



Cut-off and Escape Behaviors for Birth and Death Chains on Trees

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Abstract. We consider families of discrete time birth and death chains on trees, and show that in presence of a drift toward the root of the tree, the chains exhibit cut-off behavior along the drift and escape behavior in the opposite direction.

1. Introduction

Although they originally come from different research fields and seem apparently to be very different phenomena, cut-off and escape behaviors have been related at the level of hitting times for birth and death chains on the line (see [Barrera et al., 2009](#); [Bertoncini, 2007](#); [Bertoncini et al., 2008](#)). Cut-off behavior refers to the famous cut-off phenomenon first discovered and studied by Aldous and Diaconis in the 80's (see [Aldous, 1983](#); [Aldous and Diaconis, 1986, 1987](#)), which is characterized by an almost deterministic asymptotical abrupt convergence to equilibrium for families of Markov processes. In opposition, escape behavior is usually associated to the exit from metastability for a system trapped in local minimum of the energy profile. In that case, the transition to equilibrium occurs at “unpredictable” exponential times (see [Cassandro et al., 1984](#)).

We refer to the introduction in [Barrera et al. \(2009\)](#) for a comparative historical discussion of the phenomena. In this reference —dealing with birth-and-death chains on the line with drift toward the origin— cut-off and escape phenomena are characterized by their distinct hitting-time behavior. It is shown that, under suitable drift hypotheses, the chains exhibit both cut-off behavior toward zero and escape behavior for excursions in the opposite direction. Furthermore, as the evolutions are reversible, the law of the final escape trajectory coincides with the time reverse of the law of cut-off paths.

In the present work, we extend this study to birth-and-death chains on trees with drift toward the root of the tree. Our chains are the discrete time counterpart of the birth-and-death processes studied by Martínez and Ycart in [Martínez and Ycart \(2001\)](#). Under a “uniform” drift condition, they prove that hitting times of zero starting from any state a (denoted by $T_{a \rightarrow 0}$) exhibit cut-off at mean times when a goes to infinity (Theorem 5.1 in [Martínez and Ycart, 2001](#)). Their drift condition is expressed as an exponential decaying tail of the invariant probability measure π , in the sense that, for all a , the quantity $\pi(B_a)/\pi(a)$ is uniformly bounded by some K . In comparison, our drift condition concerns only the branch which contains the state a (see formulas (2.13) and (2.15)), and we allow the upper bound K_a to go to infinity, as long as (2.14) remains valid (see

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Section 2.4). With this assumption, we can prove the cut-off behavior of $T_{a \rightarrow 0}$ (Proposition 2.2) and that the typical time scale $\mathbb{E}[T_{a \rightarrow 0}]$ of this convergence is negligible compare to the mean escape time from zero to a (Proposition 2.3). If in addition we have a control on trajectories outside the branch which contains a (Conditions (2.17)), we get the escape behavior of $T_{0 \rightarrow a}$ (Theorem 2.4).

Note that until the drift condition is satisfied, our results apply to any tree. No particular assumptions are made on the degrees which can be non-homogeneous and non-bounded.

The paper is organized as follows. We first recall the general definition of the two types of behaviors in Section 1.1. Then, we define our birth-and-death chains model on trees (Sections 2.1, 2.2 and 2.3), and the strong drift toward the root in Section 2.4. The main results are given in Section 2.5 and we discuss the basic example of regular trees in Section 2.6. Section 3 contains exact expressions for the mean hitting times and their second moments. Finally, the main results are proven in Section 4.

1.1. *Cut-off and escape behaviors.* Cut-off and escape behaviors, will be studied at the level of hitting times. Both types of behavior are asymptotic, in the sense that they are characterized by what happens when a certain parameter a diverges. Let us recall here the relevant definitions.

Definition 1.1.

- (i) A family of random variables $U^{(a)}$ exhibits cut-off behavior at mean times if

$$\frac{U^{(a)}}{\mathbb{E}[U^{(a)}]} \xrightarrow[a \rightarrow \infty]{\text{Proba}} 1. \quad (1.1)$$

[equivalently, $\lim_{a \rightarrow \infty} \mathbb{P}(U^{(a)} > c \mathbb{E}[U^{(a)}]) = 1$ for $c < 1$ and 0 for $c > 1$].

- (ii) A family of random variables $V^{(a)}$ exhibits escape-time behavior at mean times if

$$\frac{V^{(a)}}{\mathbb{E}[V^{(a)}]} \xrightarrow[a \rightarrow \infty]{\mathcal{L}} \exp(1). \quad (1.2)$$

We refer to Barrera et al. (2009) for a discussion about the motivations of these definitions, as well as general sufficient conditions for them to occur.

2. Model and results

In order to study cut-off and escape behaviors, we will consider families of random walks $X^{(a)}(t)$ defined on a tree I_a . Each of the chains we are defining below is the discrete time counterpart of the birth-and-death processes on trees studied in Martínez and Ycart (2001).

2.1. *The trees.* A tree is an undirected connected graph $G = (I, E)$ with no “nontrivial closed loop”, where I is the set of vertices (or nodes) and E the set of edges (or links).

We define a partial order \preceq on I by choosing a node (denoted by 0) to be the root of the tree: for $x, y \in I$, we say that x is before y ($x \preceq y$) if $x = y$ or if it exists a path from y to zero containing x . Thus each element x of the tree (except the root) has a unique parent denoted by $p(x)$:

$$\forall x \in J, \exists! p(x) \text{ s.t. } p(x) \preceq x \text{ and } (p(x), x) \in E, \quad (2.1)$$

where $J := I \setminus \{0\}$.

There also exists a unique path from x to zero, and we denote by $d(x)$ its length (depth) and by $\ell(x)$ the set of vertices on that path (except 0):

$$\forall x \in J, \ell(x) := \{x_0, x_1, \dots, x_{d(x)-1}\}, \quad (2.2)$$

where $x_i := p^i(x)$ for $i = 0, \dots, d(x) - 1$. Note that according to this notation $x_0 = x$ and $x_{d(x)} = 0$. For the sake of notations, we denote by $\alpha(x)$ the last state before zero on that path: $\alpha(x) = x_{d(x)-1}$. Let us also define $\sigma(x)$ the set of the children of x , and B_x the branch stemming from x :

$$\sigma(x) := \{y : x = p(y)\}, \text{ and } B_x := \{y : x \preceq y\}. \quad (2.3)$$

In Section 3, we will make use of the following decompositions, for any $k = 0, \dots, d(a)$, of the branch B_{a_k} stemming from the k -th parent $a_k = p^k(a)$ of some $a \in J$, and its complementary \overline{B}_{a_k} (see Figure 2.1):

$$B_{a_k} = B_a \cup \left(\bigcup_{l=1}^k (\{a_l\} \cup B_c) \right), \tag{2.4}$$

$c \in \sigma(a_l)$
 $c \neq a_{l-1}$

$$\overline{B}_{a_k} = \bigcup_{l=k+1}^{d(a)} (\{a_l\} \cup B_c). \tag{2.5}$$

$c \in \sigma(a_l)$
 $c \neq a_{l-1}$

We also need to define the set C_{a_k}

$$C_{a_k} := \bigcup_{l=k+1}^{d(a)} \left(\bigcup_{\substack{c \in \sigma(a_l) \\ c \neq a_{l-1}}} B_c \right). \tag{2.6}$$

Note that C_{a_k} is the complementary of the path from a_k to zero in \overline{B}_{a_k} : $C_{a_k} = \overline{B}_{a_k} \setminus \bigcup_{l=k+1}^{d(a)} \{a_l\}$.

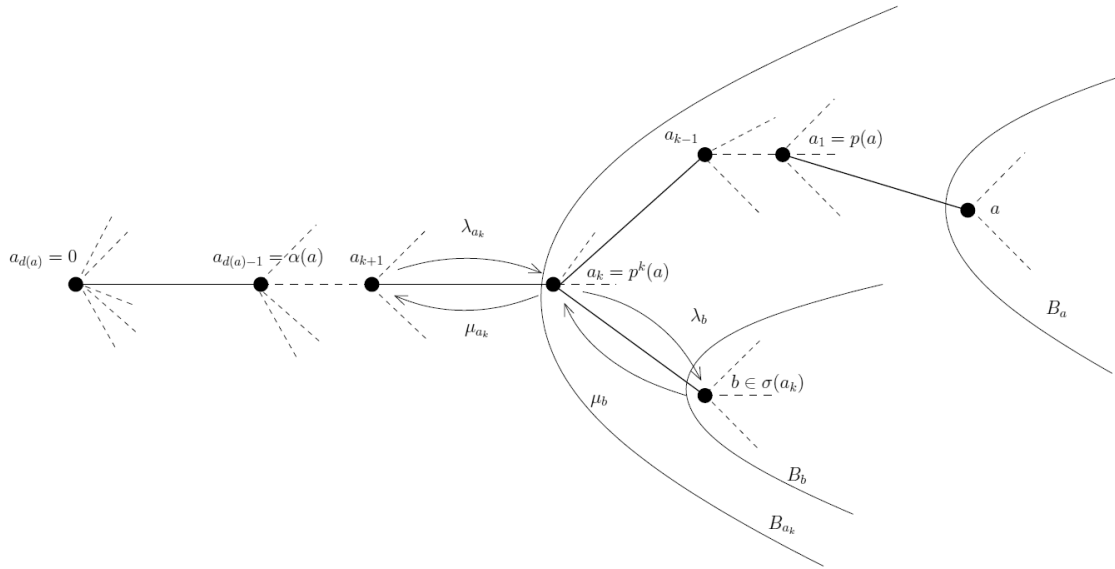


FIGURE 2.1. The path from a to zero: sites, transitions and branches

2.2. *The chains.* We consider irreducible discrete time birth and death chains $X(t)$ on I defined by transition probabilities λ_x “from parent to children” and μ_x “from children to parent” for each $x \in J$. That is $\lambda_x := P(p(x), x)$ and $\mu_x := P(x, p(x))$, where $P(x, y) := \mathbb{P}(X(t + 1) = y \mid X(t) = x)$. The λ_i ’s and μ_i ’s sum to 1 in each site:

$$\mu_x + \sum_{y \in \sigma(x)} \lambda_y = 1, \quad \forall x \in J, \quad \text{and} \quad \sum_{x \in \sigma(0)} \lambda_x = 1, \tag{2.7}$$

and since the chain is irreducible they are supposed to be non-null.

Note that, as we will consider families of chains $X^{(a)}(t)$, the λ ’s and μ ’s may vary with a .

As it doesn’t change our results and computations, we can also add (at least one) non-null “waiting probabilities” $\kappa_x := P(x, x)$ in order to make the chain aperiodic. In that case, equations (2.7) become $\mu_x + \kappa_x + \sum_{y \in \sigma(x)} \lambda_y = 1$, for all $x \in J$, and $\kappa_0 + \sum_{x \in \sigma(0)} \lambda_x = 1$.

If the tree is finite, then the chain is positive recurrent and admits a unique reversible invariant

probability measure (i.p.m.) π , which can be expressed for each $x \in J$ in terms of the transition probabilities

$$\pi(x) = \pi(0) \prod_{y \in \ell(x)} \frac{\lambda_y}{\mu_y}. \quad (2.8)$$

This expression is a consequence of the reversibility of π , which ensures that for each $x \in J$ we have $\pi(p(x)) \lambda_x = \pi(x) \mu_x$. Then, iteration along the path $\ell(x)$ from x to zero gives (2.8), where the normalization constant $\pi(0)$ is chosen such that $\sum_{x \in I} \pi(x) = 1$, that is:

$$\pi(0) = \frac{1}{\sum_{x \in I} \prod_{y \in \ell(x)} \frac{\lambda_y}{\mu_y}}. \quad (2.9)$$

In the case where the tree is not finite, we have to suppose that the quantity $\sum_{x \in I} \prod_{y \in \ell(x)} \frac{\lambda_y}{\mu_y}$ is finite, and then the chain is positive recurrent and (2.8) gives the unique i.p.m. π .

2.3. Hitting times. As mentioned above cut-off and escape behavior are studied at the level of hitting times.

More precisely, the families of random variables under consideration are the hitting times (also called first passage times in the literature) of zero starting from some $a \in J$ (resp. times from zero to a), denoted by $T_{a \rightarrow 0}$ (resp. $T_{0 \rightarrow a}$), with

$$T_{x \rightarrow y} = \min\{t \geq 0 : X^x(t) = y\}, \quad (2.10)$$

where the upper index x means that this is the initial state of the chain ($X^x(0) = x$).

Our results rely on exact expressions for the first and second moments of $T_{p^j(a) \rightarrow p^n(a)}$, for any $0 \leq j, n \leq d(a)$, which will be given in Section 3.

Let us just mention now, for all $x \in J$, the expression in terms of the invariant measure for the mean time $\mathbb{E}[T_{x \rightarrow p(x)}]$ from child to parent (see Lemma 3.1):

$$\mathbb{E}[T_{x \rightarrow p(x)}] = \frac{\pi(B_x)}{\mu_x \pi(x)}. \quad (2.11)$$

This expression is a key ingredient in our computations and allows also to interpret our drift condition as an exponential decay of π (see next section).

2.4. The drift condition. As seen in Section 1.1, cut-off and escape behaviors are defined for families of variables indexed by some diverging parameter. In the present work, we take a state $a \in J$ such that the length $d(a)$ of the path $\ell(a)$ from a to zero tends to infinity to be our asymptotical parameter.

We denote by $a \rightarrow \infty$ the limit when $d(a) \rightarrow \infty$, and we define by ℓ the path “from zero to infinity” followed by a when $d(a) \rightarrow \infty$: $\ell = \lim_{a \rightarrow \infty} \ell(a)$.

Let us give now the definition of strong drift which applies to families of previous defined chains $X^{(a)}(t)$ on trees I_a . When it is necessary, we mark with an upper index (a) the quantities corresponding to $X^{(a)}(t)$.

Definition 2.1. The family of birth-and-death chains $X^{(a)}(t)$ has a *strong drift toward 0 along the path ℓ* (we note 0^ℓ -SD) if

- (i) the transition probabilities $\mu_x^{(a)}$ satisfy

$$\inf_{b \in B_\alpha(a)} \mu_b^{(a)} =: K_\mu > 0, \text{ for all } a \in \ell, \quad (2.12)$$

- (ii) the constant

$$K_a := \sup_{b \in B_\alpha(a)} \mu_b^{(a)} \mathbb{E}[T_{b \rightarrow p(b)}^{(a)}], \quad (2.13)$$

satisfies

$$\frac{K_a^2}{\mathbb{E}[T_{a \rightarrow 0}^{(a)}]} \xrightarrow{a \rightarrow \infty} 0. \quad (2.14)$$

Condition (i) ensures that transitions to the left remain non null asymptotically in a . It also provides a useful bound (see Lemma 4.1). The drift is given by (ii) and can be interpreted in two ways. Note first that by (2.11), K_a can be written

$$K_a = \sup_{b \in B_{\alpha(a)}} \frac{\pi^{(a)}(B_b)}{\pi^{(a)}(b)}. \quad (2.15)$$

Thus, condition (ii) can be seen as an exponential decay of the i.p.m. $\pi^{(a)}$ along the branch $B_{\alpha(a)}$ which contains a : we have $\pi^{(a)}(b) \leq e^{-\gamma_a d(b)}$, with $\gamma_a = -\log(1 - 1/K_a)$, valid for all $b \in B_{\alpha(a)}$. That is for any state of any subbranch of the branch containing a , since by definition $B_{\alpha(a)} = \bigcup_{x \in \ell(a)} B_x$. In terms of energy profile, γ_a also provides a lower bound to the energy profile of any subbranch of the branch containing a . We refer to Barrera et al. (2009) for a more detailed discussion about these interpretations of our drift condition.

2.5. *Results.* For the fixed path ℓ which contains the sequence of a 's going to infinity, the 0^ℓ -SD drift condition is a sufficient condition for the cut-off behavior of hitting times $T_{a \rightarrow 0}^{(a)}$:

Proposition 2.2. *If the family $X^{(a)}(t)$ of irreducible discrete time birth and death chains on I_a satisfies the 0^ℓ -SD condition, then $\lim_{a \rightarrow \infty} \text{Var}(T_{a \rightarrow 0}^{(a)} / \mathbb{E}[T_{a \rightarrow 0}^{(a)}]) = 0$.*

As a consequence, the random variables $T_{a \rightarrow 0}^{(a)}$ exhibit cut-off behavior at mean times.

Moreover, under this drift condition the typical time-scale $\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ of the abrupt convergence toward zero is negligible comparing to the mean times $\mathbb{E}[T_{0 \rightarrow a}^{(a)}]$ of the escape from zero to a :

Proposition 2.3. *If the family $X^{(a)}(t)$ of irreducible discrete time birth and death chains on I_a satisfies the 0^ℓ -SD condition, then $\lim_{a \rightarrow \infty} \mathbb{E}[T_{a \rightarrow 0}^{(a)}] / \mathbb{E}[T_{0 \rightarrow a}^{(a)}] = 0$.*

This time-scales property is the key element of the presence of escape behavior for $T_{0 \rightarrow a}^{(a)}$. But, as seen in Section 3 of Barrera et al. (2009), this requires also that starting from any state of the state space, the mean hitting times of zero are negligible comparing to $\mathbb{E}[T_{0 \rightarrow a}^{(a)}]$, and that the sequence $T_{0 \rightarrow a}^{(a)} / \mathbb{E}[T_{0 \rightarrow a}^{(a)}]$ is uniformly integrable. The requirement that $\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}^{(a)}] / \mathbb{E}[T_{0 \rightarrow a}^{(a)}] \xrightarrow{a \rightarrow \infty} 0$ is obtained if we suppose, in addition to the 0^ℓ -SD condition, that the mean hitting times of zero are comparable. That is if the ratio $\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}^{(a)}] / \mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ remains bounded for all a . For the uniform integrability of $T_{0 \rightarrow a}^{(a)} / \mathbb{E}[T_{0 \rightarrow a}^{(a)}]$, we need to control $\mathbb{E}[T_{0 \rightarrow a}^{(a)2}]$ (see Corollary 3.5). A sufficient condition is to have $\inf_{b \in I_a} \mu_b^{(a)} > 0$ uniformly in a , and $K_a'^2 / \mathbb{E}[T_{0 \rightarrow a}^{(a)}] \xrightarrow{a \rightarrow \infty} 0$, where

$$K_a' := \sup_{b \in I_a} \mu_b^{(a)} \mathbb{E}[T_{b \rightarrow p(b)}^{(a)}] = \sup_{b \in I_a} \frac{\pi^{(a)}(B_b)}{\pi^{(a)}(b)}. \quad (2.16)$$

As above, the second condition is obtained if in addition to the 0^ℓ -SD condition, the ratio $K_a'^2 / \mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ remains bounded for all a (see Section 4.3).

Theorem 2.4. *If the family $X^{(a)}(t)$ of irreducible discrete time birth and death chains on I_a satisfies the 0^ℓ -SD condition, and if there exist K , K' and K'_μ such that*

$$\frac{\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}^{(a)}]}{\mathbb{E}[T_{a \rightarrow 0}^{(a)}]} \leq K, \quad \frac{K_a'^2}{\mathbb{E}[T_{a \rightarrow 0}^{(a)}]} \leq K', \quad \text{and} \quad \inf_{b \in I_a} \mu_b^{(a)} =: K'_\mu > 0 \quad \forall a \in \ell, \quad (2.17)$$

then:

- (1) *the random variables $T_{a \rightarrow 0}^{(a)}$ exhibit cut-off behavior at mean times.*
- (2) *the random variables $T_{0 \rightarrow a}^{(a)}$ exhibit escape-time behavior at mean times.*

Moreover, as the quantity K_a is a supremum taken over all sites of the branch $B_{\alpha(a)}$ which contains a , these assumptions are sufficient to have cut-off and escape behaviors between zero and every state b of $B_{\alpha(a)}$ for which $\mathbb{E}[T_{b \rightarrow 0}^{(a)}]$ diverges with $\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$. Let us define

$$L(a) := \{ b \in B_{\alpha(a)} : \exists C \text{ s.t. } \mathbb{E}[T_{b \rightarrow 0}^{(a)}] \geq C \mathbb{E}[T_{a \rightarrow 0}^{(a)}] \}. \quad (2.18)$$

Corollary 2.5. *Theorem 2.4 holds for $T_{b \rightarrow 0}^{(a)}$ and $T_{0 \rightarrow b}^{(a)}$ for all $b \in L(a)$.*

As we will see in Section 4 (Lemmas 4.2 and 4.3), the conditions needed to obtain the two time scales $\left(\mathbb{E}[T_{a \rightarrow 0}^{(a)}] / \mathbb{E}[T_{0 \rightarrow a}^{(a)}] \xrightarrow{a \rightarrow \infty} 0 \right)$ – which are typical of escape behavior – are actually weaker than those which imply cut-off behavior. But unlike the case when the chain is defined on \mathbb{N} where the drift condition implies both cut-off and escape behaviors (see Bertoncini et al., 2008, Theorem 2.2), here the 0^ℓ -SD condition implies only the cut-off behavior directly. As the chain started in zero could escape to other branches than $B_{\alpha(a)}$, we need in addition to have a “control” on that part of the trajectories from zero to a (conditions (2.17)) to get the escape behavior of $T_{0 \rightarrow a}^{(a)}$.

The assumption that $\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}^{(a)}] / \mathbb{E}[T_{0 \rightarrow a}^{(a)}] \xrightarrow{a \rightarrow \infty} 0$ is the analogue of conditions (2.17) in Theorem 1 of Barrera et al. (2009) for chains on \mathbb{Z} . This is a central argument of the proof of Theorem 2 in Barrera et al. (2009) which gives sufficient conditions for escape behavior of $T_{0 \rightarrow a}^{(a)}$. The idea is that there exists an intermediate time scale Δ_a between $\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ and $\mathbb{E}[T_{0 \rightarrow a}^{(a)}]$ such that the walk will almost surely (in the limit $a \rightarrow \infty$) visit zero in an interval of length Δ_a . Thus, the walk started in zero will make a great number of unfruitful attempts to escape the attraction of zero before eventually hitting a . The exponential character (randomness without memory) of $T_{0 \rightarrow a}^{(a)}$ is a consequence of this situation: each trial is followed by a return to zero, and then by Markovianness, the walk starts again in the same situation.

Remark 2.6. If the tree I_a is reduced to the branch $B_{\alpha(a)}$ which contains a , the conditions of Theorem 2.4 reduce to 0^ℓ -SD plus $\sup_{x \in B_{\alpha(a)}} \mathbb{E}[T_{x \rightarrow 0}^{(a)}] / \mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ bounded for all a .

Remark 2.7. Propositions 2.2 and 2.3 above also apply when the chains $X^{(a)}(t)$ are defined on an infinite tree I (i.e. when $I_a = I$ infinite for all a). We can also apply Theorem 2.4 in that case by considering the restrictions of the chains to $I_a := \{x \in I : d(x) \leq d(a)\}$.

Another feature of the complementarity between both phenomena, is that the successful final excursion which leads to a is “cut-off like” (see Theorem 1 and Section 7 in Barrera et al., 2009): if we consider the hitting times $\tilde{T}_{0 \rightarrow a}^{(a)}$ of a after the last visit to zero, and $\tilde{T}_{a \rightarrow 0}^{(a)}$ (hitting time of zero after last visit to a), we can show by a reversibility argument that they have the same laws. Thus, cut-off and escape behaviors are related by time inversion.

2.6. Example: biased random walk on a regular tree. A regular tree is a tree I in which each node has the same number r of children. In its infinite version (also called Bethe lattice), the degree (number of neighbors) of each node is $r + 1$, except the root which is of degree r . When we consider the finite version $I_a := \{x \in I : d(x) \leq d(a)\}$ for some a , the boundary nodes (sites x s.t. $d(x) = d(a)$) – also called leaves of the tree – are dead ends (of degree 1). In that case, we talk about a Cayley tree. We refer to Dorogovtsev et al. (2008); Dorogovtsev (2010) for discussions about the differences between Bethe lattice and Cayley tree, and to Chapter 5 of Aldous and Fill for the biased random walk on a regular tree.

We consider biased random walks $X^{(a)}(t)$ on I_a which transitions are defined by

$$\lambda_x := \frac{1}{\lambda + r} \quad \text{and} \quad \mu_x := \frac{\lambda}{\lambda + r}, \quad (2.19)$$

with $0 < \lambda < \infty$, for each site $x \neq 0$. As transitions must sum to one in each site, we add also “waiting probabilities” at root and leaves: $\kappa_0 := \lambda / (\lambda + r)$ and $\kappa_x := r / (\lambda + r)$ when $d(x) = d(a)$.

The parameter λ represents the bias of the random walk: at each step, the chain moves toward the root of the tree with probability $\lambda/(\lambda+r)$ and in the opposite direction with probability $r/(\lambda+r)$. Thus, there is a concentration phenomenon near the root level when $\lambda > r$ (localization), and near the leaves level when $\lambda < r$ (delocalization).

Due to the regular structure of the tree, we can compute the quantities involved in the 0^ℓ -SD condition, and determine when it is satisfied or not. Let us start with the invariant measure π . By (2.8), for all x we have

$$\pi(x) = \pi(0) \prod_{y \in \ell(x)} \frac{\lambda_y}{\mu_y} = \pi(0) \prod_{y \in \ell(x)} \frac{1}{\lambda+r} \frac{\lambda+r}{\lambda} = \pi(0) \frac{1}{\lambda^{d(x)}}, \tag{2.20}$$

with $\pi(0) = 1 / \sum_{x \in I_a} \frac{1}{\lambda^{d(x)}}$.

Remark 2.8. The computations of mean return times using Kac formula ($\mathbb{E}[T_x] = 1/\pi(x)$, where $T_x := \inf\{t > 0 : X^x(t) = x\}$ is the first return time to x) illustrates the localization phenomenon for the biased random walk on a regular tree:

- localization is related to positive recurrence of the walk when it takes place on an infinite tree. Lyons showed that the biased walk is recurrent (return times to any state almost surely finite) when the bias λ exceeds the branching number of the tree (here $\lambda > r$) and transient when $\lambda < r$ (see Lyons, 1990).
Moreover, when $\lambda > r$, the chain is positive recurrent (finite mean return times to any state). Using the Kac formula and since for each k there are r^k sites y in I such that $d(y) = k$, we get

$$\mathbb{E}[T_x] = \lambda^{d(x)} \sum_{y \in I} \frac{1}{\lambda^{d(y)}} = \lambda^{d(x)} \sum_{k=0}^{\infty} \left(\frac{r}{\lambda}\right)^k < \infty, \text{ if } \lambda > r. \tag{2.21}$$

- in the finite case, although the walk on I_a is always positive recurrent, the localization phenomenon is characterized by very different asymptotical behaviors of mean return times to zero and a respectively. When $\lambda > r$ the mean return time to zero remains bounded as a goes to infinity (by Kac formula, we get $\mathbb{E}[T_0^{(a)}] \sim \lambda/(\lambda-r)$ for large a). In opposition, the mean return time to a diverges with a ($\mathbb{E}[T_a^{(a)}] \sim \lambda^{d(a)+1}/(\lambda-r)$ for large a).

Let us now compute the key quantity $\frac{\pi(B_x)}{\pi(x)}$ which appears in both expressions of K_a and $\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ (see (2.15), and Proposition 3.3 below). We write

$$\frac{\pi(B_x)}{\pi(x)} = \sum_{b \in B_x} \frac{\pi(b)}{\pi(x)} = \sum_{b \in B_x} \frac{\lambda^{d(x)}}{\lambda^{d(b)}} = \sum_{k=0}^{d(a)-d(x)} \left(\frac{r}{\lambda}\right)^k, \tag{2.22}$$

since there are r^k sites b in B_x such that $d(b) = d(x) + k$.

When $\lambda \neq r$, this is equal to $\frac{\lambda}{\lambda-r} \left(1 - \left(\frac{r}{\lambda}\right)^{d(a)-d(x)+1}\right)$, and thus

$$K_a := \sup_{x \in \ell(a)} \sup_{b \in B_x} \frac{\pi(B_b)}{\pi(b)} = \frac{\lambda}{\lambda-r} \left(1 - \left(\frac{r}{\lambda}\right)^{d(a)}\right), \tag{2.23}$$

and

$$\begin{aligned}
\mathbb{E}[T_{a \rightarrow 0}^{(a)}] &= \sum_{x \in \ell(a)} \frac{\pi(B_x)}{\mu_x \pi(x)} = \frac{\lambda + r}{\lambda - r} \sum_{x \in \ell(a)} \left(1 - \left(\frac{r}{\lambda}\right)^{d(a) - d(x) + 1}\right) \\
&= \frac{\lambda + r}{\lambda - r} \left(d(a) - \left(\frac{r}{\lambda}\right)^{d(a)} \sum_{k=0}^{d(a)-1} \left(\frac{\lambda}{r}\right)^k \right) \\
&= \frac{\lambda + r}{\lambda - r} \left(d(a) + \frac{r}{\lambda - r} \left(\left(\frac{r}{\lambda}\right)^{d(a)} - 1 \right) \right).
\end{aligned} \tag{2.24}$$

In the localization case ($\lambda > r$), we have $\mathbb{E}[T_{a \rightarrow 0}^{(a)}] \sim \frac{\lambda + r}{\lambda - r} d(a)$, and $K_a \rightarrow K := \frac{\lambda}{\lambda - r}$ as $a \rightarrow \infty$.

The 0^ℓ -SD condition is then satisfied, and we have cut-off and escape behavior between the root and any leave of I_a , since for symmetry reasons, the above computations are the same for any state of I_a . This result leads to a better understanding of what happens in that case: since $T_{a \rightarrow 0}^{(a)}$ exhibits cut-off behavior at mean times $\mathbb{E}[T_{a \rightarrow 0}^{(a)}] \sim d(a)$, the walk started in a will go in an almost deterministic way to zero before eventually returning to a . Moreover, the typical time scale of cut-off trajectories is negligible comparing to that of escape trajectories (computations give $\mathbb{E}[T_{0 \rightarrow a}^{(a)}] \sim \lambda^{d(a)}$ for large a , which is of the same order than $\mathbb{E}[T_a^{(a)}]$).

In the delocalization case ($\lambda < r$), K_a and $\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ are both exponential of order $(r/\lambda)^{d(a)}$ for large a , and then there is no strong drift. The 0^ℓ -SD condition is also not satisfied when $\lambda = r$. Computations in that case show that $K_a = d(a)$ and $\mathbb{E}[T_{a \rightarrow 0}^{(a)}] = ((\lambda + r)/2\lambda) d(a)(d(a) + 1)$ and thus the ratio $K_a^2/\mathbb{E}[T_{a \rightarrow 0}^{(a)}]$ does not converge to zero when a goes to ∞ .

3. Mean hitting times

We already mentioned (without proving it) in Section 2.3, that the expression of $\mathbb{E}[T_{x \rightarrow p(x)}]$ in terms of the invariant measure π (equation (2.11)) is a key ingredient in our computations. However, we will also need exact expressions for the first and second moments of hitting times between sites along the path from zero to some fixed state $a \in J$. We give a list of these expressions in Section 3.1. We start with first and second moments for $T_{x \rightarrow p(x)}$ (Lemma 3.1), and $T_{p(x) \rightarrow x}$ (Lemma 3.2). Proposition 3.3 gives the mean hitting times between sites $a_i \in \ell(a)$ for some $a \in J$, and Proposition 3.4 their second moments. Finally we give an idea of the proofs in Section 3.2.

3.1. Exact expressions.

Lemma 3.1. *For any $x \in J$*

$$\mathbb{E}[T_{x \rightarrow p(x)}] = \frac{\pi(B_x)}{\mu_x \pi(x)}, \tag{3.1}$$

$$\mathbb{E}[T_{x \rightarrow p(x)}^2] = \frac{2}{\mu_x \pi(x)} \sum_{b \in B_x} \frac{\pi(B_b)^2}{\mu_b \pi(b)} - \mathbb{E}[T_{x \rightarrow p(x)}]. \tag{3.2}$$

Lemma 3.2. *For any $x \in J$*

$$\mathbb{E}[T_{p(x) \rightarrow x}] = \frac{1 - \pi(B_x)}{\mu_x \pi(x)}, \tag{3.3}$$

$$\mathbb{E}[T_{p(x) \rightarrow x}^2] = \frac{2}{\mu_x \pi(x)} \left(\sum_{c \in C_x} \frac{\pi(B_c)^2}{\mu_c \pi(c)} + \sum_{b \in \ell(x)} \frac{\pi(\overline{B}_b)^2}{\mu_b \pi(b)} \right) - \mathbb{E}[T_{p(x) \rightarrow x}]. \tag{3.4}$$

Proposition 3.3. For $a \in J$, recall that we denote by a_i the i -th parent of a ($a_i = p^i(a)$). Let $0 \leq j < n \leq d(a)$, we have

$$\mathbb{E}[T_{a_j \rightarrow a_n}] = \sum_{k=j}^{n-1} \frac{\pi(B_{a_k})}{\mu_{a_k} \pi(a_k)}, \quad (3.5)$$

$$\mathbb{E}[T_{a_n \rightarrow a_j}] = \sum_{k=j}^{n-1} \frac{1 - \pi(B_{a_k})}{\mu_{a_k} \pi(a_k)}. \quad (3.6)$$

And thus

$$\mathbb{E}[T_{a_j \rightarrow a_n}] + \mathbb{E}[T_{a_n \rightarrow a_j}] = \sum_{k=j}^{n-1} \frac{1}{\mu_{a_k} \pi(a_k)}. \quad (3.7)$$

Proposition 3.4. For $0 \leq j < n \leq d(a)$, we have

$$\mathbb{E}[T_{a_j \rightarrow a_n}^2] = \sum_{k=j}^{n-1} \frac{2}{\mu_{a_k} \pi(a_k)} \left(\sum_{b \in B_{a_k}} \frac{\pi(B_b)^2}{\mu_b \pi(b)} + \pi(B_{a_k}) \mathbb{E}[T_{a_{k+1} \rightarrow a_n}] \right) - \mathbb{E}[T_{a_j \rightarrow a_n}], \quad (3.8)$$

and

$$\mathbb{E}[T_{a_n \rightarrow a_j}^2] = \sum_{k=j}^{n-1} \frac{2}{\mu_{a_k} \pi(a_k)} \left(\sum_{c \in C_{a_k}} \frac{\pi(B_c)^2}{\mu_c \pi(c)} + \sum_{b \in \ell(a_k)} \frac{\pi(\bar{B}_b)^2}{\mu_b \pi(b)} + \pi(\bar{B}_{a_k}) \mathbb{E}[T_{a_k \rightarrow a_j}] \right) - \mathbb{E}[T_{a_n \rightarrow a_j}]. \quad (3.9)$$

Corollary 3.5. For $0 \leq j < n \leq d(a)$, we have

$$\mathbb{E}[T_{a_n \rightarrow a_j}^2] \leq \mathbb{E}[T_{a_n \rightarrow a_j}] \left(2 \left(\frac{K'_a{}^2}{K'_\mu} + \mathbb{E}[T_{0 \rightarrow a_j}] \right) - 1 \right). \quad (3.10)$$

Proof

This is a consequence of equation (3.9). We first use the fact that $\bar{B}_b \subseteq \bar{B}_{a_k}$ for all $b \in \ell(a_k)$ to write

$$\sum_{b \in \ell(a_k)} \frac{\pi(\bar{B}_b)^2}{\mu_b \pi(b)} \leq \pi(\bar{B}_{a_k}) \sum_{b \in \ell(a_k)} \frac{\pi(\bar{B}_b)}{\mu_b \pi(b)} = \pi(\bar{B}_{a_k}) \mathbb{E}[T_{0 \rightarrow a_k}]. \quad (3.11)$$

Then, as $C_{a_k} \subset \bar{B}_{a_k}$, and since by definition $K'_a = \sup_{b \in I_a} \frac{\pi(B_b)}{\pi(b)}$ and $K'_\mu := \inf_{b \in I_a} \mu_b^{(a)}$, we have

$$\sum_{c \in C_{a_k}} \frac{\pi(B_c)^2}{\mu_c \pi(c)} \leq \frac{K'_a{}^2}{K'_\mu} \sum_{c \in C_{a_k}} \pi(c) = \frac{K'_a{}^2}{K'_\mu} \pi(C_{a_k}) \leq \frac{K'_a{}^2}{K'_\mu} \pi(\bar{B}_{a_k}). \quad (3.12)$$

Finally, from equation (3.9)

$$\mathbb{E}[T_{a_n \rightarrow a_j}^2] \leq 2 \sum_{k=j}^{n-1} \frac{\pi(\bar{B}_{a_k})}{\mu_{a_k} \pi(a_k)} \left(\frac{K'_a{}^2}{K'_\mu} + \mathbb{E}[T_{0 \rightarrow a_k}] + \mathbb{E}[T_{a_k \rightarrow a_j}] \right) - \mathbb{E}[T_{a_n \rightarrow a_j}]. \quad (3.13)$$

Hence the result, since $T_{0 \rightarrow a_k} + T_{a_k \rightarrow a_j} = T_{0 \rightarrow a_j}$ and then using equation (3.6). \square

3.2. Proofs. The above formulas can be proven via the standard method of difference equations which rely on the following decomposition of $\mathbb{E}[F(T_{a_k \rightarrow a_n})]$ for any function F :

Lemma 3.6. For all $0 < k < d(a)$, $k \neq n$, we have

$$\mathbb{E}[F(T_{a_k \rightarrow a_n})] = \mu_{a_k} \mathbb{E}[F(T_{a_{k+1} \rightarrow a_n} + 1)] + \sum_{c \in \sigma(a_k)} \lambda_c \mathbb{E}[F(T_{c \rightarrow a_n} + 1)]. \quad (3.14)$$

Proof

This is obtained by decomposing according to the first step of the chain

$$\mathbb{E}[F(T_{a_k \rightarrow a_n})] = \mu_{a_k} \mathbb{E}[F(T_{a_k \rightarrow a_n}) \mid X(1) = a_{k+1}] + \sum_{c \in \sigma(a_k)} \lambda_c \mathbb{E}[F(T_{a_k \rightarrow a_n}) \mid X(1) = c], \quad (3.15)$$

and by Markovianness, for $x = a_{k+1}$ and $c \in \sigma(a_k)$, we have

$$\begin{aligned} \mathbb{E}[F(T_{a_k \rightarrow a_n}) \mid X(1) = x] &= \sum_{l \geq 1} F(l) \mathbb{P}(T_{a_k \rightarrow a_n} = l \mid X(1) = x) \\ &= \sum_{l \geq 1} F(l) \mathbb{P}(T_{x \rightarrow a_n} = l - 1) \\ &= \sum_{l \geq 0} F(l + 1) \mathbb{P}(T_{x \rightarrow a_n} = l) \\ &= \mathbb{E}[F(T_{x \rightarrow a_n} + 1)]. \end{aligned} \quad (3.16)$$

Hence the result. □

Together with Lemma 3.1 which is the discrete analogue of Lemma 1 of [Martínez and Ycart \(2001\)](#) for continuous chains, Lemma 3.6 allows us to prove Propositions 3.3 and 3.4: we fix $0 \leq n \leq d(a)$ and apply equation (3.14) with $F(x) = x$ (resp. $F(x) = x^2$) to obtain recursive equations for $\mathbb{E}[T_{a_k \rightarrow a_n}]$, (resp. $\mathbb{E}[T_{a_k \rightarrow a_n}^2]$) with $0 < k < d(a)$. Then we can iterate these equations in both directions up to boundary conditions – which are particular cases of (3.14) for $k = 0$ and $d(a)$ – and solve them using reversibility and equations (3.1) and (3.2).

Another way of proving both propositions is to use, together with Lemmas 3.1 and 3.2, the fact that $T_{a_j \rightarrow a_n}$ (resp. $T_{a_n \rightarrow a_j}$) is the sum of independant $T_{x \rightarrow p(x)}$ (resp. $T_{p(x) \rightarrow x}$):

$$T_{a_j \rightarrow a_n} = \sum_{k=j}^{n-1} T_{a_k \rightarrow p(a_k)} \quad \text{and} \quad T_{a_n \rightarrow a_j} = \sum_{k=j}^{n-1} T_{p(a_k) \rightarrow a_k}. \quad (3.17)$$

We give now a sketch of those proofs.

First moments:

Although they can be proven directly by difference equations, formulas (3.1) and (3.3) are proven in [Woess \(2009\)](#) (Proposition 9.8) by an alternative method. And thus (3.5) and (3.6) of Proposition 3.3 follow from (3.17).

Second moments:

The proof of (3.2) follows the same lines as the ones of Lemma 1 in [Martínez and Ycart \(2001\)](#), where the application of equation (3.14) gives the recursive equation $\mu_x t_x = 1 + \sum_{c \in \sigma(x)} \lambda_c t_c$ for $t_x = \mathbb{E}[T_{x \rightarrow p(x)}]$. We now apply (3.14) with $F(x) = x^2$, $a_k = x$ and $a_n = p(x)$ to obtain the recursive equation for $\mathbb{E}[T_{x \rightarrow p(x)}^2]$:

$$\mu_x \mathbb{E}[T_{x \rightarrow p(x)}^2] = 2 \mu_x (\mathbb{E}[T_{x \rightarrow p(x)}])^2 + \sum_{c \in \sigma(x)} \lambda_c \mathbb{E}[T_{c \rightarrow p(c)}^2] - 1. \quad (3.18)$$

Then we prove by induction that formula (3.2) satisfies the above equation. As in [Martínez and Ycart \(2001\)](#), we have first to suppose that the tree is finite, since the first step of induction is to consider a leaf x (that is when $\sigma(x) = \emptyset$). For the infinite case, we consider the limit $n \rightarrow \infty$ of restrictions to the truncated trees $I_n := \{x \in I : d(x) \leq n\}$ of level n .

For equation (3.4), we prove by induction that it satisfies the recursive equation obtained from (3.14) with $F(x) = x^2$, $a_k = p(x)$ and $a_n = x$. We don't need to restrict ourselves to finite trees in that case, since the first step of induction is to consider x such that $p(x) = 0$.

Finally, (3.8) and (3.9) of Proposition 3.4 follow from (3.17).

4. Proofs of the main results

The quantities involved in this section are defined with respect to the chain $X^{(a)}(t)$ on I_a , but for the sake of notations we will omit the upper index (a) . We will prove Propositions 2.2 and 2.3 through Lemmas 4.2 and 4.3 which are a little bit stronger. They involve the quantities

$$Q(x) := \frac{1}{\pi(B_x)} \sum_{b \in B_x} \frac{\pi(B_b)^2}{\mu_b \pi(b)} \quad ; \quad Q_a := \sup_{x \in \ell(a)} Q(x), \quad (4.1)$$

and

$$R_a := \sum_{b \in \ell(a)} \frac{\pi(B_b)^2}{\mu_b \pi(b)}, \quad (4.2)$$

which for all a satisfy the inequalities

Lemma 4.1.

$$R_a \leq Q_a \leq K_a^2 / K_\mu. \quad (4.3)$$

Proof of Lemma 4.1

As $\ell(a) \subset B_{a_{d(a)-1}}$, we have

$$R_a \leq \sum_{b \in B_{a_{d(a)-1}}} \frac{\pi(B_b)^2}{\mu_b \pi(b)} = \pi(B_{a_{d(a)-1}}) Q(a_{d(a)-1}), \quad (4.4)$$

and thus $R_a \leq Q_a$, since $\pi(B_{a_{d(a)-1}}) Q(a_{d(a)-1}) \leq Q(a_{d(a)-1}) \leq Q_a$.

For the right-hand side of (4.3), we write

$$\frac{1}{\pi(B_x)} \sum_{b \in B_x} \frac{\pi(B_b)^2}{\mu_b \pi(b)} \leq \frac{1}{\pi(B_x)} \frac{K(x)^2}{K_\mu} \sum_{b \in B_x} \pi(b) = \frac{K(x)^2}{K_\mu}, \quad (4.5)$$

where $K(x) = \sup_{b \in B_x} (\pi(B_b) / \pi(b))$.

We conclude by taking the supremum over $x \in \ell(a)$, since by (2.15), $K_a = \sup_{x \in \ell(a)} K(x)$.

□

4.1. *Proof of Proposition 2.2.* We recall that

$$\lim_{a \rightarrow \infty} \text{Var} \left(\frac{T_{a \rightarrow 0}}{\mathbb{E}[T_{a \rightarrow 0}]} \right) = 0 \quad (4.6)$$

is a sufficient condition to prove the cut-off behavior of the variables $T_{a \rightarrow 0}$ (see Barrera et al., 2009, Section 3).

Since $Q_a \leq K_a^2 / K_\mu$ (see Lemma 4.1 above), and since by the 0^ℓ -SD hypothesis the ratio $K_a^2 / \mathbb{E}[T_{a \rightarrow 0}]$ tends to zero, the following result implies Proposition 2.2.

Lemma 4.2. *If*

$$\frac{Q_a}{\mathbb{E}[T_{a \rightarrow 0}]} \xrightarrow{a \rightarrow \infty} 0, \quad (4.7)$$

then $\text{Var}(T_{a \rightarrow 0} / \mathbb{E}[T_{a \rightarrow 0}]) \xrightarrow{a \rightarrow \infty} 0$.

Proof of Lemma 4.2

For all $a \in J$, we have

$$\begin{aligned} \text{Var}(T_{a \rightarrow 0}) &= \sum_{x \in \ell(a)} \text{Var}(T_{x \rightarrow p(x)}) \\ &\leq \sum_{x \in \ell(a)} \mathbb{E}[T_{x \rightarrow p(x)}^2] \\ &\leq \sum_{x \in \ell(a)} \frac{2}{\mu_x \pi(x)} \sum_{b \in B_x} \frac{\pi(B_b)^2}{\mu_b \pi(b)}, \end{aligned} \quad (4.8)$$

where the second inequality comes from equation (3.2) of Lemma 3.1.

Thus

$$\text{Var}(T_{a \rightarrow 0}) \leq \sum_{x \in \ell(a)} \frac{2 \pi(B_x)}{\mu_x \pi(x)} Q(x) \leq 2Q_a \sum_{x \in \ell(a)} \frac{\pi(B_x)}{\mu_x \pi(x)}, \quad (4.9)$$

and by (3.1)

$$\text{Var}\left(\frac{T_{a \rightarrow 0}}{\mathbb{E}[T_{a \rightarrow 0}]}\right) \leq \frac{2Q_a}{\mathbb{E}[T_{a \rightarrow 0}]} \xrightarrow{a \rightarrow \infty} 0. \quad (4.10)$$

□

4.2. *Proof of Proposition 2.3.* As above, by Lemma 4.1 ($R_a \leq K_a^2 / K_\mu$) together with the 0^ℓ -SD condition, the following lemma implies Proposition 2.3.

Lemma 4.3. *If*

$$\frac{R_a}{\mathbb{E}[T_{a \rightarrow 0}]} \xrightarrow{a \rightarrow \infty} 0, \quad (4.11)$$

then $\mathbb{E}[T_{a \rightarrow 0}] / \mathbb{E}[T_{0 \rightarrow a}] \xrightarrow{a \rightarrow \infty} 0$.

Proof of Lemma 4.3

It is equivalent to show that

$$\Gamma_a := \frac{\mathbb{E}[T_{a \rightarrow 0}]}{\mathbb{E}[T_{a \rightarrow 0}] + \mathbb{E}[T_{0 \rightarrow a}]} \xrightarrow{a \rightarrow \infty} 0. \quad (4.12)$$

By (3.5) and (3.7), we have

$$\Gamma_a = \frac{\sum_{x \in \ell(a)} \frac{\pi(B_x)}{\mu_x \pi(x)}}{\sum_{x \in \ell(a)} \frac{1}{\mu_x \pi(x)}}. \quad (4.13)$$

We now consider the probability measure on $\ell(a)$ defined by expectations

$$\mathcal{E}(F) := \frac{\sum_{x \in \ell(a)} \frac{F(x)}{\mu_x \pi(x)}}{\sum_{x \in \ell(a)} \frac{1}{\mu_x \pi(x)}}, \quad (4.14)$$

and the inequality $\mathcal{E}(F)^2 \leq \mathcal{E}(F^2)$ for $F(x) = \pi(B_x)$ implies that $\Gamma_a \leq R_a / \mathbb{E}[T_{a \rightarrow 0}]$, which concludes the proof.

□

4.3. *Proof of Theorem 2.4.* The cut-off behavior of $T_{a \rightarrow 0}$ was already proven in Proposition 2.2. For the escape behavior of $T_{0 \rightarrow a}$, we need to prove that $\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}] / \mathbb{E}[T_{0 \rightarrow a}]$ tends to zero and that the sequence $T_{0 \rightarrow a} / \mathbb{E}[T_{0 \rightarrow a}]$ is uniformly integrable (see Barrera et al., 2009, Theorem 2). For the first point, we have

$$\frac{\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow a}]} = \frac{\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}]}{\mathbb{E}[T_{a \rightarrow 0}]} \frac{\mathbb{E}[T_{a \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow a}]} \leq K \frac{\mathbb{E}[T_{a \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow a}]} \xrightarrow{a \rightarrow \infty} 0, \quad (4.15)$$

as a consequence of (2.17) and Proposition 2.3.

For the uniform integrability, we apply Corollary 3.5 with $n = d(a)$ and $j = 0$ to get

$$\frac{\mathbb{E}[T_{0 \rightarrow a}^2]}{\mathbb{E}[T_{0 \rightarrow a}]^2} \leq 2 + \frac{2}{K'_\mu} \frac{K'_a{}^2}{\mathbb{E}[T_{0 \rightarrow a}]} \leq 2 + o(1). \quad (4.16)$$

The second inequality is due to (2.17) and Proposition 2.3, and thus the sequence $T_{0 \rightarrow a} / \mathbb{E}[T_{0 \rightarrow a}]$ is uniformly integrable.

4.4. *Proof of Corollary 2.5.* The key element of this result, is that K_b is equal to K_a , as $B_{\alpha(b)} = B_{\alpha(a)}$ for $b \in L(a)$. Then if the 0^ℓ -SD condition holds for a , it also holds for every state b of $L(a)$:

$$\frac{K_b^2}{\mathbb{E}[T_{b \rightarrow 0}]} = \frac{K_a^2}{\mathbb{E}[T_{b \rightarrow 0}]} \leq \frac{K_a^2}{C \mathbb{E}[T_{a \rightarrow 0}]} \xrightarrow{a \rightarrow \infty} 0. \quad (4.17)$$

Since equation (4.3) is valid for any state, we can apply Lemma 4.2 and Proposition 2.2 holds for all b : that is $T_{b \rightarrow 0}$ exhibits cut-off behavior for every $b \in L(a)$.

It remains to prove the escape behavior of $T_{0 \rightarrow b}$. Let $b \in L(a)$, using (2.17) we have

$$\frac{\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow b}]} = \frac{\sup_{x \in I_a} \mathbb{E}[T_{x \rightarrow 0}]}{\mathbb{E}[T_{a \rightarrow 0}]} \frac{\mathbb{E}[T_{a \rightarrow 0}]}{\mathbb{E}[T_{b \rightarrow 0}]} \frac{\mathbb{E}[T_{b \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow b}]} \leq \frac{K}{C} \frac{\mathbb{E}[T_{b \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow b}]}, \quad (4.18)$$

and this tends to zero since by Lemma 4.3, Proposition 2.3 holds for b .

Now by Corollary 3.5, we have

$$\frac{\mathbb{E}[T_{0 \rightarrow b}^2]}{\mathbb{E}[T_{0 \rightarrow b}]^2} \leq 2 + \frac{2}{K'_\mu} \frac{K'_a{}^2}{\mathbb{E}[T_{0 \rightarrow b}]} \leq 2 + o(1), \quad (4.19)$$

since by (2.17) and Proposition 2.3

$$\frac{K'_a{}^2}{\mathbb{E}[T_{0 \rightarrow b}]} = \frac{K'_a{}^2}{\mathbb{E}[T_{a \rightarrow 0}]} \frac{\mathbb{E}[T_{a \rightarrow 0}]}{\mathbb{E}[T_{b \rightarrow 0}]} \frac{\mathbb{E}[T_{b \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow b}]} \leq \frac{K'}{C} \frac{\mathbb{E}[T_{b \rightarrow 0}]}{\mathbb{E}[T_{0 \rightarrow b}]} \xrightarrow{a \rightarrow \infty} 0. \quad (4.20)$$

Thus $T_{0 \rightarrow b} / \mathbb{E}[T_{0 \rightarrow b}]$ is uniformly square integrable, and $T_{0 \rightarrow b}$ exhibits escape behavior.

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