ALEA, Lat. Am. J. Probab. Math. Stat. 9, 213–229 (2012)

Covering the whole space with Poisson random balls

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Abstract. We consider Poisson random balls in $\mathbb{R}^d$, with the pair (center, radius) given by a Poisson point process in $\mathbb{R}^d \times (0, +\infty)$. According to the intensity measure of the Poisson process, we investigate the eventuality of covering the whole space with the union of the balls. We exhibit a disjunction phenomenon between the coverage with large balls (low frequency) and the coverage with small balls (high frequency). Concerning the second type of coverage, we prove the existence of a critical regime which separates the case where coverage occurs almost surely and the case where coverage does not occur almost surely. We give an explicit value of the critical intensity and we prove that the Hausdorff measure of the set of points which are not covered by the union of balls is linked with this value. We also compare with other critical regimes appearing in continuum percolation.

1. Introduction and setting

In this paper, we adopt the following general framework. Let μ be a locally finite non-negative measure on $(0, +\infty)$. Let Φ be a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity measure ν is the product of the Lebesgue measure on $\mathbb{R}^d$ and of the measure μ. We denote by $B(x, r)$ the open Euclidean ball of $\mathbb{R}^d$ centered at x with radius r. With the previous point process Φ we associate the following random set:

$$\Xi = \bigcup_{(x,r)\in\Phi} B(x, r),$$

in which each point is covered at least once by a ball. If μ is a finite measure, we can write $\mu = \lambda m$ where λ is the total mass of μ and m is a probability measure. In this case, the set Ξ can be obtained as the union of balls of iid random radii with distribution m, centered at points of a homogeneous Poisson point process

Received by the editors April 13, 2011; accepted April 18, 2012.
2000 *Mathematics Subject Classification.* Primary: 60G60, 60G78; Secondary: 60G20, 60D05, 60G55, 60G10, 60F05..
Key words and phrases. Poisson point process, coverage, random set, Boolean model, percolation, Hausdorff dimension.

with intensity λ. It is known in stochastic geometry as the Boolean or germ-grain model (see Stoyan et al. (1987), Serra (1984), Meester and Roy (1996), Hall (1988) for instance).

Then, the space $\mathbb{R}^d$ is partitioned into two regions: the occupied region Ξ and the vacant region $\mathbb{R}^d \setminus \Xi$. Let us say that coverage occurs if the whole space $\mathbb{R}^d$ is covered by Ξ, that is, if $\mathbb{R}^d = \Xi,\ a.s.$ If coverage does not occur, then one can wonder how large (the notion will be made more precise later on) are the occupied and the vacant regions. Answering these questions is the purpose of occupancy percolation studies, respectively vacancy percolation. Several authors studied continuum percolation for Boolean models (see Hall (1985a, 1988) or Meester and Roy (1996) for an exhaustive survey). The main aim of this paper is to study coverage in our more general setting. Since percolation questions are naturally linked, we will also discuss them. In particular, we provide a new criterion for coverage which yields the existence of a critical intensity for coverage. The exact computation of this critical intensity allows us to exhibit new bounds for critical percolation intensities. We also obtain new results concerning the Hausdorff dimension of the vacant region.

Concerning the coverage question, a simple computation gives

$$\mathbb{P}(0 \notin \Xi) = \mathbb{P}\left(\{(x,r) \in \Phi; 0 \in B(x,r)\} = \emptyset\right) = \exp\left(-v_d \int_0^{+\infty} r^d \mu(dr)\right) \quad (1.1)$$

where v_d denotes the volume of the unit ball in $\mathbb{R}^d$. Therefore, we get the following necessary condition for covering:

$$\int_0^{+\infty} r^d \mu(dr) = +\infty. \quad (1.2)$$

Indeed, if the integral is finite, then $\mathbb{P}(0 \notin \Xi)$ is positive and therefore $\mathbb{R}^d$ is not almost surely covered. If the integral is infinite, then $\mathbb{P}(0 \in \Xi) = 1$. By Fubini's theorem, this ensures that the Lebesgue measure of the complement of Ξ is almost surely 0. This is not sufficient in general to ensure the almost sure coverage of $\mathbb{R}^d$. Getting a necessary and sufficient condition is an old question initiated in the sixties for the Dvoretsky's problem of covering the circle with random arcs (see Dvoretzky (1956), Gilbert (1965), Kahane (1985) and Kahane (2000) for an historical survey of this problem). To our knowledge, there are only two situations for which the problem is totally solved.

The first situation concerns the dimension $d = 1$, which was Dvoretsky's initial question. Shepp (1972) and Mandelbrot (1972) solved the problem in dimension one giving an if and only if condition for $\mathbb{R}$ to be almost surely (a.s.) covered by $\bigcup_{(x,r)\in\Phi}(x, x+r)$. In our setting, with $B(x,r) = (x-r, x+r)$, this necessary and sufficient condition is

$$\int_0^1 \exp\left(2\int_u^{+\infty} (r-u)\mu(\mathrm{d}r)\right) \mathrm{d}u = +\infty\,. \quad (1.3)$$

The second case concerns the germ-grain model, i.e. when the measure μ is finite. In that case, it is known (see Hall (1985b, 1988); Meester and Roy (1996)) that $\mathbb{R}^d$ is a.s. covered if and only if (1.2) holds, which is equivalent to saying that the balls have an infinite mean Lebesgue measure. In particular, this forbids the coverage of $\mathbb{R}^d$ with balls of an equal radius.

Apart from the random balls coverage problem, many results have been obtained on related topics. Let us mention for instance that the generalization of Dvoretsky's problem to higher dimension was considered by El Helou (1978) who gives sufficient and necessary conditions in his thesis. Kahane (1990), partially inspired by the ideas of Shepp (1972) and Janson (1986), solved the coverage problem in dimension $d > 1$ in a general setting where $B(x, r)$ is replaced by $x + r\mathcal{C}$, with $\mathcal{C}$ belonging to a certain class of open bounded convex sets of $\mathbb{R}^d$. Let us emphasize that this class of sets does not contain Euclidean balls. Actually, the restriction imposed on the convex set $\mathcal{C}$ is only required for the sufficient condition of a.s. coverage. We also mention more recent papers. Molchanov and Scherbakov (2003) are concerned with an inhomogeneous framework where the radii are random variables that depend on the center locations. In a one-dimensional setting, Barral and Fan (2004) consider the asymptotic behavior of the number of Poisson intervals which cover a point. Finally, Athreya et al. (2004) introduce a weaker notion of coverage, called *eventual coverage*, which consists in covering the orthant $(t, +\infty)^d$ for some $t \in (0, +\infty)$. Then, a critical behavior is also observed for germ-grain models, but only for dimension $d = 1$.

The content of the present paper is the following. In Section 2, we start by establishing a dichotomy result: when coverage holds, it is due to the contribution of the small balls (high frequency coverage) or the large balls (low frequency coverage). This result appears in Mandelbrot (1972) and Shepp (1972) in the one-dimensional case, as a consequence of the one-dimensional characterization (1.3). In the same section, we also quote that the low frequency coverage problem is very similar to the germ-grain coverage problem. Then a lightly modified version of (1.2) appears as a necessary and sufficient condition.

Section 3 is devoted to the existence of a critical coverage regime. Indeed, a criterion for coverage or non-coverage is exhibited. In the case of high frequency coverage, it relies on the compared asymptotics of $\int_\varepsilon^1 r^d \,\mu(\mathrm{d}r)$ and $|\ln \varepsilon|$ as ε goes to 0^+. When coverage does not hold, the Hausdorff dimension of the vacant region is computed following the ideas of El Helou (1978). Links with percolation questions, as studied in Gouéré (2009) for occupancy percolation, or in Broman and Camia (2010) for vacancy percolation, are explored.

In Section 4, we give relevant examples. A special attention is paid to the power law models with measures of the type $\mu(\mathrm{d}r) = r^{-\beta-1}\,\mathbf{1}_{(0,+\infty)}(r)\,\mathrm{d}r$. In particular, we focus on the scale invariant model, as in Broman and Camia (2010), which corresponds to $\beta = d$ and appears as a critical case. We also consider multiscale Boolean models, as studied in Menshikov et al. (2001).

The last section contains the proofs. Bringing together the proofs allows us to emphasize the links between the results of Section 2 and Section 3.

Throughout the paper, dimension d is fixed, $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^d$, and $v_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ stands for the Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^d$. The symbol $\subset$ denotes the non-strict inclusion.

2. High and low frequency coverage

Let us consider μ a locally finite non-negative measure on $(0,+\infty)$ and Φ a Poisson point process in $\mathbb{R}^d \times (0,+\infty)$ with intensity $\mathrm{d}x\mu(\mathrm{d}r)$. We set

$$\Xi = \underset{(x,r)\in\Phi}{\cup} B(x,r),$$

the occupied region. Our main goal is to find necessary or sufficient conditions on μ such that $\mathbb{R}^d$ is completely covered by Ξ. We denote by $\psi(\mu)$ the probability that $\mathbb{R}^d$ is covered by Ξ

$$\psi(\mu) = \mathbb{P}(\mathbb{R}^d \subset \Xi),$$

called "coverage probability". We start with a useful lemma. It states a zero-one law for the coverage probability, due to ergodicity of the Poisson point process of the centers of the balls, and gives some criteria for coverage.

Lemma 2.1. *Let μ be a locally finite non-negative measure on $(0,+\infty)$. Then,*

i) *$\psi(\mu)$ is either 0 or 1.*
ii) *If there exists K a compact set of $\mathbb{R}^d$ such that $\mathbb{P}(K \subset \Xi) < 1$ then $\psi(\mu) = 0$.*
iii) *If there exists K a compact set of $\mathbb{R}^d$ with non empty interior such that $\mathbb{P}(K \subset \Xi) = 1$ then $\psi(\mu) = 1$.*

Proof: *i)* The event $\{\mathbb{R}^d = \Xi\}$ is invariant under the action of the translations of $\mathbb{R}^d$ and the result follows from ergodicity arguments (see for instance Chapter 2.1 in Meester and Roy (1996)). *ii)* It is enough to note that $\mathbb{P}(\mathbb{R}^d \subset \Xi) \le \mathbb{P}(K \subset \Xi) < 1$ and to use *i)* to conclude. *iii)* Since the interior of K is non empty, one has $\mathbb{R}^d = \underset{q\in\mathbb{Q}^d}{\cup}(q+K)$. Moreover, by stationarity one has $\mathbb{P}(K+q \not\subset \Xi) = \mathbb{P}(K \not\subset \Xi)$, which equals 0 by assumption, so that

$$1-\psi(\mu) = \mathbb{P}(\mathbb{R}^d \not\subset \Xi) \le \sum_{q\in\mathbb{Q}^d} \mathbb{P}(K+q \not\subset \Xi) = 0.$$

□

One can split the random set Ξ into two independent sets, one made of small balls (radius less than 1), and the other one made of large balls (radius larger than 1):

$$\Xi = \Xi_H \cup \Xi_L \quad \text{where } \Xi_H = \underset{(x,r)\in\Phi\,;\,r<1}{\cup} B(x,r) \quad \text{and} \quad \Xi_L = \underset{(x,r)\in\Phi\,;\,r\ge 1}{\cup} B(x,r)\ .$$

Note that the radius size 1 is arbitrary and can be set to any positive real value. In order to distinguish between the contribution of the small balls and the contribution of the large balls, following Mandelbrot (1972) and Shepp (1972), we introduce the following definition.

Definition. *Let μ be a locally finite non-negative measure on $(0,+\infty)$ and let us write $\mu = \mu_H + \mu_L$ with*

$$\mu_H(\mathrm{d}r) = \mathbf{1}_{(0,1]}(r)\mu(\mathrm{d}r) \quad \text{and} \quad \mu_L(\mathrm{d}r) = \mathbf{1}_{(1,+\infty)}(r)\mu(\mathrm{d}r)\ .$$

The measure μ is said to give a high frequency coverage if $\psi(\mu_H) = 1$ and a low frequency coverage if $\psi(\mu_L) = 1$.

A straightforward generalization of the germ-grain case (see Meester and Roy (1996) or Hall (1988)) applies and concludes that the necessary and sufficient condition remains valid for low frequency coverage.

Proposition 2.2. (Necessary and sufficient condition for low frequency coverage). *Let μ be a locally finite non-negative measure on $(0,+\infty)$. Then*

$$\psi(\mu_L) = 1 \text{ if and only if } \int_1^{+\infty} r^d \mu(\mathrm{d}r) = +\infty \ .$$

Note that this condition remains valid whenever the measure μ is translated or dilated. Then, for any $a, \delta \in (0,+\infty)$, let us define

$$\Xi_{>a} = \underset{(x,r)\in\Phi\,;\,r>a}{\cup} B(x,r) \text{ and } \Xi^\delta_{>a} = \underset{(x,r)\in\Phi\,;\,r>a}{\cup} B(x,\delta r) \ .$$

Since μ is assumed to be locally finite on $(0,+\infty)$ we clearly obtain that

$$\psi(\mu_L) = \mathbb{P}\left(\mathbb{R}^d \subset \Xi_{>a}\right) = \mathbb{P}\left(\mathbb{R}^d \subset \Xi^\delta_{>a}\right) . \tag{2.1}$$

A particular and interesting consequence of this proposition is that low frequency coverage implies that a.s. any point of $\mathbb{R}^d$ is covered by an infinite number of arbitrarily large balls. Actually, it implies that

$$\psi(\mu_L) = \mathbb{P}\left(\mathbb{R}^d \subset \underset{a>0}{\cap} \Xi_{>a}\right) .$$

The main result of this section is the following theorem. It states that the coverage of $\mathbb{R}^d$ is equivalent to high or low frequency coverage.

Theorem 2.3. *Let μ be a locally finite non-negative measure on $(0,+\infty)$. Then*

$$\psi(\mu) = \max(\psi(\mu_H), \psi(\mu_L)).$$

The proof of Theorem 2.3, which is given in Section 5, relies on the characterization of low frequency coverage. Then, similarly to the low frequency feature, when high frequency coverage occurs, a.s. any point of $\mathbb{R}^d$ is covered by (an infinite number of) arbitrarily small balls. Actually, denoting $\Xi_{\le\varepsilon} = \underset{(x,r)\in\Phi\,;\,r\le\varepsilon}{\cup} B(x,r)$ for $\varepsilon > 0$ we get

$$\psi(\mu_H) = \mathbb{P}\left(\mathbb{R}^d \subset \underset{\varepsilon>0}{\cap} \Xi_{\le\varepsilon}\right) .$$

Beyond the one-dimensional case, which was solved by Shepp, high frequency covering is trickier than low frequency covering. One can give the following necessary condition on the one hand and sufficient condition on the other hand.

Proposition 2.4. (Necessary condition for high frequency coverage). *Let μ be a locally finite non-negative measure on $(0,+\infty)$.*

$$\text{If } \psi(\mu_H) = 1 \text{ then } \int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(\mathrm{d}r)\right) \mathrm{d}u = +\infty \ .$$

One notices that in the one-dimensional case the above necessary covering condition coincides with the necessary and sufficient condition (1.3). Actually, the necessary condition appears in Kahane (1990) with a proof based on a martingale convergence argument. We give an analogous proof in Section 5, based on a second moment argument.

The sufficient condition is obtained through more geometrical and combinatorial arguments, close to the ones of El Helou (1978).

Proposition 2.5. (Sufficient condition for high frequency coverage). *Let μ be a locally finite non-negative measure on $(0,+\infty)$. Then*

$$\text{If} \quad \limsup_{u\to 0} u^d \exp\left(v_d \int_u^1 (r-u)^d \mu(\mathrm{d}r)\right) = +\infty \quad \text{then} \quad \psi(\mu_{H}) = 1 \ .$$

Even though different, these two conditions allow us to exhibit a critical regime for coverage, as it often appears for percolation. We discuss this phenomenon in the next section.

3. Critical regime

3.1. *Critical intensity.* By a coupling method, one can check that the map $\lambda \mapsto \psi(\lambda\mu)$ is non-decreasing. Let us define the critical intensity for coverage by

$$\lambda^*(\mu) = \inf\{\lambda \geq 0; \psi(\lambda\mu) > 0\} \ \in [0,+\infty] \tag{3.1}$$

where, as usual, we set $+\infty$ when the set is empty. One has $\lambda^*(\mu) = 0$ if and only if $\psi(\lambda\mu) = 1$ for all $\lambda > 0$. One has $\lambda^*(\mu) = +\infty$ if and only if $\psi(\lambda\mu) = 0$ for all $\lambda > 0$. Moreover, when $\lambda^*(\mu) \in (0,+\infty)$,

$$\lambda < \lambda^*(\mu) \Rightarrow \psi(\lambda\mu) = 0 \ \text{ and } \ \lambda > \lambda^*(\mu) \Rightarrow \psi(\lambda\mu) = 1.$$

As a consequence of Propositions 2.4 and 2.5 one can derive the following explicit value of $\lambda^*(\mu)$.

Theorem 3.1. *Let μ be a locally finite non-negative measure on $(0,+\infty)$. Set*

$$\ell(\mu) = \limsup_{\varepsilon\to 0} \left(|\ln\varepsilon|^{-1}\, v_d \int_\varepsilon^1 r^d\,\mu(\mathrm{d}r)\right) \in [0,+\infty]. \tag{3.2}$$

If $\int_1^{+\infty} r^d\mu(dr) = +\infty$ then $\lambda^*(\mu) = 0$. Otherwise[2],

$$\lambda^*(\mu) = d/\ell(\mu). \tag{3.3}$$

A straightforward consequence of this theorem is the following simple condition to ensure, or not, high frequency coverage.

Remark 3.2. if $\ell(\mu) > d$ then $\psi(\mu_{H}) = 1$; if $\ell(\mu) < d$ then $\psi(\mu_{H}) = 0$.

When $\ell(\mu) = d$, different behaviors can be observed as we can see in the following examples. Let us consider the measures μ_+ and μ_- defined by:

$$\mu_+(\mathrm{d}r) = dv_d^{-1} r^{-d-1}(1 + 2|\ln(r)|^{-1})\mathbf{1}_{(0,a)}(r)\mathrm{d}r,$$

and

$$\mu_-(\mathrm{d}r) = dv_d^{-1} r^{-d-1}(1 - 2|\ln(r)|^{-1})\mathbf{1}_{(0,a)}(r)\mathrm{d}r,$$

where $a > 0$ is small enough to ensure that $1 - 2|\ln(r)|^{-1}$ is positive for $r \in (0,a)$. We easily obtain that $\ell(\mu_+) = \ell(\mu_-) = d$, so that $\lambda^*(\mu_+) = \lambda^*(\mu_-) = 1$, according to Theorem 3.1. In other words, μ_+ and μ_- are both critical. Applying Propositions 2.4 and 2.5, we get that $\psi(\mu_+) = 1$ and $\psi(\mu_-) = 0$.

An interesting behavior is observed when $\lambda^*(\mu) \in (0,+\infty)$, since it reveals a sharp transition between coverage or no coverage. For a small enough intensity λ, high frequency coverage never occurs whereas for a large enough λ, high frequency coverage always occurs. A reformulation of Theorem 3.1 yields the following necessary and sufficient condition for this to hold.

[2]We use the conventions $d/0 = +\infty$ and $d/+\infty = 0$

Remark 3.3. $\lambda^*(\mu) \in (0, +\infty)$ if and only if

$$\ell(\mu) \in (0, \infty) \text{ and } \int_1^{+\infty} r^d \mu(dr) < \infty. \tag{3.4}$$

In this case, the coverage clearly depends on the chosen intensity λ. Let us emphasize that this result can only be obtained for high frequency coverage, using not finite measures. Actually, for finite measures and associated Boolean models, the coverage can only be a low frequency coverage that does not depend on λ.

Let us mention that $\ell(\mu)$ also appears in the one-dimensional setting of Barral and Fan (2004), where it is linked to the Hausdorff dimension of the set of points which are not covered infinitely many times. In the next section, we obtain a similar result for the vacant region itself, which is valid whatever dimension d is.

3.2. *Hausdorff dimensions.* We notice that $\ell(\mu) > 0$ implies in particular condition (1.2) so that a.s. the set of points which are not high frequency covered has the Lebesgue measure 0. The aim of this section is to determine its Hausdorff dimension, which is more relevant in this situation. In the sequel, for any set A in $\mathbb{R}^d$, we denote $\dim_{\mathcal{H}}(A)$ its Hausdorff dimension and refer to Falconer (1990) for a precise definition. We will follow the proofs of El Helou (1978) which considers the Hausdorff dimension of the set of points of the torus $\mathbb{T}^d$ which are not covered infinitely many times by random open sets with a prescribed diameter. His results may be adapted to our framework giving the following preliminary properties, whose proofs are postponed in Section 5.

Proposition 3.4. *Let μ be a locally finite non-negative measure on $(0, +\infty)$ and $\ell(\mu)$ be given by (3.2). Let A be a compact set of $\mathbb{R}^d$. If $\ell(\mu) < d$ and $\dim_{\mathcal{H}}(A) > \ell(\mu)$ then with positive probability $A \not\subset \Xi_H$.*

Proposition 3.5. *Let μ be a locally finite non-negative measure on $(0, +\infty)$ and $\ell(\mu)$ be given by (3.2). Let A be a compact set of $\mathbb{R}^d$. If $\ell(\mu) > 0$ and $\dim_{\mathcal{H}}(A) < \ell(\mu)$ then almost surely $A \subset \Xi_H$.*

In particular, when $\ell(\mu) \in (0, +\infty)$, one can consider

$$\Xi_H^\lambda = \bigcup_{(x,r)\in\Phi^\lambda} B(x, r)$$

obtained from a Poisson point process Φ^λ independent from Φ with intensity $\lambda dx \mu_H(dr)$ for $\lambda > 0$. Note that $\Xi_H \cup \Xi_H^\lambda$ has the same distribution than $\Xi_H^{1+\lambda}$. Then, let us consider $F = [0, 1]^d \cap \Xi_H^c$ the compact set of points in $[0, 1]^d$ that are not covered by Ξ_H, which is already known to be empty a.s. when $\ell(\mu) > d$ according to Theorem 3.1. Note that the choice of $[0, 1]^d$ as a region of interest is arbitrary and can be replaced by any compact set with a non-empty interior. On the one hand, let us choose $\lambda > 0$ such that $\lambda\ell(\mu) < d$ and $(1 + \lambda)\ell(\mu) > d$, then $[0, 1]^d$ is almost surely covered by $\Xi_H^{1+\lambda}$ so that F is almost surely covered by Ξ_H^λ and $\dim_{\mathcal{H}}(F) \leq \lambda\ell(\mu)$ a.s. according to Proposition 3.4. This proves that $\dim_{\mathcal{H}}(F) \leq \max(d - \ell(\mu), 0)$ a.s. On the other hand, when moreover $\ell(\mu) < d$, let us choose $\lambda > 0$ such that $(1 + \lambda)\ell(\mu) < d$, then with positive probability $[0, 1]^d$ is not covered by $\Xi_H^{1+\lambda}$. Therefore $F \not\subset \Xi_H^\lambda$ and $\dim_{\mathcal{H}}(F) \geq \lambda\ell(\mu)$ with positive probability according to Proposition 3.5. This proves the following theorem.

Theorem 3.6. *Let μ be a locally finite non-negative measure on $(0,+\infty)$ such that $\ell(\mu) > 0$. Let F be the compact set of points in $[0,1]^d$ that are not covered by Ξ_H, then*

$$dim_{\mathcal{H}}(F) \leq d - \ell(\mu) \ a.s.$$

Moreover, if $\ell(\mu) \leq d$, then

$$\mathbb{P}\left(dim_{\mathcal{H}}(F) = d - \ell(\mu)\right) > 0.$$

3.3. *Link with continuum percolation.* In this section, we still consider the occupied region $\Xi = \underset{(x,r)\in\Phi}{\cup} B(x,r)$, and we set W the connected component of Ξ which contains 0. Following Meester and Roy (1996), we consider a percolation function defined by

$$\theta(\mu) = \mathbb{P}(W \text{ is unbounded}).$$

Note that in dimension 1, percolation occurs, ie $\theta(\mu) > 0$, if and only if coverage occurs. Since θ is non-decreasing, a percolation critical intensity is defined as

$$\lambda_c(\mu) = \inf\{\lambda \geq 0; \theta(\lambda\mu) > 0\}, \tag{3.5}$$

where we set $\lambda_c(\mu) = +\infty$ if there is no such λ. In dimension $d \geq 2$, assuming that μ is a non zero measure, one easily checks that the critical intensity for percolation $\lambda_c(\mu)$ is always finite. Moreover, it is proven in Gouéré (2009) that $\lambda_c(\mu)$ is positive if and only if

$$\sup_{\varepsilon>0} \varepsilon^d \mu([\varepsilon,1]) < +\infty \text{ and } \int_1^{+\infty} r^d \mu(\mathrm{d}r) < +\infty. \tag{3.6}$$

Let us remark that $\sup\limits_{\varepsilon>0} \varepsilon^d \mu([\varepsilon,1]) = +\infty$ implies that $\int_0^1 r^d \mu(\mathrm{d}r) = +\infty$.

Compared with the coverage function introduced in Section 3, we clearly have $\psi(\mu) \leq \theta(\mu)$ and hence $\lambda_c(\mu) \leq \lambda^*(\mu)$. The exact value of the critical intensity for coverage yields an upper bound for the percolation critical intensity:

$$\lambda_c(\mu) \leq \frac{d}{\ell(\mu)} \ ,$$

where $\ell(\mu)$ is given by (3.2). It is worth emphasizing that such a universal bound can be very useful for applications. Actually, percolation thresholds are usually estimated using numerical simulations (see Zuyev and Quintanilla (2003) for a theoretical basis in the framework of 2 dimensional Boolean models).

Another percolation point of view consists in considering the existence of large connected components in the complementary set of Ξ, called vacancy percolation. For this purpose, following Broman and Camia (2010) and Meester and Roy (1996), let us consider the new critical intensity $\lambda_f(\mu)$ as follows. Let $\theta_f(\mu)$ be the probability that $F = [0,1]^d \cap \Xi^c$ contains a connected component larger than one point. Then

$$\lambda_f(\mu) = \inf\{\lambda \geq 0; \theta_f(\lambda\mu) = 0\}.$$

Note that $\dim_{\mathcal{H}}(F) < 1$ implies that F is totally disconnected, which means that connected components are reduced to points (see Falconer (1990)). Therefore Theorem 3.6 yields the following upper bound

$$\lambda_f(\mu) \leq \frac{d-1}{\ell(\mu)} \ .$$

In particular, when $\ell(\mu) \in (0,+\infty)$, we clearly have $\lambda_f(\mu) < \lambda^*(\mu)$ so that, as λ increases, a phase transition of "dust" is observed in Ξ^c until it becomes empty.

We illustrate these results with typical examples in the next section.

4. Examples

4.1. *Power law type.* In a former study Biermé et al. (2010) studied random balls models with the locally finite measure μ specified to be, as $r \to 0$ or $r \to +\infty$, asymptotically of a power law type

$$r^{-\beta-1}\,\mathbf{1}_{(0,+\infty)}(r)\,\mathrm{d}r.$$

Our concern was to exhibit self-similar properties of the associated shot-noise field. Concerning the covering problem, these measures are canonical test examples and two different behaviors are observed for β less or greater than d. Indeed, applying Propositions 2.2, 2.4 and 2.5 to measures μ, that behave asymptotically like a power law yields the following.

- For $\mu(\mathrm{d}r) = f(r)\mathbf{1}_{(0,+\infty)}(r)\,\mathrm{d}r$ with $f(r) \underset{r\to+\infty}{\sim} \lambda r^{-\beta-1}$ for some $\lambda > 0$ and $\beta \in \mathbb{R}$,
$$\psi(\mu_{_L}) = 1 \text{ if and only if } \beta \le d\ .$$
- For $\mu(\mathrm{d}r) = f(r)\mathbf{1}_{(0,+\infty)}(r)\,\mathrm{d}r$ with $f(r) \underset{r\to 0}{\sim} \lambda r^{-\beta-1}$ for some $\lambda > 0$ and $\beta \in \mathbb{R}$,
$$\text{if } \beta > d \text{ then } \psi(\mu_{_H}) = 1, \text{ and if } \beta < d \text{ then } \psi(\mu_{_H}) = 0\ .$$

Let us now concentrate on the case $\beta = d$ where a phase transition appears. So let us consider

$$\mu(\mathrm{d}r) = r^{-d-1}\mathbf{1}_{(0,+\infty)}(r)\,\mathrm{d}r\ . \tag{4.1}$$

In this case the associated intensity measure $\nu(\mathrm{d}x,\mathrm{d}r) = \mathrm{d}x\mu(\mathrm{d}r)$ is scale invariant in the following sense: for measurable sets $A \subset \mathbb{R}^d \times (0,+\infty)$ with $\nu(A) < +\infty$, writing $sA = \{y \in \mathbb{R}^d \times (0,+\infty); y/s \in A\}$ we get

$$\nu(sA) = \nu(A).$$

Such an intensity is usually chosen when considering multiplicative cascades (see Barral and Mandelbrot (2002) or Chainais (2007) for instance), or Poisson random fractals as in Broman and Camia (2010). It is straightforward to get that $\lambda^*(\mu_{_L}) = 0$ while $\ell(\mu) = v_d \in (0,+\infty)$, so that, following Theorem 3.1, we obtain

$$\lambda^*(\mu_{_H}) = \frac{d}{v_d}.$$

Note that, for the one-dimensional case, the critical value (equal to 1/2) is mentioned in Kahane (2000). One can also remark that in dimension d, $v_d > d$ when $d \le 5$ whereas $v_d < d$ when $d \ge 6$. Therefore, high frequency coverage is obtained for the scale invariant intensity measure ($\lambda^*(\mu_H) < 1$) if and only if $d \le 5$. When $d \ge 6$, almost surely $\mathbb{R}^d$ is not covered by Ξ_H and considering F the vacant region inside $[0,1]^d$, its Hausdorff measure is given by $d - v_d$ with positive probability, according to Theorem 3.6.

In Broman and Camia (2010), considering the scale invariance of (4.1), the authors investigate the connectivity properties of

$$\widetilde{\Xi_{_H}} = \underset{(x,r)\in\Phi; r\le 1}{\cup} \overline{B(x,r)}.$$

More precisely they consider vacancy percolation and prove in Theorem 2.4 that there exists $\widetilde{\lambda}_f(\mu) \in (0, +\infty)$ such that with probability one the complement of $\widetilde{\Xi_H}$ in any domain of $\mathbb{R}^d$ contains connected components larger than one point if $\lambda \leq \widetilde{\lambda}_f(\mu)$ and is totally disconnected if $\lambda > \widetilde{\lambda}_f(\mu)$. Note that $\widetilde{\lambda}_f(\mu) \leq \lambda_f(\mu_H)$, where $\lambda_f(\mu_H)$ has been introduced in Section 3.3, such that

$$\widetilde{\lambda}_f(\mu) \leq \lambda_f(\mu) \leq \frac{d-1}{v_d} < \frac{d}{v_d} = \lambda^*(\mu_H).$$

Moreover, Proposition 2.4 can be adapted to prove that when $\lambda < \lambda^*(\mu_H)$ the set $\widetilde{\Xi_H}$ does not cover $\mathbb{R}^d$ almost surely. This reveals the existence of a phase transition of dust before coverage.

Finally, it is also clear that μ_H satisfies (3.6) so that $\lambda_c(\mu_H) \in (0, \lambda^*(\mu_H)]$. We conjecture that $\lambda_c(\mu_H) < \lambda^*(\mu_H)$ so that a phase transition should also be observed between percolation and coverage. Another issue should be to compare $\lambda_c(\mu_H)$ and $\lambda_f(\mu_H)$ in order to exhibit (when $d \geq 3$) or not (when $d = 2$) coexistence of unbounded connected components in the vacant and in the occupied region as it is the case in the Boolean model (see Penrose (1996)).

4.2. *Multiscale Boolean model, coverage and percolation.* For all $\rho > 1$, we consider the measure μ^ρ defined by:

$$\mu^\rho = \sum_{n \geq 0} \rho^{nd} \delta_{\rho^{-n}},$$

where $\delta_{\rho^{-n}}$ is the Dirac measure at point ρ^{-n}. The associated random set Ξ^ρ can be built as the union of independent copies of $\rho^{-n}\Xi_{=1}$, $n \geq 1$, where $\Xi_{=1}$ is the random set associated with the Dirac measure δ_1. Actually, $\Xi_{=1}$ is the classical Boolean model made of balls of identical radius 1. The random set Ξ^ρ is called Multiscale Boolean model and has been studied for instance in Chapter 8 of Meester and Roy (1996) (case $d = 2, \rho = 2$) and more generally in Menshikov et al. (2001) and Gouéré (2009).

We can simply compute $\ell(\mu^\rho) = \frac{v_d}{\ln(\rho)}$. Then, applying Theorem 3.1, we get, for all $\rho > 0$

$$\lambda^*(\mu^\rho) = \frac{d \ln(\rho)}{v_d}.$$

Note that when $\rho = 2$ and $d = 2$ one has $\lambda^*(\mu^\rho) = \frac{2\ln(2)}{\pi}$. This significantly improves Meester and Roy (1996) that only gives a lower bound equal to $\frac{2\ln(2)}{45}$ and an upper bound equal to $8\ln(2)$. Moreover, a consequence of Theorem 8.1 of Meester and Roy (1996) and Theorem 3.6 is that $0 < \lambda_f(\mu^\rho) \leq \frac{\ln(2)}{\pi}$, so that we also obtain a phase transition of dust. Note that in the general case $d \geq 2$ and $\rho > 1$, one can still observe a phase of dust since

$$\lambda_f(\mu^\rho) \leq \frac{(d-1)\ln(\rho)}{v_d} < \lambda^*(\mu^\rho)\ .$$

For $d \geq 2$, concerning the occupancy percolation, one can check that $\sup_{\varepsilon>0} \varepsilon^d \mu^\rho([\varepsilon, 1]) = \frac{1}{d\ln(\rho)}$ such that μ^ρ satisfies (3.6). Moreover, it is clear that $\lambda_c(\mu^\rho) \leq \lambda_c(\delta_1)$. Then, according to Menshikov et al. (2001, Theorem 1.1), we get

$$\lim_{\rho \to +\infty} \lambda_c(\mu^\rho) = \lambda_c(\delta_1).$$

Remark that $\lim_{\rho\to+\infty} \lambda^*(\mu^\rho) = +\infty = \lambda^*(\delta_1)$. Hence, for ρ large enough, $\lambda_c(\mu^\rho) < \lambda^*(\mu^\rho)$ and for any $\lambda \in (\lambda_c(\mu^\rho), \lambda^*(\mu^\rho))$, the measure $\lambda\mu^\rho$ gives percolation without covering the whole space.

5. Proofs

5.1. *Proof of Theorem 2.3.* The only thing to prove is that both asumptions $\psi(\mu_L) = 0$ and $\psi(\mu) = 1$ imply $\psi(\mu_H) = 1$. We start with the following lemma, which states that if $\mathbb{R}^d$ is not covered by Ξ_L then no given ball intersects Ξ_L with positive probability.

Lemma 5.1. *If low frequency coverage does not hold, i.e. if $\psi(\mu_L) = 0$, then*

$$\forall y \in \mathbb{R}^d,\ \forall \rho > 0,\ \mathbb{P}\left(B(y,\rho) \cap \Xi_L = \emptyset\right) > 0\ .$$

Proof: Assume that $\mathbb{P}\left(\mathbb{R}^d \subset \Xi_L\right) = 0$. By stationarity the result follows if we prove that

$$\forall \rho > 0,\ \mathbb{P}\left(B(0,\rho) \cap \Xi_L = \emptyset\right) > 0.$$

Let $\rho > 0$ and denote $\delta = 1 + 2\rho$. By (2.1), $\mathbb{P}\left(\mathbb{R}^d \subset \Xi^\delta_{>1}\right) = 0$, and according to Lemma 2.1 *iii*),

$$\mathbb{P}\left(\overline{B(0,\rho)} \not\subseteq \Xi^\delta_{>1}\right) > 0\ .$$

Let $y \in \overline{B(0,\rho)}$. Assume that $y \notin \Xi^\delta_{>1}$. Then for any $(x, r) \in \Phi$ with $r > 1$,

$$|x| \geq |x - y| - |y| \geq \delta r - \rho \geq r + \rho$$

which implies that $B(0,\rho) \cap B(x,r) = \emptyset$. Hence $B(0,\rho) \cap \Xi_L = \emptyset$. We have established that

$$\{\overline{B(0,\rho)} \not\subseteq \Xi^\delta_{>1}\} \subset \{B(0,\rho) \cap \Xi_L = \emptyset\}$$

and Lemma 5.1 follows. □

Let us come back to the proof of Theorem 2.3 and suppose that low frequency coverage does not hold whereas coverage does. By Lemma 5.1, $\mathbb{P}\left(B(0,1) \cap \Xi_L = \emptyset\right) > 0$. Denoting by $\eta > 0$ this probability and using the coverage assumption we get

$$\begin{aligned}
\eta &= \mathbb{P}\big((B(0,1) \cap \Xi_L = \emptyset) \cap (B(0,1) \subset \Xi)\big) \\
&= \mathbb{P}\big((B(0,1) \cap \Xi_L = \emptyset) \cap (B(0,1) \subset \Xi_H)\big) \\
&= \mathbb{P}(B(0,1) \cap \Xi_L = \emptyset) \times \mathbb{P}(B(0,1) \subset \Xi_H) \\
&= \eta \times \mathbb{P}(B(0,1) \subset \Xi_H)
\end{aligned}$$

and therefore $\mathbb{P}(B(0,1) \subset \Xi_H) = 1$. Lemma 2.1 *iii*) implies that high frequency coverage occurs, which concludes the proof of Theorem 2.3.

5.2. *Proof of Propositions 2.4 and 3.4.* We can assume that the measure μ is supported by $(0,1]$, so that $\mu = \mu_H$, and consider A a compact set of $[0,1]^d$. We will establish that one of the following assumptions

- either $\displaystyle\int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(\mathrm{d}r)\right) \mathrm{d}u < +\infty$,
- or $\ell(\mu) < d$ and $\dim_{\mathcal{H}}(A) > \ell(\mu)$,

implies $\mathbb{P}(A \not\subset \Xi) > 0$. This will prove either Proposition 2.4 (using Lemma 2.1 *ii*)) or Proposition 3.4.

Hence, let m be a probability measure supported by A. We recall that for $\varepsilon > 0$, $\Xi_{\geq\varepsilon}$ denotes $\underset{(x,r)\in\Phi; r\geq\varepsilon}{\cup} B(x,r)$. We consider

$$m_\varepsilon = m\left(A \cap \Xi^c_{\geq\varepsilon}\right) = \int_A \mathbf{1}_{y\notin\Xi_{\geq\varepsilon}} m(\mathrm{d}y).$$

Then, by Fubini,

$$\mathbb{E}(m_\varepsilon) = \int_A \mathbb{P}(y \notin \Xi_{\geq\varepsilon}) m(\mathrm{d}y),$$

and with a similar computation as for (1.1), we get for any $y \in \mathbb{R}^d$,

$$\mathbb{P}(y \notin \Xi_{\geq\varepsilon}) = e^{-\kappa_\varepsilon},$$

where $\kappa_\varepsilon := v_d \int_\varepsilon^1 r^d \mu(\mathrm{d}r)$. Since m is a probability measure on A, we get

$$\mathbb{E}(m_\varepsilon) = e^{-\kappa_\varepsilon}.$$

Moreover, by Fubini again,

$$\mathbb{E}(m_\varepsilon^2) = \int_{A\times A} \mathbb{P}(y \notin \Xi_{\geq\varepsilon}, z \notin \Xi_{\geq\varepsilon})\, m(\mathrm{d}y) m(\mathrm{d}z).$$

Lemma 5.2. *There exists some positive constant $b \leq 1/2$ such that for any $y, z \in \mathbb{R}^d$,*

$$\mathbb{P}(y \notin \Xi_{\geq\varepsilon}, z \notin \Xi_{\geq\varepsilon}) \leq e^{-2\kappa_\varepsilon} \exp\left(v_d \int_\varepsilon^1 r^{d-1}\,(r - b|y-z|)_+\, \mu(\mathrm{d}r)\right)$$

Proof of the lemma: For any $y, z \in \mathbb{R}^d$, we write

$$\begin{aligned}
\mathbb{P}(y \notin \Xi_{\geq\varepsilon}, z \notin \Xi_{\geq\varepsilon}) &= \exp\left(-\int_{\mathbb{R}^d}\int_\varepsilon^1 \mathbf{1}_{B(y,r)\cup B(z,r)}(x)\mathrm{d}x\mu(\mathrm{d}r)\right) \\
&= \exp\left(-2\kappa_\varepsilon + \int_{\mathbb{R}^d}\int_\varepsilon^1 \mathbf{1}_{B(y,r)\cap B(z,r)}(x)\mathrm{d}x\mu(\mathrm{d}r)\right) \\
&= e^{-2\kappa_\varepsilon} \exp\left(\int_\varepsilon^1 \gamma_d(|y-z|, r)\mu(\mathrm{d}r)\right)
\end{aligned}$$

where for $u, r > 0$, $\gamma_d(u, r)$ denotes the Lebesgue measure of the intersection of two balls in $\mathbb{R}^d$ with a common radius r and whose centers are at distance u (in Kahane (1990), γ_d is called "pagoda" function). We will prove that γ_d satisfies the following: there exists a constant $b \in (0, 1/2]$ such that for any $u \geq 0, r > 0$

$$\gamma_d(u, r) \leq v_d\, r^{d-1}(r - bu)_+ \tag{5.1}$$

where $(x)_+$ denotes the positive part of any real x, ie $(x)_+ = x\mathbf{1}_{x\geq 0}$. First let us assume that $d = 1$ and remark that $\gamma_1(u, r) = 2\,(r - u/2)_+ = v_1\,(r - u/2)_+$ so that (5.1) is satisfied with $b = 1/2$. In the general case $d \geq 2$, on the one hand we notice that $\gamma_d(u, r) = 0$ for all $u \geq 2r$ so that (5.1) holds in this case whatever the constant $b \leq 1/2$ is. On the other hand, for $u < 2r$, let us write for $e \in \mathbb{R}^d$ a fixed direction

$$\begin{aligned}\gamma_d(u,r) &= r^d \int_{\mathbb{R}^d} \mathbf{1}_{B((u/r)e,1)\cap B(0,1)}(\xi)\mathrm{d}\xi \\ &= r^d \int_{z\in\mathbb{R}^{d-1};|z|<1} \gamma_1\left(u/r, \sqrt{1-|z|^2}\right)\mathrm{d}z \\ &= r^d \int_{z\in\mathbb{R}^{d-1};|z|<1} \left(2\sqrt{1-|z|^2} - u/r\right)_+ \mathrm{d}z = v_d r^d - \delta(u,r),\end{aligned}$$

where

$$\delta(u,r) = r^d \left(\int_{z\in\mathbb{R}^{d-1};2\sqrt{1-|z|^2}\leq u/r} 2\sqrt{1-|z|^2}\,\mathrm{d}z + \frac{u}{r}\int_{z\in\mathbb{R}^{d-1};2\sqrt{1-|z|^2}>u/r} \mathrm{d}z \right) .$$

In order to establish inequality (5.1), we will prove that $\delta(u,r) \geq Cur^{d-1}$ for some positive constant C, distinguishing the two cases: u/r less or greater than 1. If $u/r < 1$ then

$$\delta(u,r) \geq r^d\,\frac{u}{r}\left(\int_{z\in\mathbb{R}^{d-1};2\sqrt{1-|z|^2}>1} \mathrm{d}z\right) ,$$

whereas, if $1 \leq u/r < 2$ then

$$\delta(u,r) \geq r^d\,\frac{u}{r}\left(\int_{z\in\mathbb{R}^{d-1};2\sqrt{1-|z|^2}\leq 1} \sqrt{1-|z|^2}\,\mathrm{d}z\right) .$$

Hence, taking $b = \min(1/2\,, C)$ concludes the proof of Lemma 5.2. □

Therefore,

$$\mathbb{E}(m_\varepsilon^2) \leq e^{-2\kappa_\varepsilon} \int_{A\times A} \exp\left(v_d \int_\varepsilon^1 r^{d-1}\,(r - b|y-z|)_+\,\mu(\mathrm{d}r)\right) m(\mathrm{d}y)m(\mathrm{d}z). \quad (5.2)$$

<u>First case:</u> Let us consider $A = [0,1]^d$ and m the Lebesgue measure on A. By translation invariance and change in polar coordinates we get

$$\begin{aligned}\mathbb{E}(m_\varepsilon^2) &\leq e^{-2\kappa_\varepsilon} \int_{\mathbb{R}^d} \mathbf{1}_{|z|\leq\sqrt{d}} \exp\left(v_d \int_\varepsilon^1 r^{d-1}\,(r-b|z|)_+\,\mu(\mathrm{d}r)\right)\mathrm{d}z \\ &= e^{-2\kappa_\varepsilon}\, dv_d\, b^{-d} \int_0^{b\sqrt{d}} u^{d-1} \exp\left(v_d \int_\varepsilon^1 r^{d-1}\,(r-u)_+\,\mu(\mathrm{d}r)\right)\mathrm{d}u \\ &\leq J_d(\mu) e^{-2\kappa_\varepsilon} ,\end{aligned}$$

where $J_d(\mu) = dv_d\, b^{-d} \int_0^{b\sqrt{d}} u^{d-1} \exp\left(v_d \int_0^1 r^{d-1}\,(r-u)_+\,\mu(\mathrm{d}r)\right)\mathrm{d}u \in (0,+\infty)$ when assuming that

$$\int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(\mathrm{d}r)\right)\mathrm{d}u < +\infty .$$

<u>Second case:</u> Let us assume that $\ell(\mu) < d$ and consider A such that $\dim_{\mathcal{H}}(A) > \ell(\mu)$. Then, let us choose $\delta > 0$ such that $\dim_{\mathcal{H}}(A) > \ell(\mu) + 2\delta > \ell(\mu)$. According to Frostman's lemma (see Falconer (1990) for instance), one can choose for m a measure carried by A such that

$$I_{\ell(\mu)+\delta}(m) := \int_{A\times A} |y-z|^{-(\ell(\mu)+\delta)} m(\mathrm{d}y)m(\mathrm{d}z) \in (0,+\infty).$$

According to the definition of $\ell(\mu)$ (see (3.2)), for all $y,z\in[0,1]^d$ such that $b|y-z|$ is small enough,

$$v_d\int_{b|y-z|}^1 r^d\,\mu(\mathrm{d}r)\le-(\ell(\mu)+\delta)\,\ln(b|y-z|)\ .$$

Then, using (5.2), one can find a positive constant C_δ such that

$$\mathbb{E}(m_\varepsilon^2)\le C_\delta e^{-2\kappa_\varepsilon}I_{\ell(\mu)+\delta}(m).$$

Therefore, in both cases, using Cauchy-Schwarz inequality we obtain

$$\mathbb{P}(m_\varepsilon>0)\ge\frac{\mathbb{E}(m_\varepsilon)^2}{\mathbb{E}(m_\varepsilon^2)}\ge c\ ,$$

with $c=1/J_d(\mu)>0$ in the first case and $c=1/(C_\delta I_{\ell(\mu)+\delta}(m))>0$ in the second case. Hence,

$$\mathbb{P}(A\not\subset\Xi_\varepsilon)\ge c.$$

Taking the limit as ε tends to 0 we get

$$\mathbb{P}\left(\bigcap_{\varepsilon>0}\{A\not\subset\Xi_\varepsilon\}\right)\ge c.$$

Using the compactness of A we finally obtain

$$\mathbb{P}\left(A\not\subset\Xi\right)\ge c>0.$$

This concludes the proof of Proposition 2.4 and Proposition 3.4.

5.3. *Proof of Propositions 2.5 and 3.5.* We can again assume that the measure μ is supported by $(0,1]$. Let us remark that for $r>\varepsilon>0$

$$\overline{B(y,\varepsilon)}\subset B(x,r)\Leftrightarrow y\in B(x,r-\varepsilon),$$

and that $y+[-\varepsilon/\sqrt{d},\varepsilon/\sqrt{d}]^d\subset\overline{B(y,\varepsilon)}$. Therefore,

$$\begin{aligned}\mathbb{P}(y+[-\varepsilon/\sqrt{d},\varepsilon/\sqrt{d}]^d\nsubseteq\Xi)&\le\mathbb{P}(\forall(x,r)\in\Phi\text{ with }r>\varepsilon,\ [-\varepsilon/\sqrt{d},\varepsilon/\sqrt{d}]^d\nsubseteq B(x,r))\\&\le\mathbb{P}(\forall(x,r)\in\Phi\text{ with }r>\varepsilon,\ 0\notin B(x,r-\varepsilon))\\&\le\exp\left(-v_d\int_\varepsilon^1(r-\varepsilon)^d\mu(\mathrm{d}r)\right).\end{aligned}\tag{5.3}$$

<u>First case:</u> (Proof of Proposition 2.5). Let us assume that

$$\limsup_{\varepsilon\to0}\varepsilon^d\exp\left(v_d\int_\varepsilon^1(r-\varepsilon)^d\mu(\mathrm{d}r)\right)=+\infty.\tag{5.4}$$

Then for any $\varepsilon>0$,

$$\begin{aligned}\mathbb{P}([0,1]^d\nsubseteq\Xi)&\le\sqrt{d}^d\varepsilon^{-d}\,\mathbb{P}([-\varepsilon/\sqrt{d},\varepsilon/\sqrt{d}]^d\nsubseteq\Xi)\\&\le\sqrt{d}^d\varepsilon^{-d}\exp\left(-v_d\int_\varepsilon^1(r-\varepsilon)^d\mu(\mathrm{d}r)\right).\end{aligned}$$

Hence, according to (5.4) we can choose $\varepsilon\to0$ in an appropriate way such that we get $\mathbb{P}([0,1]^d\nsubseteq\Xi)=0$ and Lemma 2.1 allows us to conclude that $\mathbb{R}^d$ is almost surely covered by Ξ.

Second case: (Proof of Proposition 3.5). Let us assume that $\ell(\mu) > 0$ and A is a compact set of $[0,1]^d$ with $\dim_{\mathcal{H}}(A) < \ell(\mu)$. Note that for any $\alpha > 1$ and any small enough $\varepsilon > 0$, the following inequality holds

$$\int_{\varepsilon}^{1} (r-\varepsilon)^d \, \mu(\mathrm{d}r) \geq \int_{\alpha\varepsilon}^{1} (r-\varepsilon)^d \, \mu(\mathrm{d}r) \geq (1-1/\alpha)^d \int_{\alpha\varepsilon}^{1} r^d \, \mu(\mathrm{d}r) \ . \tag{5.5}$$

Let us choose $\alpha > 1$ and $l \in (0, \ell(\mu))$ such that $l\,(1-1/\alpha)^d > \dim_{\mathcal{H}}(A)$. Then, one can find ε arbitrarily small such that

$$\left(|\ln(\alpha\varepsilon)|^{-1} \int_{\alpha\varepsilon}^{1} v_d r^d \, \mu(\mathrm{d}r) \right) > l \ .$$

We get for an arbitrary small ε,

$$\exp\left(-v_d \int_{\varepsilon}^{1} (r-\varepsilon)^d \mu(\mathrm{d}r) \right) \leq \exp\left(-l\,(1-1/\alpha)^d |\ln(\alpha\varepsilon)| \right) = \alpha^{l(1-1/\alpha)^d} \varepsilon^{l\,(1-1/\alpha)^d}. \tag{5.6}$$

Let $\eta > 0$ be fixed. Since $\dim_{\mathcal{H}}(A) < l\,(1-1/\alpha)^d$, we can choose an appropriate covering of A with N hypercubes $Q_1, \ldots, Q_N$ of side size $2\varepsilon_1/\sqrt{d}, \ldots, 2\varepsilon_N/\sqrt{d}$ such that

$$\sum_{1 \leq i \leq N} (\varepsilon_i)^{l\,(1-1/\alpha)^d} \leq \eta \ .$$

Using (5.3) and (5.6) for ε_i, we get

$$\mathbb{P}(A \nsubseteq \Xi) \leq \sum_{1 \leq i \leq N} \mathbb{P}(Q_i \nsubseteq \Xi) \leq \alpha^{l(1-1/\alpha)^d} \eta \ .$$

Hence $\mathbb{P}(A \nsubseteq \Xi) = 0$, which proves that A is almost surely covered by Ξ.

5.4. *Proof of Theorem 3.1.* By Theorem 2.3, $\psi(\mu) = \max(\psi(\mu_H), \psi(\mu_L))$, and therefore

$$\lambda^*(\mu) = \min(\lambda^*(\mu_H), \lambda^*(\mu_L)).$$

According to Proposition 2.2, we have $\lambda^*(\mu_L) \in \{0, +\infty\}$ with $\lambda^*(\mu_L) = +\infty$ if and only if $\int_1^{+\infty} r^d \mu(\mathrm{d}r) < +\infty$. Hence, if this last integral is infinite, then $\lambda^*(\mu_L) = 0$.

Let us now be concerned with $\lambda^*(\mu_H)$ and let us recall that the quantity $\ell(\mu)$ is introduced in (3.2). The following simple criterion follows from the necessary and sufficient conditions of Propositions 2.4 and 2.5.

Lemma 5.3. *Let μ be a locally finite non-negative measure on $(0, +\infty)$.*

(i) *If $\ell(\mu) > d$ then $\psi(\mu_H) = 1$.*
(ii) *If $\ell(\mu) < d$ then $\psi(\mu_H) = 0$.*

Proof: (i) Let us assume that $\ell(\mu) > d$. Let $l \in (d, \ell(\mu))$ and choose $\alpha > 1$ such that $l\,(1-1/\alpha)^d > d$. Then, one can find u arbitrarily small such that

$$\left(|\ln(\alpha u)|^{-1} \int_{\alpha u}^{1} v_d r^d \, \mu(\mathrm{d}r) \right) > l \ .$$

Using similar inequalities as in (5.5), we get for an arbitrary small u,

$$u^d \exp\left(v_d \int_{u}^{1} (r-u)^d \mu(\mathrm{d}r) \right) \geq u^d \exp\left(l\,(1-1/\alpha)^d |\ln(\alpha u)| \right) \ .$$

Writing the right hand side term as $u^{d-l\,(1-1/\alpha)^d}\alpha^{-l(1-1/\alpha)^d}$ proves that the sufficient coverage condition of Proposition 2.5 holds and hence $\psi(\mu_H)=1$. (ii) follows easily from the obvious inequality $\int_u^1 r^{d-1}(r-u)\,\mu(\mathrm{d}r)\le\int_u^1 r^d\,\mu(\mathrm{d}r)$ and Proposition 2.4. □

Since $\ell(\lambda\mu)=\lambda\ell(\mu)$, for any $\lambda>0$, we obtain that $\lambda^*(\mu_H)=d/\ell(\mu)$ with $\lambda^*(\mu_H)=0$ if and only if $\ell(\mu)=+\infty$, and $\lambda^*(\mu_H)=+\infty$ if and only if $\ell(\mu)=0$.

Acknowledgment. The authors are very grateful to Jean-Baptiste Gouéré, for his valuable work to improve a first version of this paper. They would like also to thank Professor Jean-Pierre Kahane for letting them know about the work of El Helou (1978).

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