \cap F''') \triangle F') &= P((F'' \cap F''') \triangle (F' \cap F')) \\ &\le P(F'' \triangle F') + P(F''' \triangle F') \le 4\varepsilon. \end{aligned}$$

Hence by a standard simple argument, $P(F'' \cap F''') \ge P(F') - 4\varepsilon$. Hence by (10.82), (10.6), and (10.81),

$$\begin{aligned} P(F') - 4\varepsilon \le P(F'' \cap F''') &= P(F'') \cdot P(F''') \\ &\le [P(F') + \varepsilon] \cdot [P(F') + 3\varepsilon]. \end{aligned}$$

However, that contradicts (10.3). Hence (10.2) must be false, and $\mathcal{T}_{\mathrm{double}}(X)$ is trivial after all. That completes the proof of Property (D) in Theorem 1.1, and of Theorem 1.1 itself.

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