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Fat fractal percolation and k-fractal percolation

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Abstract. We consider two variations on the Mandelbrot fractal percolation model. In the *k-fractal percolation model*, the d-dimensional unit cube is divided in N^d equal subcubes, k of which are retained while the others are discarded. The procedure is then iterated inside the retained cubes at all smaller scales. We show that the (properly rescaled) percolation critical value of this model converges to the critical value of ordinary site percolation on a particular d-dimensional lattice as $N \to \infty$. This is analogous to the result of Falconer and Grimmett (1992) that the critical value for Mandelbrot fractal percolation converges to the critical value of site percolation on the same d-dimensional lattice.

In the *fat fractal percolation model*, subcubes are retained with probability p_n at step n of the construction, where $(p_n)_{n\geq 1}$ is a non-decreasing sequence with $\prod_{n=1}^{\infty} p_n > 0$. The Lebesgue measure of the limit set is positive a.s. given non-extinction. We prove that either the set of connected components larger than one point has Lebesgue measure zero a.s. or its complement in the limit set has Lebesgue measure zero a.s.

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1. Introduction

Mandelbrot (1982) introduced the following fractal percolation model. Let $N \geq 2, d \geq 2$ be integers and consider the unit cube $[0,1]^d$. Divide the unit cube into N^d subcubes of side length $1/N$. Each subcube is retained with probability p and discarded with probability $1-p$, independently of other subcubes. The closure of the union of the retained subcubes forms a random subset D_p^1 of $[0,1]^d$. Next, each retained subcube in D_p^1 is divided into N^d cubes of side length $1/N^2$. Again, each smaller subcube is retained with probability p and discarded with probability $1-p$, independently of other cubes. We obtain a new random set $D_p^2 \subset D_p^1$. Iterating this procedure in every retained cube at every smaller scale yields an infinite decreasing sequence of random subsets $D_p^1 \supset D_p^2 \supset D_p^3 \supset \cdots$ of $[0,1]^d$. We define the limit set $D_p := \bigcap_{n=1}^{\infty} D_p^n$. We will refer to this model as the Mandelbrot fractal percolation (MFP) model with parameter p.

It is easy to extend and generalize the classical Mandelbrot model in ways that preserve at least a certain amount of statistical self-similarity and generate random fractal sets. It is interesting to study such models to obtain a better understanding of general fractal percolation processes and explore possible new features that are not present in the MFP model. In this paper we are concerned with two natural extensions which have previously appeared in the literature, as we mention below. We will next introduce the models and state our main results.

1.1. *k-fractal percolation.* Let $N \geq 2$ be an integer and divide the unit cube $[0,1]^d$, $d \geq 2$, into N^d subcubes of side length $1/N$. Fix an integer $0 < k \leq N^d$ and retain k subcubes in a uniform way, that is, all configurations where k cubes are retained have equal probability, other configurations have probability 0. Let D_k^1 denote the random set which is obtained by taking the closure of the union of all retained cubes. Iterating the described procedure in retained cubes and on all smaller scales yields a decreasing sequence of random sets $D_k^1 \supset D_k^2 \supset D_k^3 \supset \cdots$. We are mainly interested in the connectivity properties of the limiting set $D_k := \bigcap_{n=1}^{\infty} D_k^n$. This model was called the *micro-canonical fractal percolation process* by Chayes (1995) and both *correlated fractal percolation* and *k out of N^d fractal percolation* by Dekking and Don (2010). We will adopt the terms *k-fractal percolation* and *k-model*.

For $F \subset [0,1]^d$, we say that the unit cube is *crossed by F* if there exists a connected component of F which intersects both $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. Define $\theta(k,N,d)$ as the probability that $[0,1]^d$ is crossed by D_k. Similarly, $\sigma(p,N,d)$ denotes the probability that $[0,1]^d$ is crossed by D_p. Let us define the critical probability $p_c(N,d)$ for the MFP model and the critical threshold value $k_c(N,d)$ for the k-model by

$$p_c(N,d) := \inf\{p : \sigma(p,N,d) > 0\}, \quad k_c(N,d) := \min\{k : \theta(k,N,d) > 0\}.$$

Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $(x_1,\ldots,x_d) = x \sim y = (y_1,\ldots,y_d)$ if and only if $x \neq y$, $|x_i - y_i| \leq 1$ for all i and $x_i = y_i$ for at least one value of i. Let $p_c(d)$ denote the critical probability for site percolation on $\mathbb{L}^d$. It is known (see Falconer and Grimmett (1992)) that $p_c(N,d) \to p_c(d)$ as $N \to \infty$. We have the following analogous result for the k-model.

Theorem 1.1. *For all $d \geq 2$, we have that*

$$\lim_{N\to\infty} \frac{k_c(N,d)}{N^d} = p_c(d).$$

Remark 1.2. Note that the choice for the unit cube in the definitions of $\theta(k,N,d)$ and $\sigma(p,N,d)$ (and thus implicitly also in the definitions of $k_c(N,d)$ and $p_c(N,d)$) is rather arbitrary: We could define them in terms of crossings of other shapes such as annuli, for example, and obtain the same conclusion, i.e. $k_c(N,d)/N^d \to p_c(d)$ as $N \to \infty$, where $\theta(k,N,d)$ and $k_c(N,d)$ are defined using the probability that D_k crosses an annulus. One advantage of using annuli is that the percolation function $\sigma(p,N,d)$ is known to have a discontinuity at $p_c(N,d)$ for all N,d and any choice of annulus (Broman and Camia (2010, Corollary 2.6)). (This is known to be the case also when $p_c(N,d)$ is defined using the unit cube if $d = 2$ (Chayes et al. (1988); Dekking and Meester (1990)), but for $d \geq 3$ it is proven only for N sufficiently large (Broman and Camia (2008)).) In the present paper we stick to the "traditional" choice of the unit cube.

Remark 1.3. For the MFP model it is the case that, for $p > p_c(d)$,

$$\sigma(p,N,d) \to 1, \tag{1.1}$$

as $N \to \infty$. This is part (b) of Theorem 2 in Falconer and Grimmett (1992). During the course of the proof of Theorem 1.1 we will prove a similar result for the k-model, see Theorem 3.2.

Next, consider the following generalization of both the k-model and the MFP model. Let $d \geq 2, N \geq 2$ be integers and let $Y = Y(N,d)$ be a random variable taking values in $\{0,\ldots,N^d\}$. Divide the unit cube into N^d smaller cubes of side length $1/N$. Draw a realization y according to Y and retain y cubes uniformly. Let D_Y^1 denote the closure of the union of the retained cubes. Next, every retained cube is divided into N^d smaller subcubes of side length $1/N^2$. Then, for every subcube C in D_Y^1 (where we slightly abuse notation by viewing D_Y^1 as the set of retained cubes in the first iteration step) draw a new (independent) realization $y(C)$ of Y and retain $y(C)$ subcubes in C uniformly, independently of all other subcubes. Denote the closure of the union of retained subcubes by D_Y^2. Repeat this procedure in every retained subcube at every smaller scale and define the limit set $D_Y := \bigcap_{n=1}^{\infty} D_Y^n$. We will call this model the *generalized fractal percolation model* (GFP model) *with generator* Y. Define $\phi(Y,N,d)$ as the probability of the event that $[0,1]^d$ is crossed by D_Y.

By taking Y equal to an integer k, resp. to a binomially distributed random variable with parameters N^d and p, we obtain the k-model, resp. the MFP model with parameter p. If Y is stochastically dominated by a binomial random variable with parameters N^d and p, where $p < p_c(N,d)$, then by standard coupling techniques it follows that $\phi(Y,N,d) = 0$. Likewise, if $Y(N,d)$ dominates a binomial random variable with parameters N^d and p, where $p > p_c(d)$, then $\phi(Y(N,d),N,d) \geq \sigma(p,N,d) \to 1$ as $N \to \infty$, as mentioned in Remark 1.3. The following theorem, which generalizes (1.1), shows that the latter conclusion still holds if for some $p > p_c(d)$, $\mathbb{P}(Y(N,d) \geq pN^d) \to 1$ as $N \to \infty$.

Theorem 1.4. *Consider the GFP model with generator $Y(N,d)$. Let $p > p_c(d)$. Suppose that $\mathbb{P}(Y(N,d) \geq pN^d) \to 1$ as $N \to \infty$. Then*

$$\lim_{N\to\infty} \phi(Y(N,d),N,d) = 1.$$

Remark 1.5. Observe that by Chebyshev's inequality the condition of Theorem 1.4 is satisfied if, for some $p > p_c(d)$, $\mathbb{E}Y(N,d) \geq pN^d$ for all $N \geq 2$ and $\mathbb{V}\mathrm{ar}(Y(N,d))/N^{2d} \to 0$ as $N \to \infty$.

Open problem 1.6. It is a natural question to ask whether a "symmetric version" of Theorem 1.4 is true. That is, if e.g. $\mathbb{P}(Y(N,d) \leq pN^d) \to 1$ as $N \to \infty$, for some $p < p_c(d)$, implies $\phi(Y(N,d), N, d) \to 0$ as $N \to \infty$. The proof of Theorem 1.4 can not be adapted to this situation.

1.2. *Fat fractal percolation.* Let $(p_n)_{n\geq 1}$ be a non-decreasing sequence in $(0,1]$ such that $\prod_{n=1}^{\infty} p_n > 0$. We call *fat fractal percolation* a model analogous to the MFP model, but where at every iteration step n a subcube is retained with probability p_n and discarded with probability $1 - p_n$, independently of other subcubes. Iterating this procedure yields a decreasing sequence of random subsets $D^1_{\mathrm{fat}} \supset D^2_{\mathrm{fat}} \supset D^3_{\mathrm{fat}} \supset \cdots$ and we will mainly study connectivity properties of the limit set $D_{\mathrm{fat}} := \bigcap_{n=1}^{\infty} D^n_{\mathrm{fat}}$. In Chayes et al. (1997) it is shown that if $p_n \to 1$ and $\prod_{n=1}^{\infty} p_n = 0$, then the limit set does not contain a directed crossing from left to right.

For a point $x \in D_{\mathrm{fat}}$, let C^x_{fat} denote its *connected component*:

$$C^x_{\mathrm{fat}} := \{y \in D_{\mathrm{fat}} : y \text{ connected to } x \text{ in } D_{\mathrm{fat}}\}.$$

We define the set of "dust" points by $D^d_{\mathrm{fat}} := \{x \in D_{\mathrm{fat}} : C^x_{\mathrm{fat}} = \{x\}\}$. Define $D^c_{\mathrm{fat}} := D_{\mathrm{fat}} \setminus D^d_{\mathrm{fat}}$, which is the union of connected components larger than one point. Let λ denote the d-dimensional Lebesgue measure. It is easy to prove that $\lambda(D_{\mathrm{fat}}) > 0$ with positive probability, see Proposition 4.1. Moreover, we can show that the Lebesgue measure of the limit set is positive a.s. given non-extinction, i.e. $D_{\mathrm{fat}} \neq \emptyset$.

Theorem 1.7. *We have that $\lambda(D_{fat}) > 0$ a.s. given non-extinction.*

It is a natural question to ask whether both D^c_{fat} and D^d_{fat} have positive Lebesgue measure. The following theorem shows that they cannot simultaneously have positive Lebesgue measure.

Theorem 1.8. *Given non-extinction of the fat fractal process, it is the case that either*

$$\lambda(D^d_{fat}) = 0 \text{ and } \lambda(D^c_{fat}) > 0 \text{ a.s.} \tag{1.2}$$

or

$$\lambda(D^d_{fat}) > 0 \text{ and } \lambda(D^c_{fat}) = 0 \text{ a.s.} \tag{1.3}$$

Part (ii) of the following theorem gives a sufficient condition under which (1.2) holds. Furthermore, the theorem shows that the limit set either has an empty interior a.s. or can be written as the union of finitely many cubes a.s.

Theorem 1.9. *We have that*

- *(i) If $\prod_{n=1}^{\infty} p_n^{N^{dn}} = 0$, then D_{fat} has an empty interior a.s.;*
- *(ii) If $\prod_{n=1}^{\infty} p_n^{N^n} > 0$, then $\lambda(D^d_{fat}) = 0$ a.s.;*
- *(iii) If $\prod_{n=1}^{\infty} p_n^{N^{dn}} > 0$, then D_{fat} can be written as the union of finitely many cubes a.s.*

Open problem 1.10. Part (ii) of Theorem 1.9 shows that if $\prod_{n=1}^{\infty} p_n^{N^n} > 0$, then (1.2) holds. However, we do not have an example for which (1.3) holds, and we do not know whether (1.3) is possible at all.

In two dimensions, we have the following characterizations of $\lambda(D_{\text{fat}}^c)$ being positive a.s. given non-extinction of the fat fractal process.

Theorem 1.11. *Let $d = 2$. The following statements are equivalent.*

(i) $\lambda(D_{fat}^c) > 0$ *a.s., given non-extinction of the fat fractal process;*
(ii) *There exists a set $U \subset [0,1]^2$ with $\lambda(U) > 0$ such that for all $x, y \in U$ it is the case that $\mathbb{P}(x$ is in the same connected component as $y) > 0$;*
(iii) *There exists a set $U \subset [0,1]^2$ with $\lambda(U) = 1$ such that for all $x, y \in U$ it is the case that $\mathbb{P}(x$ is in the same connected component as $y) > 0$.*

Let us now outline the rest of the paper. The next section will be devoted to a formal introduction of the fractal percolation processes in the unit cube. We also define an ordering on the subcubes which will facilitate the proofs of Theorems 1.1 and 1.4 in Section 3. In Section 4 we prove our results concerning fat fractal percolation.

2. Preliminaries

In this section we set up an ordering for the subcubes of the fractal processes in the unit cube which will turn out to be very useful during the course of the proofs. We also give a formal probabilistic definition of the different fractal percolation models. We follow Falconer and Grimmett (1992) almost verbatim in this section; a simple reference to Falconer and Grimmett (1992) would however not be very useful for the reader, so we repeat some definitions here.

Order $J^d := \{0, 1, \ldots, N-1\}^d$ in some way, say lexicographically by coordinates. For a positive integer n, write $J^{d,n} := \{(\mathbf{i}_1, \ldots, \mathbf{i}_n) : \mathbf{i}_j \in J^d, 1 \leq j \leq n\}$ for the set of n-vectors with entries in J^d. Set $J^{d,0} := \{\emptyset\}$. With $\mathbf{I} = (\mathbf{i}_1, \ldots, \mathbf{i}_n) = ((i_{1,1}, \ldots, i_{1,d}), \ldots, (i_{n,1}, \ldots, i_{n,d}))$ we associate the subcube of $[0,1]^d$ given by

$$C(\mathbf{I}) = c(\mathbf{I}) + [0, N^{-n}]^d,$$

where

$$c(\mathbf{I}) = \left(\sum_{j=1}^{n} N^{-j} i_{j,1}, \ldots, \sum_{j=1}^{n} N^{-j} i_{j,d} \right)$$

and $c(\emptyset)$ is defined to be the origin. Such a cube $C(\mathbf{I})$ is called a *level-n cube* and we write $|\mathbf{I}| = n$. A concatenation of $\mathbf{I} \in J^{d,n}$ and $\mathbf{j} \in J^d$ is denoted by $(\mathbf{I}, \mathbf{j})$, which is in $J^{d,n+1}$. We define the set of indices for all cubes until (inclusive) level-n as $\mathcal{J}^{(n)} := J^{d,0} \cup J^{d,1} \cup \cdots \cup J^{d,n}$ and we order them in the following way. We declare $\mathbf{I} = (\mathbf{i}_1, \ldots, \mathbf{i}_a) < \mathbf{I}' = (\mathbf{i}'_1, \ldots, \mathbf{i}'_b)$ if and only if

- *either* $\mathbf{i}_r < \mathbf{i}'_r$ (according to the order on J^d) where $r \leq \min\{a, b\}$ is the smallest index so that $\mathbf{i}_r \neq \mathbf{i}'_r$ holds;
- *or* $a > b$ and $\mathbf{i}_r = \mathbf{i}'_r$ for $r = 1, \ldots, b$.

To clarify this ordering we give a short example, see Figure 2.1. Suppose $N = 2$, $d = 2$ and J^2 is ordered by $(1,1) > (1,0) > (0,1) > (0,0)$, then the ordering of

$\mathcal{J}^{(2)}$ starts with

$$\begin{aligned}\emptyset \quad &> \quad ((1,1)) > ((1,1),(1,1)) > ((1,1),(1,0)) \\ &> \quad ((1,1),(0,1)) > ((1,1),(0,0)) > ((1,0)) > \ldots\end{aligned}$$

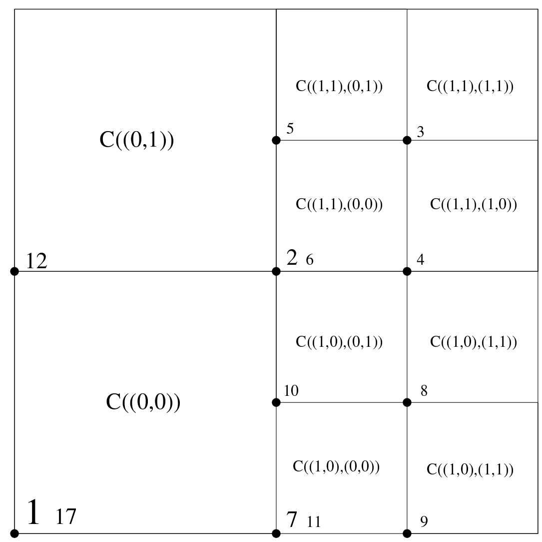

FIGURE 2.1. Illustration of the ordering of subcubes in $\mathcal{J}^{(2)}$, for $N = 2$ and $d = 2$. A black dot denotes the corner point $c(\mathbf{I})$ of a subcube $C(\mathbf{I})$. The number in the lower left corner of a subcube indicates the rank of the subcube in the ordering: e.g. the unit cube, i.e. $C(\emptyset)$, has rank 1 and $C((0,0))$ has rank 17.

We introduce the following formal probabilistic definition of the fractal percolation models. As noted before, the k-model and MFP model can be obtained from the GFP model with generator Y by setting $Y \equiv k$, resp. Y binomially distributed with parameters N^d and $p \in [0,1]$. Therefore, we only provide a formal probabilistic definition of the GFP model and the fat fractal percolation model. Define the index set $\mathcal{J} := \bigcup_{n=0}^{\infty} J^{d,n}$. We define a family of random variables $\{Z_{\text{model}}(\mathbf{I})\}$, where $\mathbf{I} \in \mathcal{J}$ and – here as well as in the rest of the section – "model" stands for either p, fat, k or Y.

1. GFP model with generator Y: For every $\mathbf{I} \in \mathcal{J}$, let $y(\mathbf{I})$ denote a realization of Y, independently of other $\mathbf{I}'$. We define $J(\mathbf{I})$ as a uniform choice of $y(\mathbf{I})$ different indices of J^d, independently of other $J(\mathbf{I}')$. For $\mathbf{j} \in J^d$ define
$$Z_Y(\mathbf{I},\mathbf{j}) = \begin{cases} 1, & \mathbf{j} \in J(\mathbf{I}), \\ 0, & \text{otherwise.} \end{cases}$$
2. Fat fractal percolation with parameters $(p_n)_{n\geq 1}$: For every $\mathbf{I} \in \mathcal{J}$ and $\mathbf{j} \in J^d$, let $n = |\mathbf{I}|$ and define
$$Z_{\text{fat}}(\mathbf{I},\mathbf{j}) = \begin{cases} 1, & \text{with probability } p_{n+1}, \\ 0, & \text{with probability } 1 - p_{n+1}, \end{cases}$$
independently of all other $Z_{\text{fat}}(\mathbf{I}')$.

For each $\mathbf{I} \in \mathcal{J}$ we define the indicator function $1_{\text{model}}(\mathbf{I})$ by

$$1_{\text{model}}(\emptyset) = 1, \quad 1_{\text{model}}(\mathbf{I}) = Z_{\text{model}}(\mathbf{i}_1)Z_{\text{model}}(\mathbf{i}_1, \mathbf{i}_2) \cdots Z_{\text{model}}(\mathbf{I}),$$

where $\mathbf{I} = (\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_n) \in J^{d,n}$. We retain the subcube $C(\mathbf{I})$ if $1_{\text{model}}(\mathbf{I}) = 1$ and we write D^n_{model} for the set of retained level-n cubes. Note that $D^1_{\text{model}}, D^2_{\text{model}}, \ldots$ correspond to the sets informally constructed in the introduction. We denote by $\mathbb{P}_{\text{model}}$ the distribution of the corresponding model on $\Omega = \{0,1\}^{\mathcal{C}}$, where $\mathcal{C} := \{C(\mathbf{I}) : \mathbf{I} \in \mathcal{J}\}$ denotes the collection of all subcubes, endowed with the usual sigma algebra generated by the cylinder events. To simplify the notation, we will drop the subscripts fat, k, p, Y when there is no danger of confusion.

3. Proofs of the k-fractal results

In this section we prove Theorem 1.1 and Theorem 1.4. The proof of Theorem 1.1 is divided in two parts. First we treat the subcritical case and show that $\liminf_{N\to\infty} k_c(N,d)/N^d \geq p_c(d)$.

Theorem 3.1. *Consider the k-model. We have*

$$\liminf_{N\to\infty} k_c(N,d)/N^d \geq p_c(d).$$

In the supercritical case, we prove that the crossing probability converges to 1 as $N \to \infty$. Again, for future reference we state this as a theorem.

Theorem 3.2. *Let $p > p_c(d)$ and let $(k(N))_{N\geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$, for all $N \geq 2$. We have*

$$\lim_{N\to\infty} \theta(k(N), N, d) = 1.$$

Theorem 1.1 follows immediately from these two theorems.

We prove Theorems 3.1 and 3.2 in Sections 3.1 and 3.2, respectively. In Section 3.3 we prove Theorem 1.4, using the idea of the proof of Theorem 3.1 and the result of Theorem 3.2.

3.1. *Proof of Theorem 3.1.* Let $p < p_c(d)$ and consider a sequence $(k(N))_{N\geq 2}$ such that $k(N)/N^d \leq p$, for all $N \geq 2$, and $k(N)/N^d \to p$ as $N \to \infty$. Our goal is to show that the probability that the unit cube is crossed by $D_{k(N)}$, is equal to zero for all N large enough. Let $N \geq 2$ and let D_{p_0} be the limit set of an MFP process with parameters p_0 and N, where $p < p_0 < p_c(d)$. First, part (a) of Theorem 2 in Falconer and Grimmett (1992) states that

$$p_c(d) \leq p_c(N,d), \tag{3.1}$$

for all N. Hence, the MFP process with parameter $p_0 < p_c(d)$ is subcritical. Therefore, a natural approach to prove that the probability that $D_{k(N)}$ crosses the unit cube equals zero for N large enough would be to couple the limit set $D_{k(N)}$ to the limit set D_{p_0} in such a way that $D_{k(N)} \subset D_{p_0}$. However, a "direct" coupling between the limit sets $D_{k(N)}$ and D_{p_0} is not possible, since with fixed positive probability at each iteration of the MFP process the number of retained subcubes is less than $k(N)$. We therefore need to find a more refined coupling.

The following is an informal strategy of the proof. We will define an event E on which the MFP process contains an infinite tree of retained subcubes, such that each subcube in this tree contains at least $k(N)$ retained subcubes in the tree.

Next, we perform a construction of two auxiliary random subsets of the unit cube, from which it will follow that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E. In particular, the probability that $D_{k(N)}$ crosses $[0,1]^d$ is less than or equal to the conditional probability that D_{p_0} crosses the unit cube, given E. The latter probability is zero for N large enough, since the event E has positive probability for N large enough and the MFP process is subcritical.

Let us start by defining the event E. Consider an MFP process with parameters p_0 and N. For notational convenience we call the unit cube the level-0 cube. A level-n cube, $n \geq 0$, is declared *0-good* if it is retained and contains at least $k(N)$ retained level-$(n+1)$ subcubes. (We adopt the convention that $[0,1]^d$ is automatically retained.) Recursively, we define the notion *m-good*, for $m \geq 0$. A level-n cube, for $n \geq 0$, is $(m+1)$-good if it is retained and contains at least $k(N)$ m-good subcubes. We say that the unit cube is *∞-good* if it is m-good for every $m \geq 0$. Define the following events

$$\begin{aligned} E_m &:= \{[0,1]^d \text{ is } m\text{-good}\}, \\ E &:= \{[0,1]^d \text{ is } \infty\text{-good}\}. \end{aligned} \tag{3.2}$$

The following lemma states that we can make the probability of E arbitrary close to 1, for N large enough. In particular, E has positive probability for large enough N, which will be sufficient for the proof of Theorem 3.1.

Lemma 3.3. *Let $p_0 < p_c(d)$. Let $(k(N))_{N\geq 2}$ be a sequence of integers satisfying $\limsup_{N\to\infty} k(N)/N^d < p_0$. Consider an MFP model with parameters p_0 and N. For all $\epsilon > 0$ there exists N_0 such that $\mathbb{P}_{p_0}(E) > 1-\epsilon$ for all $N \geq N_0$.*

Proof: Let $\delta > 0$ and N_0 be such that $k(N)/N^d \leq p_0 - 2\delta =: p$ for all $N \geq N_0$. Choose $N_1 \geq N_0$ so large that $p_0/(4\delta^2 N^d) < \delta$ for $N \geq N_1$. We will show that

$$\mathbb{P}_{p_0}(E_m) \geq 1 - \frac{1}{4\delta^2 N^d}, \tag{3.3}$$

for all $m \geq 0$ and $N \geq N_1$. Since E_m decreases to E as $m \to \infty$, it follows that

$$\mathbb{P}_{p_0}(E) = \lim_{m\to\infty} \mathbb{P}_{p_0}(E_m) \geq 1 - \frac{1}{4\delta^2 N^d},$$

for $N \geq N_1$. Now take $N_2 \geq N_1$ so large that $1 - \frac{1}{4\delta^2 N^d} > 1 - \epsilon$ for all $N \geq N_2$. It remains to show (3.3).

We prove (3.3) by induction on m. Consider the event E_0, i.e. the event that the unit cube contains at least $k(N)$ retained level-1 subcubes. Let $X(n,p)$ denote a binomially distributed random variable with parameters $n \in \mathbb{N}$ and $p \in [0,1]$. Since the number of retained level-1 cubes has a binomial distribution with parameters N^d and p_0, it follows from Chebyshev's inequality that, for every $N \geq N_1$, we have (writing $\mathbb{P}$ for the probability measure governing the binomially distributed random variables)

$$\begin{aligned}\mathbb{P}_{p_0}(E_0) &= \mathbb{P}(X(N^d, p_0) \geq k(N)) \\ &\geq \mathbb{P}(X(N^d, p_0) \geq pN^d) \\ &\geq 1 - \frac{\mathbb{V}\mathrm{ar}X(N^d, p_0)}{4\delta^2 N^{2d}} \\ &= 1 - \frac{p_0(1-p_0)N^d}{4\delta^2 N^{2d}} \\ &\geq 1 - \frac{1}{4\delta^2 N^d}.\end{aligned}$$

Next, let $m \geq 0$ and $N \geq N_1$ and suppose that (3.3) holds for this m and N. Recall that E_{m+1} is the event that the unit cube contains at least $k(N)$ m-good level-1 cubes. The probability that a level-1 cube is m-good, given that it is retained, is equal to $\mathbb{P}_{p_0}(E_m)$. Using the induction hypothesis, we get

$$\begin{aligned}\mathbb{P}_{p_0}(E_{m+1}) &= \mathbb{P}(X(N^d, p_0\mathbb{P}_{p_0}(E_m)) \geq k(N)) \\ &\geq \mathbb{P}(X(N^d, p_0(1 - \tfrac{1}{4\delta^2 N^d})) \geq k(N)).\end{aligned}$$

By our choices for δ and N it follows that $p_0(1 - \frac{1}{4\delta^2 N^d}) > p + \delta$. Hence, using Chebyshev's inequality, we get

$$\begin{aligned}\mathbb{P}(X(N^d, p_0(1 - \tfrac{1}{4\delta^2 N^d})) \geq k(N)) &\geq \mathbb{P}(X(N^d, p+\delta) \geq k(N)) \\ &\geq \mathbb{P}(X(N^d, p+\delta) \geq pN^d) \\ &\geq 1 - \frac{\mathbb{V}\mathrm{ar}X(N^d, p+\delta)}{\delta^2 N^{2d}} \\ &\geq 1 - \frac{1}{4\delta^2 N^d}.\end{aligned}$$

Therefore, the induction step is valid and we have proved (3.3). □

Proof of Theorem 3.1: Let p, p_0 be such that $p < p_0 < p_c(d)$. Let $(k(N))_{N\geq 2}$ be a sequence such that $k(N)/N^d \leq p$, for all $N \geq 2$, and $k(N)/N^d \to p$ as $N \to \infty$. Consider an MFP model with parameters p_0 and N and define the event E as in (3.2). Henceforth, we assume that N is so large that $\mathbb{P}_{p_0}(E) > 0$, which is possible by Lemma 3.3. In order to prove Theorem 3.1 we will use E to construct two random subsets, $\tilde{D}_{p_0}$ and $\tilde{D}_{k(N)}$, of the unit cube, on a common probability space and with the following properties:

(i) $\tilde{D}_{k(N)} \subset \tilde{D}_{p_0}$;

(ii) the law of $\tilde{D}_{p_0}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E;

(iii) the law of $\tilde{D}_{k(N)}$ is the same as the law of $D_{k(N)}$.

It follows that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E. Hence, the probability that the unit cube is crossed by $D_{k(N)}$ is at most the conditional probability that D_{p_0} crosses the unit cube, conditioned on the event E. By (3.1) the MFP process with parameter p_0 is subcritical, thus the latter probability equals zero. Using the fact that $k(N)/N^d \to p$ as $N \to \infty$, we conclude that

$$\liminf_{N\to\infty} \frac{k_c(N,d)}{N^d} \geq p.$$

Since $p < p_c(d)$ was arbitrary, we get

$$\liminf_{N\to\infty} \frac{k_c(N,d)}{N^d} \geq p_c(d).$$

It remains to construct random sets $\tilde{D}_{p_0}, \tilde{D}_{k(N)}$ with the properties (i)-(iii). First we construct two sequences $(\tilde{D}^n_{p_0})_{n\geq 1}, (\tilde{D}^n_{k(N)})_{n\geq 1}$ of decreasing random subsets. Let $\mathcal{L}$ be the conditional law of the number of ∞-good level-1 cubes of the MFP process, conditioned on the event E. Note that the support of $\mathcal{L}$ is $\{k(N), k(N)+1, \ldots, N^d\}$. Furthermore, for a fixed level-n cube $C(\mathbf{I})$, $\mathcal{L}$ is also equal to the conditional law of the number of ∞-good level-$(n+1)$ subcubes in $C(\mathbf{I})$, conditioned on $C(\mathbf{I})$ being ∞-good.

Choose an integer l according to $\mathcal{L}$ and choose l level-1 cubes uniformly. Define $\tilde{D}^1_{p_0}$ as the closure of the union of these l level-1 cubes. Choose $k(N)$ out of these l cubes in a uniform way and define $\tilde{D}^1_{k(N)}$ as the closure of the union of these $k(N)$ cubes. For each level-1 cube $C(\mathbf{I}) \subset \tilde{D}^1_{p_0}$, pick an integer $l(\mathbf{I})$ according to $\mathcal{L}$, independently of other cubes, and choose $l(\mathbf{I})$ level-2 subcubes of $C(\mathbf{I})$ in a uniform way. Define $\tilde{D}^2_{p_0}$ as the closure of the union of all selected level-2 cubes. For each level-1 cube $C(\mathbf{I}) \subset \tilde{D}^1_{k(N)}$, uniformly choose $k(N)$ out of the $l(\mathbf{I})$ selected level-2 subcubes. Define $\tilde{D}^2_{k(N)}$ as the closure of the union of the $k(N)^2$ selected level-2 cubes of $C(\mathbf{I})$. Iterating this procedure yields two infinite decreasing sequences of random subsets $(\tilde{D}^n_{p_0})_{n\geq 1}, (\tilde{D}^n_{k(N)})_{n\geq 1}$.

Now define

$$\tilde{D}_{p_0} := \bigcap_{n=1}^{\infty} \tilde{D}^n_{p_0}, \quad \tilde{D}_{k(N)} := \bigcap_{n=1}^{\infty} \tilde{D}^n_{k(N)}.$$

By construction, for each $n \geq 1$, we have that (1) $\tilde{D}^n_{k(N)} \subset \tilde{D}^n_{p_0}$, (2) the law of $\tilde{D}^n_{p_0}$ is stochastically dominated by the conditional law of $D^n_{p_0}$ given E and (3) the law of $\tilde{D}^n_{k(N)}$ is equal to the law of $D^n_{k(N)}$. It follows that the limit sets $\tilde{D}_{p_0}, \tilde{D}_{k(N)}$ satisfy properties (i)-(iii). □

3.2. *Proof of Theorem 3.2*. Let us start by outlining the proof. The first part consists mainly of setting up the framework, where we use the notation of Falconer and Grimmett (1992), which will enable us in the second part to prove that the subcubes of the fractal process satisfy certain "good" properties with probability arbitrarily close to 1 as $N \to \infty$. Informally, a subcube is good when there exist many connections inside the cube between its faces and when it is also connected to other good subcubes. Therefore, the probability of crossing the unit cube converges to 1 as $N \to \infty$.

Although we will partly follow Falconer and Grimmett (1992), it does not seem possible to use Theorem 2.2 of Falconer and Grimmett (1992) directly. First, we state (a slightly adapted version of) Lemma 2 of Falconer and Grimmett (1992), which concerns site percolation with parameter π on $\mathbb{L}^d$. We let every vertex of $\mathbb{L}^d$ be colored *black* with probability π and *white* otherwise, independently of other vertices. We write P_π for the ensuing product measure with density $\pi \in [0,1]$. We call a subset C of $\mathbb{L}^d$ a *black cluster* if it is a maximal connected subset (with respect to the adjacency relation on $\mathbb{L}^d$) of black vertices. Denote the cube with vertex set $\{1,2,\ldots,N\}^d$ by B_N. Let $\mathcal{L}$ be the set of edges of the unit cube $[0,1]^d$, that is $\mathcal{L}$ contains all sets of the form

$$L_r(\mathbf{a}) = \{a_1\} \times \{a_2\} \times \cdots \times \{a_{r-1}\} \times [0,1] \times \{a_{r+1}\} \times \cdots \times \{a_d\}$$

as r ranges over $\{1,\dots,d\}$ and $\mathbf{a} = (a_1, a_2, \dots, a_d)$ ranges over $\{0,1\}^d$. For each $L = L_r(\mathbf{a}) \in \mathcal{L}$ we write

$$L_N = \{\mathbf{x} \in B_N : x_i = \max\{1, a_i N\} \text{ for } 1 \le i \le d, i \ne r\}$$

for the corresponding edge of B_N.

Lemma 3.4. *Suppose $\pi > p_c(d), \epsilon > 0$ and let q be a positive integer. There exist positive integers u and N_1 such that the following holds for all $N \ge N_1$. Let $U(1), \dots, U(q)$ be subsets of vertices of B_N such that for each $r \in \{1,\dots,q\}$, (i) $|U(r)| \ge u$ and (ii) there exists $L \in \mathcal{L}$ such that $U(r) \subset L_N$. Then,*

$$P_\pi \left(\begin{array}{c} \text{there exists a black cluster } C_N \text{ such that } |C_N \cap L_N| \ge u \\ \text{for all } L \in \mathcal{L}, \text{ and } |C_N \cap U(r)| \ge 1, \text{ for all } r \in \{1,\dots,q\} \end{array} \right) \ge 1 - \frac{\epsilon}{2}. \quad (3.4)$$

Our goal is to show that the following holds uniformly in n: With probability arbitrarily close to 1 as $N \to \infty$, there is a sequence of cubes in $D^n_{k(N)}$, each with at least one edge in common with the next, which crosses the unit cube. In order to prove this we examine the cubes $C(\mathbf{I})$, for $\mathbf{I} \in \mathcal{J}^{(n)}$, in turn according to the ordering on $\mathcal{J}^{(n)}$, and declare some of them to be good according to the rule given below. Since the probabilistic bounds on the goodness of cubes will hold uniformly in n, the desired conclusion follows.

Fix integers $n, u, k \ge 1$ until Lemma 3.7. For $m \ge 1$, identify a level-m cube with a vertex in $B_{N^m} \subset \mathbb{L}^d$ in the canonical way. A set of level-m cubes $\{C(\mathbf{I}_1), \dots, C(\mathbf{I}_l)\}$ is called *edge-connected* if they form a connected set with respect to the adjacency relation of $\mathbb{L}^d$. Whether a cube $C(\mathbf{I})$, for $\mathbf{I} \in \mathcal{J}^{(n)}$, is called (n,u)-*good* or not, is determined by the following inductive procedure. Let $\mathbf{I} \in \mathcal{J}^{(n)}$, and assume that the goodness of $C(\mathbf{I}')$ has been decided for all $\mathbf{I}' < \mathbf{I}$. We have the following possibilities:

(a) $|\mathbf{I}| = n$. Then $C(\mathbf{I})$ is always declared (n,u)-good.
(b) $0 \le |\mathbf{I}| = m < n$.

In the latter case we act as follows. Note that the subcubes $C(\mathbf{I}, \mathbf{j})$ with $\mathbf{j} \in J^d$ have already been examined, since $(\mathbf{I}, \mathbf{j}) < \mathbf{I}$. Define the following set of level-$(m+1)$ subcubes of $C(\mathbf{I})$,

$$\mathcal{D}(\mathbf{I}) := \{C(\mathbf{I}, \mathbf{j}) : \mathbf{j} \in J^d \text{ with } C(\mathbf{I}, \mathbf{j})\ (n,u)\text{-good and } Z_k(\mathbf{I}, \mathbf{j}) = 1\}. \quad (3.5)$$

We declare $C(\mathbf{I})$ to be (n,u)-good if there exists an edge-connected set $\mathcal{H}(\mathbf{I}) \subset \mathcal{D}(\mathbf{I})$ such that

(i) Each edge of $C(\mathbf{I})$ intersects at least u cubes of $\mathcal{H}(\mathbf{I})$;
(ii) For every (n,u)-good level-m cube $C(\mathbf{I}')$ with $\mathbf{I}' < \mathbf{I}$ that has (at least) one edge in common with $C(\mathbf{I})$, there are a cube of $\mathcal{H}(\mathbf{I}')$ and a cube of $\mathcal{H}(\mathbf{I})$ with a common edge.

(If there is more than one candidate for $\mathcal{H}(\mathbf{I})$ we use some deterministic rule to choose one of them.) This procedure determines whether $C(\mathbf{I})$ is (n,u)-good for each $\mathbf{I}$ in turn. Note that it is easier for higher level cubes to be (n,u)-good than for lower level cubes. In particular, for the unit cube, i.e. $C(\emptyset)$, it is the hardest to be (n,u)-good.

The next lemma shows that if the unit cube is (n,u)-good then there is a sequence of cubes in D_k^n, each with at least one edge in common with the next, which connects the "left-hand side" of $[0,1]^d$ with its "right-hand side". If such a sequence of cubes exists in D_k^n we say that *percolation occurs in* D_k^n.

Lemma 3.5. *Suppose* $[0,1]^d$ *is* (n,u)*-good, then percolation occurs in* D_k^n*.*

Proof: Assume that the unit cube, i.e. $C(\emptyset)$, is (n,u)-good. We will show, with a recursive argument, that for $1 \le m \le n$ there exists an edge-connected chain of retained (n,u)-good level-m cubes which joins $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. In particular, this holds for $m = n$ and hence percolation occurs in D_k^n.

Since the unit cube is assumed to be (n,u)-good, $\mathcal{D}(\emptyset)$ contains by definition an edge-connected subset $\mathcal{H}(\emptyset)$ of retained (n,u)-good level-1 subcubes, such that each edge of $C(\emptyset)$ intersects at least u cubes of $\mathcal{H}(\emptyset)$. In particular, there is a sequence of retained (n,u)-good edge-connected level-1 cubes that connects the left-hand side of $[0,1]^d$ with its right-hand side.

Let $1 \le m < n$ and assume there exists an edge-connected chain $C(\mathbf{I}_1), \dots, C(\mathbf{I}_l)$ of retained (n,u)-good level-m cubes which connects the left-hand side of $[0,1]^d$ with its right-hand side. For each i, $1 \le i \le l$, either $\mathbf{I}_i < \mathbf{I}_{i+1}$ or $\mathbf{I}_{i+1} < \mathbf{I}_i$. By condition (ii), there exist level-$(m+1)$ cubes of $\mathcal{H}(\mathbf{I}_{i+1})$ which are edge-connected to level-$(m+1)$ cubes of $\mathcal{H}(\mathbf{I}_i)$. These level-$(m+1)$ cubes $C(\mathbf{J})$ are all (n,u)-good and have $Z_k(\mathbf{J}) = 1$, by (3.5) and the definition of the $\mathcal{H}(\mathbf{I})$. It follows that there is an edge-connected chain of retained (n,u)-good level-$(m+1)$ cubes $C(\mathbf{J})$ which joins $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. □

For $\mathbf{I} \in \mathcal{J}^{(n)}$, define the index $\mathbf{I}^- \in \mathcal{J}^{(n)}$ by

$$\mathbf{I}^- = \max\{\mathbf{I}' : \mathbf{I}' < \mathbf{I} \text{ and } |\mathbf{I}'| \le |\mathbf{I}|\}.$$

If there is no such index, $\mathbf{I}^-$ is left undefined. For each $\mathbf{I} \in \mathcal{J}^{(n)}$ we let $\mathcal{F}(\mathbf{I})$ denote the σ-field

$$\mathcal{F}(\mathbf{I}) = \sigma(Z_k(\mathbf{I}', \mathbf{j}) : |\mathbf{I}'| \le n-1, \mathbf{I}' \le \mathbf{I}, \mathbf{j} \in J^d).$$

If $\mathbf{I}^-$ is undefined, we take $\mathcal{F}(\mathbf{I}^-)$ to be the trivial σ-field. Note that $\mathcal{F}(\mathbf{I})$ is generated by those Z_k that have been examined prior to deciding whether $C(\mathbf{I})$ is (n,u)-good. In particular, by virtue of the ordering on the cubes as introduced in Section 2, $\mathcal{F}(\mathbf{I}^-)$ does *not* contain any information about subcubes of $\mathbf{I}$.

Let $p > p_c(d)$ and let $(k(N))_{N \ge 2}$ be a sequence such that $k(N)/N^d \ge p$, for all $N \ge 2$. We want to prove that, for every $\epsilon > 0$, the probability that $[0,1]^d$ is (n,u)-good in the $k(N)$-model is at least $1-\epsilon$, for $N \ge N_0$, where N_0 is an integer which has to be taken sufficiently large to satisfy certain probabilistic bounds but is independent of n.

Let us first give a sketch of the proof. Fix $N \ge N_0$ and consider the $k(N)$-model. We use a recursive argument. The smallest level-n cube according to the ordering on $\mathcal{J}^{(n)}$ is by definition (n,u)-good. Let $\mathbf{I} \in \mathcal{J}^{(n)}$ and assume that $\mathbb{P}_{k(N)}(C(\mathbf{I}') \text{ is } (n,u)\text{-good} \mid \mathcal{F}(\mathbf{I}'^-)) \ge 1-\epsilon$ for all $\mathbf{I}' < \mathbf{I}$. We prove that, given $\mathcal{F}(\mathbf{I}^-)$, $C(\mathbf{I})$ is (n,u)-good with probability at least $1-\epsilon$. The proof of this consists of a coupling between a product measure with density $\pi \in (p_c(d), (1-\epsilon)p)$ in the box B_N and the law of the set of subcubes $C(\mathbf{I}, \mathbf{j})$ of $C(\mathbf{I})$ which are (n,u)-good and satisfy $Z_{k(N)}(\mathbf{I}, \mathbf{j}) = 1$. Applying Lemma 3.4 to the product measure combined with the coupling yields that the subcubes satisfy properties (i) and (ii) with probability at least $1-\epsilon$. Therefore, given $\mathcal{F}(\mathbf{I}^-)$, $C(\mathbf{I})$ is (n,u)-good with probability at least

$1-\epsilon$. Iterating this argument then yields that the unit cube is (n,u)-good with probability at least $1-\epsilon$, for $N \geq N_0$.

The proof in Falconer and Grimmett (1992) of the analogous result that, for $p > p_c(d)$, $\sigma(p,N,d) \to 1$ as $N \to \infty$ is considerably less involved. In the context of Falconer and Grimmett (1992), subcubes are retained with probability p independently of other cubes, which is not the case in k-fractal percolation. Therefore, they can directly show that there exists $\pi > p_c(d)$ such that, for $\mathbf{I} \in \mathcal{J}^{(n)}$, the law of the set of subcubes $C(\mathbf{I},\mathbf{j})$ of $C(\mathbf{I})$ which are good and satisfy $Z_p(\mathbf{I},\mathbf{j}) = 1$, dominates an i.i.d. process on the box B_N with density π.

We need the following result for binomially distributed random variables, which we state as a lemma for future reference. Since the result follows easily from Chebyshev's inequality, we omit the proof.

Lemma 3.6. *Let $p > p_c(d)$ and let $(k(N))_{N\geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$ for all $N \geq 2$. Let $\epsilon > 0$ be such that $(1-\epsilon)p > p_c(d)$, let $\pi \in (p_c(d),(1-\epsilon)p)$ and define $M := ((1-\epsilon)p+\pi)N^d/2$. There exists N_2 such that*

$$\mathbb{P}(\{X(k(N),1-\epsilon) \geq M\} \cap \{X'(N^d,\pi) \leq M\}) \geq 1-\epsilon/2,$$

for $N \geq N_2$, where X and X' are independent, binomially distributed random variables with the indicated parameters.

We now prove that, for any $\epsilon > 0$, the unit cube is (n,u)-good with probability at least $1-\epsilon$, for N large enough but independent of n.

Lemma 3.7. *Let $p > p_c(d)$ and let $(k(N))_{N\geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$, for all $N \geq 2$. Let $\epsilon > 0$ be such that $(1-\epsilon)p > p_c(d)$. Take $\pi \in (p_c(d),(1-\epsilon)p)$ and set $q = 3^d$. Let u and N_1 be given by Lemma 3.4. Let N_2 be given by Lemma 3.6. Set $N_0 = \max\{N_1,N_2\}$. Then, for all $n \geq 1$,*

$$\mathbb{P}_{k(N)}([0,1]^d \text{ is } (n,u)\text{-good}) \geq 1-\epsilon, \tag{3.6}$$

for all $N \geq N_0$.

Proof: Fix $N \geq N_0$ and $n \geq 1$ and consider the $k(N)$-fractal model. Our aim is to show that

$$\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n,u)\text{-good} \mid \mathcal{F}(\mathbf{I}^-)) \geq 1-\epsilon \tag{3.7}$$

holds for all $\mathbf{I} \in \mathcal{J}^{(n)}$. Taking $\mathbf{I} = \emptyset$ then yields (3.6). We prove this with a recursive argument. Let $\mathbf{I}_0$ be the smallest index in $\mathcal{J}^{(n)}$, according to the ordering on $\mathcal{J}^{(n)}$. By virtue of the ordering, we have $|\mathbf{I}_0| = n$. Hence, by definition, $C(\mathbf{I}_0)$ is (n,u)-good. In particular, (3.7) holds for $\mathbf{I}_0$.

The recursive step is as follows. Take an index $\mathbf{I} \in \mathcal{J}^{(n)}$ and assume that

$$\mathbb{P}_{k(N)}(C(\mathbf{I}') \text{ is } (n,u)\text{-good} \mid \mathcal{F}(\mathbf{I}'^-)) \geq 1-\epsilon, \tag{3.8}$$

has been established for all indices $\mathbf{I}'$ in $\mathcal{J}^{(n)}$ less than $\mathbf{I}$. We have to show that (3.7) holds for $\mathbf{I}$ given this assumption. We have two cases:

(a) $|\mathbf{I}| = n$; then $\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n,u)\text{-good}) = 1$ and (3.7) is true.
(b) $0 \leq |\mathbf{I}| = m < n$.

For case (b), given $\mathcal{F}(\mathbf{I}^-)$, the goodness of $C(\mathbf{I}')$ is determined (in particular) for all $\mathbf{I}' < \mathbf{I}$ with $|\mathbf{I}| = m$. Let

$$\mathcal{Q} = \left\{ \begin{array}{ll} \mathbf{I}': & \mathbf{I}' < \mathbf{I} \text{ and } C(\mathbf{I}') \text{ is an } (n,u)\text{-good level-}m \\ & \text{cube with an edge in common with } C(\mathbf{I}) \end{array} \right\}.$$

For each $\mathbf{I}' \in \mathcal{Q}$, let $E(\mathbf{I}')$ be some common edge of $C(\mathbf{I})$ and $C(\mathbf{I}')$. Since $C(\mathbf{I}')$ is (n,u)-good, there are at least u level-$(m+1)$ subcubes in $\mathcal{H}(\mathbf{I}')$ which intersect $E(\mathbf{I}')$; call this set of subcubes $\mathcal{U}(\mathbf{I}')$. To see whether $C(\mathbf{I})$ is (n,u)-good, we look at $C(\mathbf{I},\mathbf{j}(l))$ where $\mathbf{j}(l), 1 \leq l \leq N^d$, are the vectors of J^d arranged in order. We have $(\mathbf{I},\mathbf{j}(l)) < \mathbf{I}$, so by the induction hypothesis (3.8) we have

$$\mathbb{P}_{k(N)}(C(\mathbf{I},\mathbf{j}(l)) \text{ is } (n,u)\text{-good} \mid \mathcal{F}((\mathbf{I},\mathbf{j}(l))^-)) \geq 1-\epsilon, \tag{3.9}$$

for all l. Note that $\mathcal{F}((\mathbf{I},\mathbf{j}(1))^-) = \mathcal{F}(\mathbf{I}^-)$.

We identify each subcube of $C(\mathbf{I})$ in the canonical way with a vertex in B_N. We will construct three random subsets G_1, G_2, G_3 of B_N on a common probability space with the following properties:

(I) the law of G_1 equals the law of the set of subcubes $C(\mathbf{I},\mathbf{j})$ of $C(\mathbf{I})$ which are (n,u)-good and satisfy $Z_{k(N)}(\mathbf{I},\mathbf{j}) = 1$;

(II) G_2 is obtained by first selecting $k(N)$ vertices of B_N uniformly and then retaining each selected vertex with probability $1-\epsilon$, independently of other vertices;

(III) the law of G_3 is the Bernoulli product measure with density π on B_N;

(IV) $G_1 \supset G_2$;

(V) $\mathbb{P}(G_2 \supset G_3) \geq 1-\epsilon/2$.

From (3.9) and a standard coupling technique, sometimes referred to as sequential coupling (see e.g. Liggett and Steif (2006)), the construction of G_1 and G_2 with properties (I), (II) and (IV) is straightforward. The construction of G_3 such that properties (III) and (V) hold is given below. Let $|G_2|$ denote the cardinality of the set G_2. Define $M = ((1-\epsilon)p + \pi)N^d/2$ and let R be a number drawn from a binomial distribution with parameters N^d and π, independently of G_1 and G_2. If $|G_2| \geq M$ and $M \geq R$ we select R vertices uniformly out of the $|G_2|$ retained vertices of G_2 and call this set G_3. Otherwise, we select, independently of G_1 and G_2, R vertices of B_N in a uniform way and call this set G_3. From the construction (note that also G_2 was obtained in a uniform way) it is clear that G_3 satisfies property (III). Observe that $|G_2|$ has a binomial distribution with parameters $k(N)$ and $1-\epsilon$. From Lemma 3.6 it follows that

$$\mathbb{P}(\{|G_2| \geq M\} \cap \{R \leq M\}) \geq 1-\epsilon/2.$$

Hence, property (V) also holds.

Let us now return to the goodness of $C(\mathbf{I})$. As before, we identify the random subsets G_1, G_2, G_3 of B_N with the corresponding sets of subcubes of $C(\mathbf{I})$ in the canonical way. It then follows from property (III) and Lemma 3.4 (note that $\mathcal{Q}$ has cardinality at most $3^d = q$) that G_3 has an edge-connected subset which satisfies the following properties with probability at least $1-\epsilon/2$:

(i) intersects every edge of $C(\mathbf{I})$ with at least u cubes;

(ii) contains a cube that is edge-connected to a cube of $\mathcal{U}(\mathbf{I}')$, for all $\mathbf{I}' \in \mathcal{Q}$.

Combining properties (IV), (V) and the previous paragraph we obtain

$$\begin{aligned}
&\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n,u)\text{-good} \mid \mathcal{F}(\mathbf{I}^-)) \\
&\quad \geq \mathbb{P}(\{G_1 \supset G_3\} \cap \{G_3 \text{ satisfies properties (i) and (ii)}\}) \\
&\quad \geq 1-\epsilon.
\end{aligned}$$

Therefore, (3.7) holds for the index $\mathbf{I}$ given that (3.8) holds for all indices $\mathbf{I}' < \mathbf{I}$. A recursive use of this argument – recall that (3.7) is valid for $\mathbf{I}_0$ (the smallest index according to the ordering) – yields that (3.7) holds for all $\mathbf{I}$. Taking $\mathbf{I} = \emptyset$ in (3.7) proves the lemma. □

We are now able to conclude the proof of Theorem 3.2.

Proof of Theorem 3.2: Let $p > p_c(d)$ and consider a sequence $(k(N))_{N\geq 2}$ such that $k(N)/N^d \geq p$, for all $N \geq 2$. We get, using both Lemma 3.7 and Lemma 3.5, that for any $\epsilon > 0$ such that $(1-\epsilon)p > p_c(d)$, there exists N_0, depending on ϵ, such that

$$\mathbb{P}_{k(N)}(\text{percolation in } D^n_{k(N)}) \geq \mathbb{P}_{k(N)}([0,1]^d \text{ is } (n,u)\text{-good}) \geq 1-\epsilon, \tag{3.10}$$

for $N \geq N_0$. It is well known (see e.g. Falconer and Grimmett (1992)) that

$$\{[0,1]^d \text{ is crossed by } D_{k(N)}\} = \bigcap_{n=1}^{\infty} \{\text{percolation in } D^n_{k(N)}\}.$$

Hence, taking the limit $n \to \infty$ in (3.10) yields that for $\epsilon > 0$ small enough

$$\mathbb{P}_{k(N)}([0,1]^d \text{ is crossed by } D_{k(N)}) \geq 1-\epsilon, \tag{3.11}$$

for $N \geq N_0$. Therefore,

$$\theta(k(N), N, d) \to 1,$$

as $N \to \infty$. □

3.3. *Proof of Theorem 1.4.*

Proof of Theorem 1.4: We use the idea of the proof of Theorem 3.1 and the result of Theorem 3.2. Fix some p_0 such that $p_c(d) < p_0 < p$ and set $k(N) := \lfloor p_0 N^d \rfloor$. Consider the event F that in the GFP model with generator $Y = Y(N,d)$ there exists an infinite tree of retained subcubes such that each subcube in the tree contains at least $k(N)$ retained subcubes in the tree. Similar to the proof of Lemma 3.3, we prove that $\mathbb{P}(F) \to 1$ as $N \to \infty$. We then show that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_Y, conditioned on the event F. By Theorem 3.2 we can then conclude that $\phi(Y(N,d), N, d) \to 1$ as $N \to \infty$.

Consider the construction of D_Y. We will use the same definition of m-good as in Section 3.1, that is, if a level-n cube is retained and contains at least $k(N)$ retained subcubes, we call this level-n cube 0-good. Recursively, we say that a level-n cube is $(m+1)$-good if it is retained and contains at least $k(N)$ m-good level-$(n+1)$ subcubes. We call the unit cube ∞-good if it is m-good for every $m \geq 0$. Define the following events

$$F_m := \{[0,1]^d \text{ is } m\text{-good}\},$$
$$F := \{[0,1]^d \text{ is } \infty\text{-good}\}.$$

We will show that for every $\epsilon > 0$ such that $(1-\epsilon)p > p_0$ there exists $N_0 = N_0(\epsilon)$ such that, for all $m \geq 0$,

$$\mathbb{P}(F_m) > 1-\epsilon, \quad \text{for all } N \geq N_0. \tag{3.12}$$

The proof of (3.12) is similar to the proof of Lemma 3.3. Let $\epsilon > 0$ be such that $(1-\epsilon)p > p_0$. Take $\delta > 0$ such that $(1-\epsilon)p > p_0 + \delta$. Then, take N_0 so large that

$$1 - \frac{1}{4\delta^2 N} > 1 - \epsilon/2 \quad \text{and} \tag{3.13}$$

$$\mathbb{P}(Y \geq pN^d) > 1 - \epsilon/2, \tag{3.14}$$

for all $N \geq N_0$. We prove that (3.12) holds for this N_0 and all $m \geq 0$, by induction on m. Since $k(N) = \lfloor p_0 N^d \rfloor \leq pN^d$ it follows from (3.14) that $\mathbb{P}(F_0) > 1 - \epsilon$, for all $N \geq N_0$.

Next, assume that (3.12) holds for some $m \geq 0$. The probability that a level-1 cube is m-good, given that it is retained, is equal to $\mathbb{P}(F_m)$. It follows that, given that the number of retained level-1 cubes equals y, the number of m-good level-1 cubes has a binomial distribution with parameters y and $\mathbb{P}(F_m)$. By our choices for N_0 and δ we get

$$\begin{aligned}
\mathbb{P}(F_{m+1}) &= \sum_{y \geq k(N)} \mathbb{P}(X(y, \mathbb{P}(F_m)) \geq k(N))\, \mathbb{P}(Y = y) \\
&\geq \mathbb{P}(X(\lfloor pN^d \rfloor, \mathbb{P}(F_m)) \geq p_0 N^d)\, \mathbb{P}(Y \geq \lfloor pN^d \rfloor) \\
&\geq \mathbb{P}(X(\lfloor pN^d \rfloor, 1 - \epsilon) \geq p_0 N^d)(1 - \epsilon/2) \\
&\geq \left(1 - \frac{\mathbb{V}\mathrm{ar} X(\lfloor pN^d \rfloor, 1-\epsilon)}{(p_0 - (1-\epsilon)p)^2 N^{2d}}\right)(1 - \epsilon/2) \\
&\geq \left(1 - \frac{(1-\epsilon)\epsilon p N^d}{\delta^2 N^{2d}}\right)(1 - \epsilon/2) \\
&\geq \left(1 - \frac{1}{4\delta^2 N^d}\right)(1 - \epsilon/2) \\
&\geq (1 - \epsilon/2)(1 - \epsilon/2) > 1 - \epsilon,
\end{aligned}$$

for all $N \geq N_0$. Hence, the induction step is valid.

Analogously to the proof of Theorem 3.1 we use the event $F = \bigcap_{m=1}^{\infty} F_m$ to construct two random subsets $\tilde{D}_{k(N)}$ and $\tilde{D}_Y$ on a common probability space, with the following properties:

(i) $\tilde{D}_{k(N)} \subset \tilde{D}_Y$;
(ii) the law of $\tilde{D}_Y$ is stochastically dominated by the conditional law of D_Y, conditioned on the event F;
(iii) the law of $\tilde{D}_{k(N)}$ is equal to the law of $D_{k(N)}$.

This construction is the same (modulo replacing the binomial distribution with Y) as in the proof of Theorem 3.1 and is therefore omitted.

From properties (i)-(iii) and Theorem 3.2 we get

$$\begin{aligned}
&\mathbb{P}([0,1]^d \text{ is crossed by } D_{Y(N,d)} | F) \\
&\quad \geq \mathbb{P}([0,1]^d \text{ is crossed by } \tilde{D}_{Y(N,d)}) \\
&\quad \geq \mathbb{P}([0,1]^d \text{ is crossed by } \tilde{D}_{k(N)}) \\
&\quad = \mathbb{P}([0,1]^d \text{ is crossed by } D_{k(N)}) \to 1,
\end{aligned}$$

as $N \to \infty$. Since (3.12) implies that $\mathbb{P}(F) \to 1$ as $N \to \infty$, we obtain

$$\mathbb{P}([0,1]^d \text{ is crossed by } D_{Y(N,d)}) \to 1,$$

as $N \to \infty$. □

4. Proofs of the fat fractal results

In this section we prove our results concerning fat fractal percolation. First, we state an elementary property of the fat fractal percolation model; it follows immediately from Fubini's theorem and we omit the proof.

Proposition 4.1. *The expected Lebesgue measure of the limit set of fat fractal percolation is given by*

$$\mathbb{E}\lambda(D_{fat}) = \prod_{n=1}^{\infty} p_n.$$

4.1. *Proof of Theorem 1.7.* Since $\prod_{n=1}^{\infty} p_n > 0$ it follows from Proposition 4.1 that with positive probability the limit set has positive Lebesgue measure given $D_{\text{fat}} \neq \emptyset$. Theorem 1.7 states that the latter holds with probability 1.

Proof of Theorem 1.7: Let Z_n denote the number of retained level-n cubes after iteration step n and set $Z_0 := 1$. Since the retention probabilities p_n vary with n, the process $(Z_n)_{n\geq 1}$ is a so-called branching process in a time-varying environment. Following the notation of Lyons (1992) let L_n be a random variable, having the distribution of Z_n given that $Z_{n-1} = 1$. Note that L_n has a binomial distribution with parameters N^d and p_n.

Define the process $(W_n)_{n\geq 1}$ by

$$W_n := \frac{Z_n}{\prod_{i=1}^{n} p_i N^d}.$$

It is straightforward to show that $(W_n)_{n\geq 1}$ is a martingale:

$$\begin{aligned} \mathbb{E}[W_n|W_{n-1}] &= \frac{\mathbb{E}[Z_n|Z_{n-1}]}{\prod_{i=1}^{n} p_i N^d} = \frac{Z_{n-1}}{\prod_{i=1}^{n} p_i N^d}\mathbb{E}[Z_n|Z_{n-1}=1] \\ &= \frac{Z_{n-1} p_n N^d}{\prod_{i=1}^{n} p_i N^d} = W_{n-1}. \end{aligned}$$

The Martingale Convergence Theorem tells us that W_n converges almost surely to a random variable W. Theorem 4.14 of Lyons (1992) states that if

$$A := \sup_n ||L_n||_\infty < \infty,$$

then $W > 0$ a.s. given non-extinction. It is clearly the case that $A < \infty$, because L_n can take at most the value N^d. Therefore, W_n converges to a random variable W which is stricly positive a.s. given non-extinction.

The Lebesgue measure of the retained cubes at each iteration step n is equal to Z_n/N^{dn}. We have

$$\lambda(D_{\text{fat}}^n) = \frac{Z_n}{N^{dn}} = \frac{\left(\prod_{i=1}^{n} p_i N^d\right) W_n}{N^{dn}} = \left(\prod_{i=1}^{n} p_i\right) W_n. \tag{4.1}$$

Letting $n \to \infty$ in (4.1) yields $\lambda(D_{\text{fat}}) = (\prod_{i=1}^{\infty} p_i)W$. Since $\prod_{i=1}^{\infty} p_i > 0$ and $W > 0$ a.s. given non-extinction, we get the desired result. □

4.2. *Proof of Theorem 1.8.* We start with a heuristic strategy for the proof. For a fixed configuration $\omega \in \Omega$, let us call a point x in the unit cube *conditionally connected* if the following property holds: If we change ω by retaining all cubes that contain x, then x is contained in a connected component larger than one point. We show that for almost all points x it is the case that x is conditionally connected with probability 0 or 1. We define an ergodic transformation T on the unit cube. The transformation T enables us to prove that the probability for a point x to be conditionally connected has the same value for λ-almost all x. From this we can then conclude that either the set of dust points or the set of connected components contains all Lebesgue measure.

Proof of Theorem 1.8: First, we have to introduce some notation. Let U be the collection of points in $[0,1]^d$ not on the boundary of a subcube. For each $x \in U$ there exists a unique sequence $(C(\mathbf{x}_1,\ldots,\mathbf{x}_n))_{n\geq 1}$ of cubes of the fractal process, where $\mathbf{x}_j \in J^d$ for all j, such that $\bigcap_{n\geq 1} C(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \{x\}$. Therefore, we can define an invertible transformation $\phi : U \to (J^d)^{\mathbb{N}}$ by $\phi(x) = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$. For each $n \in \mathbb{N}$ let μ_n be the uniform measure on $(X_n, \mathcal{F}_n)$, where $X_n = J^d$ and $\mathcal{F}_n$ is the power set of X_n. Let $(X, \mathcal{F}, \mu) = \bigotimes_{n=1}^{\infty}(X_n, \mathcal{F}_n, \mu_n)$ be the product space. Since $\phi : (U, \mathcal{B}(U), \lambda) \to (X, \mathcal{F}, \mu)$ is an invertible measure-preserving transformation, we have that $(X, \mathcal{F}, \mu)$ is by definition isomorphic to $(U, \mathcal{B}(U), \lambda)$. Here $\mathcal{B}(U)$ denotes the Borel σ-algebra.

Next, we define the transformation $T : U \to U$, which will play a crucial role in the rest of the proof. Define the auxiliary shift transformation $T^* : X \to X$ by $T^*((\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)) = (\mathbf{x}_2, \mathbf{x}_3, \ldots)$, for $(\mathbf{x}_1, \mathbf{x}_2, \ldots) \in X$. The transformation T^* is measure preserving with respect to the measure μ and also ergodic, see for instance Walters (1982). Let $T := \phi^{-1} \circ T^* \circ \phi$ be the induced transformation on U and note that T is isomorphic to T^* and hence also ergodic. Informally, T sends a point $x \in U$ to the point Tx, in such a way that the relative position of Tx in the unit cube is the same as the relative position of x in its level-1 cube $C(\mathbf{x}_1)$; see Figure 4.2.

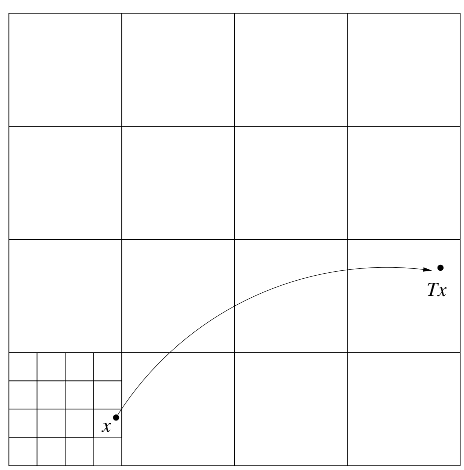

FIGURE 4.2. Illustration of the transformation T. Note that the relative position of x in the level-1 cube is the same as the relative position of Tx in the unit cube.

Recall that $\omega \in \Omega$ denotes a particular realization of the fat fractal percolation process. For $x \in U$, we define the following event.

$$A^x := \{\omega : \text{if we set } \omega(C(\mathbf{x}_1,\ldots,\mathbf{x}_n)) = 1 \text{ for all } n \geq 1, \text{ then } C^x_{\text{fat}} \neq \{x\}\}.$$

In other words, A^x consists of those configurations ω such that when we change the configuration by retaining all $C(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, then in this new configuration, x is in the same connected component as some $y \neq x$. Observe that

$$A^x \cap \{x \in D_{\text{fat}}\} = \{x \in D^c_{\text{fat}}\}. \tag{4.2}$$

It is easy to see that A^x is a tail event. Hence, by Kolmogorov's 0-1 law we get $\mathbb{P}(A^x) \in \{0,1\}$ for all $x \in U$.

However, a priori it is not clear that for almost all x in the unit cube $\mathbb{P}(A^x)$ has the same value. To this end, define the set $V := \{x \in U : \mathbb{P}(A^x) = 0\}$. We will show that $\lambda(V) \in \{0,1\}$. Recall that the relative position of Tx in the unit cube is the same as the relative position of x in the level-1 cube $C(\mathbf{x}_1)$. It is possible to construct a coupling between the fractal process in the unit cube and the fractal process in $C(\mathbf{x}_1)$, given that $C(\mathbf{x}_1)$ is retained, with the following property: For every cube $C(\mathbf{I})$ in $C(\mathbf{x}_1)$, it is the case that if $TC(\mathbf{I})$ is retained in the fractal process in the unit cube, then $C(\mathbf{I})$ is also retained in the fractal process in $C(\mathbf{x}_1)$, given that $C(\mathbf{x}_1)$ is retained. It is straightforward that such a coupling exists since the retention probabilities p_n are non-decreasing in n. Hence,

$$\mathbb{P}(A^{Tx}) \leq \mathbb{P}(A^x | C(\mathbf{x}_1) \text{ is retained}). \tag{4.3}$$

Furthermore, since A^x is a tail event, we have

$$\mathbb{P}(A^x) = \mathbb{P}(A^x | C(\mathbf{x}_1) \text{ is retained}). \tag{4.4}$$

It follows from (4.3) and (4.4) that $\mathbb{P}(A^{Tx}) \leq \mathbb{P}(A^x)$ for all x. This implies that $V \subset T^{-1}V$. Because T is measure preserving it follows that

$$\lambda(V \Delta T^{-1}V) = \lambda(V \setminus T^{-1}V) + \lambda(T^{-1}V \setminus V) = 0 + \lambda(T^{-1}V) - \lambda(V) = 0.$$

Ergodicity of T now yields that $\lambda(V) \in \{0,1\}$.

Suppose $\lambda(V) = 0$. Then $\mathbb{P}(x \in D^d_{\text{fat}}) = \mathbb{P}(\{x \in D_{\text{fat}}\} \setminus A^x) = 0$ for almost all $x \in [0,1]^d$, by (4.2). Applying Fubini's theorem gives

$$\begin{aligned} \mathbb{E}\lambda(D^d_{\text{fat}}) &= \int_\Omega \int_{[0,1]^d} 1_{D^d_{\text{fat}}}(x,\omega) d\lambda d\mathbb{P} \\ &= \int_{[0,1]^d} \int_\Omega 1_{D^d_{\text{fat}}}(x,\omega) d\mathbb{P} d\lambda \\ &= \int_{[0,1]^d} \mathbb{P}(x \in D^d_{\text{fat}}) d\lambda = 0. \end{aligned}$$

Therefore $\lambda(D^d_{\text{fat}}) = 0$ a.s. By Theorem 1.7 we have $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction.

Next suppose that $\lambda(V) = 1$. Then with a similar argument we can show that $\lambda(D^c_{\text{fat}}) = 0$ and $\lambda(D^d_{\text{fat}}) > 0$ a.s. given non-extinction. □

4.3. *Proof of Theorem 1.9.*

Proof of Theorem 1.9: (i) Suppose that D_{fat} has a non-empty interior with positive probability. Then we have

$$\begin{aligned} 0 &< \mathbb{P}(D_{\text{fat}} \text{ has non-empty interior}) \\ &= \mathbb{P}(\exists n, \exists \mathbf{i}_1, \ldots, \mathbf{i}_n : C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}) \\ &\leq \sum_{n, \mathbf{i}_1, \ldots, \mathbf{i}_n} \mathbb{P}(C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}). \end{aligned}$$

Since we sum over countably many cubes, there must exist n and $\mathbf{i}_1,\dots,\mathbf{i}_n$ such that $\mathbb{P}(C(\mathbf{i}_1,\dots,\mathbf{i}_n)\subset D_{\text{fat}})>0$. Hence, by translation invariance, $\mathbb{P}(C(\mathbf{i}_1,\dots,\mathbf{i}_n)\subset D_{\text{fat}})>0$ for this specific n and all $\mathbf{i}_1,\dots,\mathbf{i}_n$. We can apply the FKG inequality to obtain $\mathbb{P}(D_{\text{fat}}=[0,1]^d)=\mathbb{P}(C(\mathbf{i}_1,\dots,\mathbf{i}_n)\subset D_{\text{fat}}\ \forall \mathbf{i}_1,\dots,\mathbf{i}_n)>0$. Since $\mathbb{P}(D_{\text{fat}}=[0,1]^d)=\prod_{n=1}^{\infty}p_n^{N^{dn}}$, this proves the first part of the theorem.

(ii) Suppose $\prod_{n=1}^{\infty}p_n^{N^n}>0$. Then for each $x\in[0,1]^{d-1}$ we have $\mathbb{P}(\{x\}\times[0,1]\subset D_{\text{fat}})\geq\prod_{n=1}^{\infty}p_n^{N^n}>0$. Let λ_{d-1} denote $(d-1)$-dimensional Lebesgue measure. Applying Fubini's theorem gives

$$\begin{aligned}
&\mathbb{E}\lambda_{d-1}(\{x\in[0,1]^{d-1}:\{x\}\times[0,1]\subset D_{\text{fat}}\})\\
=&\int_{\Omega}\int_{[0,1]^{d-1}}1_{\{x\}\times[0,1]\subset D_{\text{fat}}}\,d\lambda_{d-1}d\mathbb{P}\\
=&\int_{[0,1]^{d-1}}\int_{\Omega}1_{\{x\}\times[0,1]\subset D_{\text{fat}}}\,d\mathbb{P}d\lambda_{d-1}\\
=&\int_{[0,1]^{d-1}}\mathbb{P}(\{x\}\times[0,1]\subset D_{\text{fat}})d\lambda_{d-1}>0.
\end{aligned}$$

Hence,

$$\lambda_{d-1}(\{x\in[0,1]^{d-1}:\{x\}\times[0,1]\subset D_{\text{fat}}\})>0 \tag{4.5}$$

with positive probability. Observe that

$$D_{\text{fat}}^c\supset\bigcup_{x\in[0,1]^{d-1}:\{x\}\times[0,1]\subset D_{\text{fat}}}\{x\}\times[0,1].$$

In particular,

$$\lambda(D_{\text{fat}}^c)\geq\lambda_{d-1}(\{x\in[0,1]^{d-1}:\{x\}\times[0,1]\subset D_{\text{fat}}\}).$$

From (4.5) we conclude that $\lambda(D_{\text{fat}}^c)>0$ with positive probability. It now follows from Theorem 1.8 that the Lebesgue measure of the dust set is 0 a.s.

(iii) Next assume that $\prod_{n=1}^{\infty}p_n^{N^{dn}}>0$. For each level n, we have $\mathbb{P}(D_{\text{fat}}^n=D_{\text{fat}}^{n-1})\geq p_n^{N^{dn}}$. Since $\prod_{n=1}^{\infty}p_n^{N^{dn}}>0$ is equivalent to $\sum_{n=1}^{\infty}(1-p_n^{N^{dn}})<\infty$, we have

$$\sum_{n=1}^{\infty}\mathbb{P}(D_{\text{fat}}^n\neq D_{\text{fat}}^{n-1})\leq\sum_{n=1}^{\infty}(1-p_n^{N^{dn}})<\infty.$$

Applying the Borel-Cantelli lemma gives that, with probability 1, $\{D_{\text{fat}}^n\neq D_{\text{fat}}^{n-1}\}$ occurs for only finitely many n. Hence, with probability 1 there exists an n such that D_{fat} can be written as the union of level-n cubes. □

4.4. *Proof of Theorem 1.11.*

Proof of Theorem 1.11: $(iii)\Rightarrow(ii)$. Trivial.

$(ii) \Rightarrow (i)$. Suppose $\mathbb{P}(x \text{ connected to } y) > 0$ for all $x, y \in U$, for some set $U \subset [0,1]^2$ with $\lambda(U) > 0$. Fix $y \in U$. By Fubini's theorem

$$\begin{aligned} \mathbb{E}\lambda(D^c_{\text{fat}}) &= \int_\Omega \int_{[0,1]^2} 1_{D^c_{\text{fat}}}(x,\omega) d\lambda(x) d\mathbb{P}(\omega) \\ &= \int_{[0,1]^2} \int_\Omega 1_{D^c_{\text{fat}}}(x,\omega) d\mathbb{P}(\omega) d\lambda(x) \\ &= \int_{[0,1]^2} \mathbb{P}(x \in D^c_{\text{fat}}) d\lambda(x) \\ &\geq \int_{U\setminus\{y\}} \mathbb{P}(x \text{ connected to } y) d\lambda(x) > 0. \end{aligned}$$

Hence $\lambda(D^c_{\text{fat}}) > 0$ with positive probability. By Theorem 1.8 it follows that $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction of the fat fractal process.

$(i) \Rightarrow (iii)$. Next suppose that $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction of the fat fractal process. For points $x \in [0,1]^2$ not on the boundary of a subcube, define the event A^x as in the proof of Theorem 1.8. It follows from the proof of Theorem 1.8 that $\mathbb{P}(A^x) = 1$ for all $x \in V$, for some set $V \subset [0,1]^2$ with $\lambda(V) = 1$. By (4.2) we have for all $x \in V$

$$\mathbb{P}(x \in D^c_{\text{fat}}) = \mathbb{P}(x \in D_{\text{fat}}) > 0.$$

Let $x \in V$. Then

$$0 < \mathbb{P}(x \in D^c_{\text{fat}}) \leq \sum_{n=1}^{\infty} \mathbb{P}(\text{diam}(C^x_{\text{fat}}) > \tfrac{1}{n}),$$

where $\text{diam}(C^x_{\text{fat}})$ denotes the diameter of the set C^x_{fat}. So there exists a natural number n_x such that $\mathbb{P}(\text{diam}(C^x_{\text{fat}}) > \frac{1}{n_x}) > 0$. Hence

$$\mathbb{P}(x \text{ connected to } S(x, \tfrac{1}{2n_x})) > 0,$$

where $S(x, \frac{1}{2n_x})$ is a circle centered at x with radius $\frac{1}{2n_x}$. Write $x = (x_1, x_2)$ and define the following subsets of $\mathbb{R}^2$

$$\begin{aligned} H_1 &= [0,1] \times [x_2 - \tfrac{1}{4n_x}, x_2], \\ H_2 &= [0,1] \times [x_2, x_2 + \tfrac{1}{4n_x}], \\ V_1 &= [x_1 - \tfrac{1}{4n_x}, x_1] \times [0,1], \\ V_2 &= [x_1, x_1 + \tfrac{1}{4n_x}] \times [0,1]. \end{aligned}$$

Note that for every $x \in [0,1]^2$ it is the case that at least one horizontal strip H_i and at least one vertical strip V_j is entirely contained in $[0,1]^2$. Define the event Γ_x by

$$\begin{aligned} \Gamma_x = & \bigcap_{i\in\{1,2\}:H_i\subset[0,1]^2} \{\text{horizontal crossing in } H_i\} \\ \cap & \bigcap_{j\in\{1,2\}:V_j\subset[0,1]^2} \{\text{vertical crossing in } V_j\}. \end{aligned}$$

See Figure 4.3 for an illustration of the event Γ_x. From Theorem 2 in Chayes et al. (1988) it follows that in the MFP model with parameter $p \geq p_c(N,2)$, the limit set D_p connects the left-hand side of $[0,1]^2$ with its right-hand side with positive probability. It then follows from the RSW lemma (e.g. Lemma 5.1 in Dekking and Meester (1990)) and the FKG inequality that $\mathbb{P}_p(\Gamma_x) > 0$. Let A_n denote the event of complete retention until level n, i.e. $\omega(C(\mathbf{I})) = 1$ for all $\mathbf{I} \in \mathcal{J}^{(n-1)}$. Since $\prod_{n=1}^{\infty} p_n > 0$ there exists an integer n_0 such that $p_n \geq p_c(N,2)$ for all $n \geq n_0$. Hence, the probability measure $\mathbb{P}_{\text{fat}}(\cdot|A_{n_0})$ dominates $\mathbb{P}_{p_c(N,2)}(\cdot)$. Since $\mathbb{P}_{\text{fat}}(A_{n_0}) > 0$ it follows that $\mathbb{P}_{\text{fat}}(\Gamma_x) > 0$.

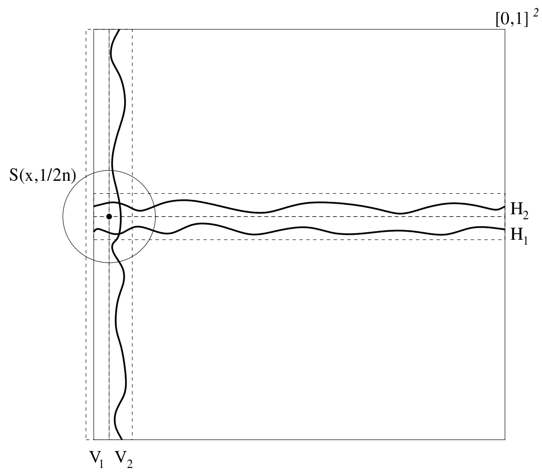

FIGURE 4.3. Realization of the event Γ_x.

Observe that for $x, y \in V$

$$\{x \text{ connected to } y\} \supset \{x \text{ connected to } S(x, \tfrac{1}{2n_x})\} \cap \Gamma_x \cap \{y \text{ connected to } S(y, \tfrac{1}{2n_y})\} \cap \Gamma_y.$$

Since all four events on the right-hand side are increasing and have positive probability, we can apply the FKG inequality to conclude that for all $x, y \in V$ we have $\mathbb{P}(x \text{ connected to } y) > 0$. □

References

E. I. Broman and F. Camia. Large-N limit of crossing probabilities, discontinuity, and asymptotic behavior of threshold values in Mandelbrot's fractal percolation process. *Electron. J. Probab.* **13**, no. 33, 980–999 (2008). MR2413292.

E. I. Broman and F. Camia. Universal behavior of connectivity properties in fractal percolation models. *Electron. J. Probab.* **15**, 1394–1414 (2010). MR2721051.

J. T. Chayes, L. Chayes and R. Durrett. Connectivity properties of Mandelbrot's percolation process. *Probab. Theory Related Fields* **77** (3), 307–324 (1988). MR931500.

L. Chayes. Aspects of the fractal percolation process. In *Fractal geometry and stochastics (Finsterbergen, 1994)*, volume 37 of *Progr. Probab.*, pages 113–143. Birkhäuser, Basel (1995). MR1391973.

L. Chayes, R. Pemantle and Y. Peres. No directed fractal percolation in zero area. *J. Statist. Phys.* **88** (5-6), 1353–1362 (1997). MR1478072.

F. M. Dekking and R. W. J. Meester. On the structure of Mandelbrot's percolation process and other random Cantor sets. *J. Statist. Phys.* **58** (5-6), 1109–1126 (1990). MR1049059.

M. Dekking and H. Don. Correlated fractal percolation and the Palis conjecture. *J. Stat. Phys.* **139** (2), 307–325 (2010). MR2602964.

K. J. Falconer and G. R. Grimmett. On the geometry of random Cantor sets and fractal percolation. *J. Theoret. Probab.* **5** (3), 465–485 (1992). MR1176432.

T. M. Liggett and J. E. Steif. Stochastic domination: the contact process, Ising models and FKG measures. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2), 223–243 (2006). MR2199800.

R. Lyons. Random walks, capacity and percolation on trees. *Ann. Probab.* **20** (4), 2043–2088 (1992). MR1188053.

B. B. Mandelbrot. *The fractal geometry of nature*. W. H. Freeman and Co., San Francisco, Calif. (1982). ISBN 0-7167-1186-9. Schriftenreihe für den Referenten. [Series for the Referee]. MR665254.

P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (1982). ISBN 0-387-90599-5. MR648108.