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Scaling limits and aging for asymmetric trap models on the complete graph and K processes

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Abstract. We obtain scaling limit results for asymmetric trap models and their infinite volume counterparts, namely asymmetric K processes. Aging results for the latter processes are derived therefrom.

1. Introduction

The long time behavior of trap models and related processes with disordered parameters has been the theme of several papers in the recent literature. From the inaugurating work of Bouchaud (1992), where the case of the complete graph was shown to exhibit *aging*, the same as well as other cases were analysed. The model on the complete graph was further studied in Bovier and Faggionato (2005) and Fontes and Mathieu (2008), with different points of view, and considering distinct time scales. And more recently, Gayrard (2012) took up the asymmetric case, which is also the model we study here.

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The trap model in the complete graph is sometimes also called REM-like trap model, due to its resemblance to a dynamics for the Random Energy Model (REM Derrida (1980)). Such a dynamics for the REM, on the hypercube rather than the complete graph, was studied in Ben Arous et al. (2003a,b), where aging results comparable to the ones of Bouchaud were derived. See also Černý (2009); Fontes and Lima (2006); Gayrard (2010). Trap and trap-like models associated to correlated energy (mean field) spin glasses have been the object of more recent work: a dynamics for the p-spin model was studied in Ben Arous et al. (2008); Bovier and Gayrard (2010), and results on the GREM-like trap model were obtained in Fontes et al. (2011).

Trap models on $\mathbb{Z}^d$ have also attracted a lot of interest, in connection with aging as well as with *localization*; see Fontes et al. (2002); Ben Arous and Černý (2005, 2007); Ben Arous et al. (2006); Fontes and Mathieu (2010) – results on the asymmetric case were obtained recently in Barlow and Černý (2011); Černý (2011); Mourrat (2011). Analyses on tori were performed in Ben Arous and Černý (2006); Jara et al. (2011).

In this paper, we revisit the trap model in the complete graph, described briefly below in this introduction, and in full in Section 3. Our goal is twofold:

(1) to propose a representation of the model – in terms of trap depth, rather than location – for which scaling limits can be derived in a unified manner in different scaling regimes;
(2) and to introduce the infinite volume processes which result from these scaling limits, in particular the asymmetric K process.

Let us now briefly describe the asymmetric trap model in the complete graph with n vertices. This is a continuous time Markov chain on the vertices of that graph, whose mean jump time at site x is given by τ_x^{1-a}, where $a \in [0,1]$ is an asymmetry parameter, and whose transition probability from any site x to any site y is proportional to τ_y^a, where $\{\tau_x\}$ are iid positive random variables in the domain of attraction of an α-stable law. The random variable τ_x^{1-a} may be interpreted as the depth of the trap at site x. One readily checks that this dynamics is reversible with respect to the measure whose weights are given by $\{\tau_x\}$. The case $a = 0$ is that of the *symmetric* model. We call the general case where $a \in [0,1]$ the *asymmetric* model. Let $Y_n(t)$ denote the site visited at time t.

This paper is more immediately related to Fontes and Mathieu (2008) and Gayrard (2012), so let us briefly outline our results here against the background of the ones of those papers. Whereas in the former reference a scaling limit was derived for the symmetric model at times of the order of the deepest trap in the landscape, and then aging results were derived for a class of two-time correlation functions of the limit model at vanishing times, in here we present similar limit results for the asymmetric model. Rather than looking at $Y_n(t)$ however, we consider $Z_n(t) = \tau_{Y_n(t)}^{1-a}$, the depth of the currently visited trap. As explained below, this is a convenient representation for taking scaling limits, not only at times of the order of the deepest trap in the landscape, which we do here using this representation (see Theorem 3.1), obtaining a limiting process which we denote by Z, but at shorter time scales as well. We call Z the *asymmetric K process*, in allusion to the K process introduced in Fontes and Mathieu (2008). We further derive a scaling limit result for Z at vanishing times (see Theorem 3.2), obtaining a limiting process $\hat{Z}$ which is self similar of index 1. The latter fact may be interpreted as a fuller aging result for Z, involving the dynamics itself, not only a class of correlation functions thereof. Other scaling regimes of Z_n may be analysed with the same approach, with similar results.

Scaling limits of asymmetric trap models in the complete graph are also the main theme of Gayrard (2012). In that work scaling limits of the *clock process* are derived in several scaling regimes (essentially all of them: from "order 1", where the volume limit is taken first, and then the time limit, to the scale where the model is virtually at equilibrium, including scales in between, in particular the ones treated here); occurrence of aging and other dynamical phenomena are discussed for each regime.

One reason to consider a representation like Z_n, as we do here, rather than the clock process, is that, besides the information on the jump times given by the latter process, Z_n provides also location information, absent in that process. For, say, correlation functions which depend only on jump times (like the Π functions discussed on Subsection 3.3 below; see (3.66-3.69)), the clock process is enough. But other ones require location information, and in those cases the clock process is no longer enough on its own. We discuss two such examples in Subsection 3.3 below.

Z_n and Z, as well as their rescaled versions, and $\hat{Z}$ also, can be described as functions of two related subordinators, the second being obtained as the integral of an independent iid family of mean 1 exponential random variables with respect to the first one. Once we obtain the limit of the first subordinator in a given scaling regime, a continuity property of the above mentioned function implies a limit result for the original process. Section 2 below is devoted to establishing that continuity property (see Lemma 2.1) in a somewhat abstract setting, which may turn out to be the setting of similar processes.

In Section 3 we describe our trap models and K processes in more detail and then, applying the auxiliary result of Section 2, we derive scaling limit results for them, as anticipated above, the one for the trap model in Subsection 3.1, and the one for the K process in Subsection 3.2. In the closing Subsection 3.3 we discuss the derivation of aging results for three particular two-time correlation functions of Z as corollaries to Theorem 3.2.

2. A continuity lemma about a class of trajectories in D

Let D be the space of càdlàg real trajectories on $\mathbb{R}^+ = [0,\infty)$ equipped with the J_1 Skorohod metric (see e.g. Ethier and Kurtz (1986) Chapter 3, Section 5). Let $\mathbb{N}^* = \{1, 2, \ldots\}$ denote the positive integers.

Let S, S^ε, $\varepsilon > 0$, be nonnegative nondecreasing jump functions in D, i.e., suppose that there exist (countable) subsets $A^\varepsilon = \{x_i^\varepsilon,\, i \in \mathbb{N}^*\}$ and $A = \{x_i,\, i \in \mathbb{N}^*\}$ of $\mathbb{R}^+$ and positive number sequences $\{\gamma^\varepsilon_{x_i^\varepsilon},\, i \in \mathbb{N}^*\}$ and $\{\gamma_{x_i},\, i \in \mathbb{N}^*\}$ such that

$$S_r^\varepsilon = \sum_{i:\, x_i^\varepsilon \in [0,r]} \gamma^\varepsilon_{x_i^\varepsilon} < \infty, \quad S_r = \sum_{i:\, x_i \in [0,r]} \gamma_{x_i} < \infty, \quad r \geq 0. \tag{2.1}$$

Consider $\{T_i,\, i \in \mathbb{N}^*\}$, a family of i.i.d. exponential random variables of mean 1, and let

$$\Gamma_r^\varepsilon = \sum_{i:\, x_i^\varepsilon \in [0,r]} \gamma^\varepsilon_{x_i^\varepsilon} T_i, \quad \Gamma_r = \sum_{i:\, x_i \in [0,r]} \gamma_{x_i} T_i, \quad r \geq 0, \tag{2.2}$$

$$Z_t^\varepsilon = \begin{cases} \gamma^\varepsilon_{x^\varepsilon_{i_0}}, & \text{if } t \in [\Gamma^\varepsilon_{x^\varepsilon_{i_0}-}, \Gamma^\varepsilon_{x^\varepsilon_{i_0}}) \text{ for some } i_0, \\ 0, & \text{if } t \notin [\Gamma^\varepsilon_{x^\varepsilon_i-}, \Gamma^\varepsilon_{x^\varepsilon_i}) \text{ for any } i, \end{cases} \tag{2.3}$$

and

$$Z_t = \begin{cases} \gamma_{x_{i_0}}, & \text{if } t \in [\Gamma_{x_{i_0}-}, \Gamma_{x_{i_0}}) \text{ for some } i_0, \\ 0, & \text{if } t \notin [\Gamma_{x_i-}, \Gamma_{x_i}) \text{ for any } i. \end{cases} \tag{2.4}$$

Below, we will use the symbol $\xrightarrow{J_1}$ to denote (strong) convergence on (D, J_1), while $\xrightarrow{J_1,P}$ will denote weak convergence on (D, J_1) with respect to a given probability measure P.

For the moment let P denote the probability measure on (D, J_1) induced by the distribution of $\{T_i,\, i \in \mathbb{N}^*\}$.

Lemma 2.1. *Let S^ε, S, Z^ε, Z be as above. As $\varepsilon \to 0$, if $S^\varepsilon \xrightarrow{J_1} S$, then $Z^\varepsilon \xrightarrow{J_1,P} Z$.*

Remark 1. From equations (2.1-2.4), we see that $Z = \Xi(S, \{T_i,\, i \in \mathbb{N}^*\})$ and $Z^\varepsilon = \Xi(S^\varepsilon, \{T_i,\, i \in \mathbb{N}^*\})$, where Ξ is the composition underlying the above definitions. Lemma 2.1 then establishes a continuity property of the distribution of Ξ in its first argument.

Proof of Lemma 2.1

We will assume that there exists $R' \in \mathbb{R}^+$ such that $|A \cap [0, R']| = \infty$. Other cases may be argued similarly, yet more simply.

Let Γ^{-1} be the (right continuous) inverse of Γ. Let us fix $T > 0$. Then one readily checks that, given $\delta > 0$, there exists $R \notin A$, $R \geq R'$, such that

$$\mathbb{P}(\Gamma^{-1}(T) \geq R) \leq \delta. \tag{2.5}$$

Given $\eta > 0$, we may choose $\delta' > 0$ be such that

$$S_{R+\delta'} - S_R < \eta. \tag{2.6}$$

Let us now enumerate $A \cap [0, R] = \{x_1, x_2, \ldots\}$ such that $\gamma_{x_1} \geq \gamma_{x_2} \geq \ldots$. From the hypothesis, there exists $m = m(\varepsilon)$, with $m \to \infty$ as $\varepsilon \to 0$, and an enumeration of $A^\varepsilon \cap [0, R] = \{x^\varepsilon_1, x^\varepsilon_2, \ldots\}$ such that as $\varepsilon \to 0$

$$\left(\sup_{1\leq i\leq m} |x^\varepsilon_i - x_i|\right) \vee \left(m \sup_{1\leq i\leq m} |\gamma^\varepsilon_{x^\varepsilon_i} - \gamma_{x_i}|\right) \to 0. \tag{2.7}$$

It follows from this and the hypothesis that, given $\eta > 0$, for all small enough ε and $1 \leq k \leq m$

$$\sum_{i>k} \gamma^\varepsilon_{x^\varepsilon_i} = S^\varepsilon_R - \sum_{i=1}^k \gamma^\varepsilon_{x^\varepsilon_i} \leq S_{R+\delta'} - \sum_{i=1}^k \gamma_{x_i} + \eta = S_{R+\delta'} - S_R + \sum_{i>k} \gamma_{x_i} + \eta \leq \sum_{i>k} \gamma_{x_i} + 2\eta. \tag{2.8}$$

We now recall that in the J_1 topology, functions are close if they are uniformly close inside arbitrary bounded intervals, after allowing small time distortions (for details see e.g. Ethier and Kurtz (1986) Chapter 3, Section 5).

Now, given $k \geq 1$ arbitrary but fixed, independent of ε, let $\{\bar{x}_1, \ldots, \bar{x}_k\}$ be an enumeration of $\{x_1, \ldots, x_k\}$ such that $\{\bar{x}_1 < \ldots < \bar{x}_k\}$. This leads to an enumeration $\{\bar{x}^\varepsilon_1, \ldots, \bar{x}^\varepsilon_k\}$ of $\{x^\varepsilon_1, \ldots, x^\varepsilon_k\}$ such that for $1 \leq i \leq k$

$$\bar{x}^\varepsilon_i \to \bar{x}_i \text{ and } \gamma^\varepsilon_{\bar{x}^\varepsilon_i} \to \gamma_{\bar{x}_i} \tag{2.9}$$

(see paragraph of (2.7) above). At this point we relabel $\{T_i\}$ so that $T_1, \ldots, T_k$ are attached to $\bar{x}_1 < \ldots < \bar{x}_k$ and commonly to $x_1^\varepsilon, \ldots, x_k^\varepsilon$, respectively, which does not change distributions. Let $Z^{(k)}$ and $Z^{(k,\varepsilon)}$ be the respective versions of Z and Z^ε with the relabeled $\{T_i\}$.

Let us now take a family of temporal distortions $(\lambda^\varepsilon) = (\lambda_k^\varepsilon)$ as follows. For $1 \le i \le k$, we consider the time intervals $I_i = [t_i^-, t_i]$, where $t_i = \Gamma_{\bar{x}_i}$ and $t_i^- = \Gamma_{\bar{x}_i -}$, and $[t_i^{\varepsilon-}, t_i^\varepsilon]$, where $t_i^\varepsilon = \Gamma^\varepsilon_{\bar{x}_i^\varepsilon}$ and $t_i^{\varepsilon-} = \Gamma^\varepsilon_{\bar{x}_i^\varepsilon -}$, and then define

$$\lambda^\varepsilon(s) = \begin{cases} \frac{t_1^{\varepsilon-}}{t_1^-} s, & \text{if } 0 \le s \le t_1^-, \\ \frac{t_i^\varepsilon - t_i^{\varepsilon-}}{t_i - t_i^-}(s - t_i^-) + t_i^{\varepsilon-}, & \text{if } t_i^- \le s \le t_i, \\ \frac{t_{i+1}^{\varepsilon-} - t_i^\varepsilon}{t_{i+1}^- - t_i}(s - t_i) + t_i^\varepsilon, & \text{if } t_i \le s \le t_{i+1}^-, \\ (s - t_{k+1}^-) + t_{k+1}^{\varepsilon-}, & \text{if } s \ge t_{k+1}^-, \end{cases} \tag{2.10}$$

where $t_{k+1}^- := \Gamma_R$, $t_{k+1}^{\varepsilon-} := \Gamma_R^\varepsilon$.

At this point, we have two tasks: the first one is to control the slopes of the functions λ^ε and the second one is to control the sup norm of the difference $Z^{(k,\varepsilon)}_{\lambda^\varepsilon(t)} - Z_t^{(k)}$.

We start by the second task. Let $\mathcal{M} = \cup_{i=1}^k I_i$. If $t \in \mathcal{M}$, then

$$|Z_t^{(k)} - Z^{(k,\varepsilon)}_{\lambda^\varepsilon(t)}| \le \max_{1 \le i \le k} |\gamma_{\bar{x}_i} - \gamma^\varepsilon_{\bar{x}_i^\varepsilon}|, \tag{2.11}$$

which goes to zero as ε goes to zero by (2.9).

If $t \in [0, t_{k+1}^-] \setminus \mathcal{M}$, then we have that $Z_t^{(k)} \le \gamma_{x_{k+1}}$ and $Z^{(k,\varepsilon)}_{\lambda^\varepsilon(t)} \le \max_{i>k} \gamma^\varepsilon_{x_i^\varepsilon}$. Hence,

$$|Z_t^{(k)} - Z^{(k,\varepsilon)}_{\lambda^\varepsilon(t)}| \le \gamma_{x_{k+1}} \vee \max_{i>k} \gamma^\varepsilon_{x_i^\varepsilon} \le \gamma_{x_{k+1}} \vee \sum_{i>k} \gamma^\varepsilon_{x_i^\varepsilon} \le \sum_{i>k} \gamma_{x_i} + 2\eta, \tag{2.12}$$

for all small enough ε, by (2.8).

Now, we solve the first problem by considering two cases:

1) If $s \in [t_i^-, t_i]$ for some $1 \le i \le k$, then the slope of λ^ε is given by

$$\frac{t_i^\varepsilon - t_i^{\varepsilon-}}{t_i - t_i^-} = \frac{\gamma^\varepsilon_{\bar{x}_i^\varepsilon} T_i}{\gamma_{\bar{x}_i} T_i} = \frac{\gamma^\varepsilon_{\bar{x}_i^\varepsilon}}{\gamma_{\bar{x}_i}} \to 1 \tag{2.13}$$

as $\varepsilon \to 0$, by (2.9).

2) If $s \in [t_i, t_{i+1}^-]$ for some $0 \le i \le k$, where $t_0 := 0$, then it suffices to prove that

$$t_i^\varepsilon \to t_i \tag{2.14}$$

$$t_i^{\varepsilon-} \to t_i^- \tag{2.15}$$

as $\varepsilon \to 0$ in probability.

In all cases, the absolute value of the difference of right and left hand sides is bounded above by

$$\sum_{i=1}^m |\gamma^\varepsilon_{x_i^\varepsilon} - \gamma_{x_i}| T_i + \sum_{i>m} |\gamma^\varepsilon_{x_i^\varepsilon} - \gamma_{x_i}| T_i. \tag{2.16}$$

The first term vanishes almost surely as $\varepsilon \to 0$ by (2.7) and the Law of Large Numbers, and, given $\eta > 0$, the expected value of the second term is bounded above by

$$\sum_{i>m}\gamma^{\varepsilon}_{x^{\varepsilon}_i}+\sum_{i>m}\gamma_{x_i}\le 2\sum_{i>m}\gamma_{x_i}+\eta, \tag{2.17}$$

for all small enough ε, where use is made of (2.8) in the latter inequality, and (2.14, 2.15) follow since $m\to\infty$ as $\varepsilon\to 0$ and η is arbitrary.

To conclude, given $0<\zeta<1,\delta>0$, choose $T>-\log(\zeta/2)$, and then R satisfying (2.5), and then δ' satisfying (2.6) with $\eta=\zeta/4$, and then k such that $\sum_{i>k}\gamma_{x_i}<\zeta/2$. Choosing now λ^{ε} as in (2.10), we conclude that

$$\limsup_{\varepsilon\to 0}\mathbb{P}(d(Z^{(k,\varepsilon)},Z^{(k)})>\zeta)\le\delta, \tag{2.18}$$

where d is the J_1 Skorohod distance on D (see Ethier and Kurtz (1986) Chapter 3, Section 5). Since $Z^{(k,\varepsilon)}=Z^{\varepsilon}$ and $Z^{(k)}=Z$ in distribution for all fixed k and ε small enough, the result follows. □

Let us now explain how Lemma 2.1 will be used in the sequel. Our aim is to apply it to a case where S^{ε} and S are random objects, in fact subordinators, with parameters that are themselves random, which we call *environment*. Both S^{ε} and S, as well as their respective environments, will be independent of $\{T_i\}$, and the convergence $S^{\varepsilon}\to S$ will hold only in distribution: either 1) the joint distribution of the environment and the subordinators, or 2) the distributions of subordinators given the environment, for almost every realization of the environment. In both cases, we may use the Skorohod representation theorem (see e.g. Whitt (2002) Theorem 3.2.2). In case 1) we will first explicitly choose a convenient version of the environment, for which the distribution of the subordinator, given the environment, converges for almost every realization of the environment; with the modified environment, we are effectively in case 2. We can then, by Skorohod representation, in both cases, for each choice of the environment, choose versions of the subordinators that converge almost surely, and then we are in the setting of Lemma 2.1. It is clear that the conclusion of the lemma holds for the original subordinator, where the distribution referred to in the lemma is the joint distribution of $\{T_i\}$ and the subordinators given the original environment in case 2, and the modified environment in case 1, for almost every realization of that environment in each case. In case 1, the result of the lemma will then hold for the overall joint distribution of $\{T_i\}$, the subordinators given the environment, and the environment.

Establishing the convergence in distribution of the subordinators is done by verifying the convergence of the respective Laplace exponents.

3. Application to trap models on the complete graph and K processes

We will apply the lemma above to show scaling limit results for trap models in the complete graph and for K processes. We introduce these two processes next.

We first consider the trap model on the complete graph

$$K_n=\{\{1,\dots,n\},\{(x,y),\,x,y=1,\dots,n\}\} \tag{3.1}$$

with n vertices (differently from the usual definition, here we include self loops, for convenience – this should not matter in the convergence results below): $Y_n=(Y_n(t))_{t\ge 0}$, which is a continuous time Markov chain with jump rate at site x given by

$$\tau_x^{-(1-a)}, \tag{3.2}$$

and transition probability from site x to site y given by

$$\frac{\tau_y^a}{\sum_{z=1}^n \tau_z^a}, \tag{3.3}$$

where $a \in [0,1]$ is a parameter, and

$$\tau := \{\tau_x,\, x = 1, 2, \dots\} \tag{3.4}$$

is an independent family of positive random variables with common distribution in the domain of attraction of a stable law of degree $0 < \alpha < 1$, that is,

$$\mathbb{P}(\tau_1 > t) = \frac{L(t)}{t^\alpha},\ t > 0, \tag{3.5}$$

where L is a slowly varying function at infinity.

We call Y_n an *asymmetric* or *weighted trap model on the complete graph* with asymmetry parameter a, mean jump time parameters $\{\tau_x^{1-a},\, x = 1, \dots, n\}$ and weights $\{\tau_x^a,\, x = 1, \dots, n\}$. The latter set of parameters may indeed be seen as unnormalized weights of the transition probabilities of Y_n. Notice that the $a = 0$ (symmetric) case corresponds to uniform weights.

We will consider the following construction of Y_n. Let

$$\mathcal{N} = \{N^{(x)} := (N_r^{(x)})_{r\geq 0},\, x \in \mathbb{N}^*\} \tag{3.6}$$

be a family of independent Poisson counting processes such that the rate of $N^{(x)}$ is τ_x^a. Let $\sigma_j^{(x)}$ the j-th event time of $N^{(x)}$, $j \geq 1$. Let also

$$\mathcal{T} = \{T_i^{(x)},\, x \in \mathbb{N}^*\} \tag{3.7}$$

be independent mean 1 exponential random variables, independent of $\mathcal{N}$ and τ, and define for $r \geq 0$

$$S_n(r) = \sum_{x=1}^n \tau_x^{1-a}\, N_r^{(x)}, \quad \Gamma_n(r) = \sum_{x=1}^n \tau_x^{1-a} \sum_{i=1}^{N_r^{(x)}} T_i^{(x)}. \tag{3.8}$$

Then

$$Y_n(t) = x,\ \text{if } \Gamma_n(\sigma_j^{(x)}-) \leq t < \Gamma_n(\sigma_j^{(x)}) \ \text{ for some } x, j \geq 1. \tag{3.9}$$

is a construction of Y_n as above described, with initial state distributed on $\{1, \dots, n\}$ in such a way that site x has probability weight proportional to τ_x^a, $x \in \{1, \dots, n\}$.

Remark 2. Regarding the latter point, notice that the initial state of Y_n is the one whose Poisson mark is the earliest, so it corresponds to the minimum of n independent exponential random interarrival times with rates τ_x^a, $x \in \{1, \dots, n\}$, and it is well known that the probability that the minimum of n independent exponential random variables is a given such random variable is proportional to its rate.

Below we will be interested in

$$Z_n(t) = \tau_{Y_n(t)}^{1-a}. \tag{3.10}$$

This is the representation for the process aluded to at the introduction above. It has been considered in Bovier and Faggionato (2005), where the symmetric ($a = 0$) case was studied, and a (single time) scaling limit result was derived for it, first taking the volume, and then the time, to infinity (see Proposition 2.10 in that reference) – this is an aging regime not considered in this paper, but rather in Gayrard (2012).

Remark 3. Z_n and Y_n may be seen as processes in random environment, where τ is the set of random parameters acting as environment. Indeed, given τ, both are Markovian (this should be clear for Y_n, but a moment's thought reveals that it is true for Z_n as well, even when there are same values for τ_i's with distinct i's). Notice also that τ is an environment for S_n as well, which for each $n \geq 1$ is a subordinator for every fixed such environment (recall the discussion at the end of Section 2.1). This aspect, which is characteristic of the complete graph, makes our approach particularly suitable, since by an application of (the continuity) Lemma 2.1, we are left with establishing convergence of subordinators (in the Skorohod topology), which reduces to showing convergence of Laplace exponents (in the topology of real numbers), which is relatively simple, as we will see below.

Remark 4. Given S_n, Z_n may be identified in distribution to $\Xi(S_n, \{T_i,\, i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 1.

We now turn to K processes, which is a Markov process in continuous time on $\bar{\mathbb{N}}^* = \{1, 2, \ldots, \infty\}$ constructed in a similar way as Y_n was above, as follows. Let $\gamma = \{\gamma_x,\, x \in [0, \infty)\}$ be the increments of an α-stable subordinator in $[0, \infty)$ given by a Poisson process $\mathcal{P}$ in $(0, \infty) \times (0, \infty)$ with intensity measure

$$\alpha x^{-1-\alpha}\, dx\, dy. \tag{3.11}$$

It is well known that the *nonzero* set $\{x \in [0, \infty) : \gamma_x > 0\}$ is countable, so in particular the sums over $[0, 1]$ below have a countable number of nonzero terms only, and thus make the usual sense, almost surely.

Let

$$\hat{\mathcal{N}} = \{\hat{N}^{(x)} := (\hat{N}^{(x)}_r)_{r \geq 0},\, x \in [0, 1]\} \tag{3.12}$$

be a family of independent Poisson counting processes such that the rate of $\hat{N}^{(x)}$ is γ_x^a, where $\hat{N}^{(x)} \equiv 0$ whenever $\gamma_x = 0$. Let $\hat{\sigma}_j^{(x)}$ the j-th event time of $\hat{N}^{(x)}$, $j \geq 1$. Let also

$$\hat{\mathcal{T}} = \{\hat{T}_i^{(x)},\, x \in [0, 1]\} \tag{3.13}$$

be a family iid mean 1 exponential random variables independent of $\hat{\mathcal{N}}$.

Define for $r \geq 0$

$$S(r) = \sum_{x \in [0,1]} \gamma_x^{1-a} \hat{N}_r^{(x)}, \quad \Gamma(r) = \sum_{x \in [0,1]} \gamma_x^{1-a} \sum_{i=1}^{\hat{N}_r^{(x)}} \hat{T}_i^{(x)}, \tag{3.14}$$

and then make

$$Y_t = \begin{cases} x, & \text{if } \Gamma(\hat{\sigma}_j^{(x)}-) \leq t < \Gamma(\hat{\sigma}_j^{(x)}) \text{ for some } x, j \geq 1, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.15}$$

Remark 5. It can be verified that when $a > \alpha$, then Y is a jump process, and so there is almost surely no t for which $Y(t) = \infty$ (since in this case $\cup_{j,x}[\Gamma(\hat{\sigma}_j^{(x)}-), \Gamma(\hat{\sigma}_j^{(x)})) = [0, \infty)$). And in the case where $a \leq \alpha$, there almost surely exist t's for which $Y(t) = \infty$. (One way to check these claims is by verifying that when $a > \alpha$, $\{\hat{\sigma}_j^{(x)};\, j \geq 1, x \in [0, 1]\}$ is a discrete subset of $[0, \infty)$ almost surely, and when $a \leq \alpha$, it is almost surely dense in $[0, \infty)$, and these in turn follow from the fact that $\sum_{x \in [0,1]} \gamma_x^a$ is almost surely finite in the former case, and infinite in the latter one.)

Let

$$Z_t = \gamma_{Y_t}^{1-a}, \tag{3.16}$$

where γ_∞ should be interpreted as 0.

Remark 6. Z and Y may be seen as processes in random environment, where γ (more specifically, $\gamma|_{[0,1]} = \{\gamma_x,\, x \in [0,1]\}$) is the environment. Indeed, given γ, both are Markovian. $\gamma|_{[0,1]}$ is also an environment for S, which is a subordinator for every fixed such environment (recall the discussion at the end of Section 2.1).

Remark 7. Given S, Z may be identified in distribution to $\Xi(S, \{T_i,\, i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 1.

Remark 8. In Fontes and Mathieu (2008) and other references the representations used for the trap model and K process are the ones given here by $Y_n(t)$ and $Y(t)$, $t \geq 0$, respectively (see (3.9) and (3.15) above). The alternative representation $Z_n(t)$ and $Z(t)$, $t \geq 0$, we adopt here (see (3.10) and (3.16) above) has the advantage of leading to a unifying approach for taking the scaling limits of those processes, as explained in the introduction and will be done in detail in Subsections 3.1 and 3.2 below.

In the next subsection, we will consider a particular scaling regime for Z_n and establish a scaling limit result under which Z_n converges to the K process. Then, in the following subsection we will derive a scaling limit result satisfied by Z. All proofs will rely on Lemma 2.1 above to get the results from the convergence of the appropriate S^ε in each case (see statement of that lemma and its preliminaries above). In order to obtain the latter convergence, since we have subordinators in all cases, it will suffice to establish convergence of the associated Laplace exponents. The last subsection is devoted to a discussion on aging results (for two-time correlation functions) satisfied by Z as a consequence of Theorem 3.2 and other results.

3.1. *Scaling limit for Z_n at large times.*

For $r \geq 0$, let

$$U(r) = \sum_{x \in [0,r]} \gamma_x. \tag{3.17}$$

Given a sequence $(c_n)_{n\geq 1}$, set

$$Z_t^{(n)} = c_n^{1-a} Z_n(t/c_n^{1-a}),\ t \geq 0. \tag{3.18}$$

Let P_1 denote the probability measure induced on (D, J_1) by the joint distribution of τ, $\mathcal{N}$ and $\mathcal{T}$ – given above in respectively (3.4), (3.6) and (3.7).

Theorem 3.1. *There exists a deterministic sequence $(c_n)_{n\geq 1}$ such that*

$$(Z_t^{(n)})_{t\geq 0} \xrightarrow{J_1, P_1} (Z_t)_{t\geq 0}. \tag{3.19}$$

as $n \to \infty$.

The sequence (c_n) will be exhibited explicitly in the proof below (see 3.22).

Proof:

By Lemma 2.1, and recalling the discussion at the end of Section 2, it is enough to establish the limit

$$S^{(n)} \xrightarrow{J_1, P_1} S, \tag{3.20}$$

where

$$S_r^{(n)} := c_n^{1-a} S_n(c_n^a r) = \sum_{x=1}^{n} (c_n \tau_x)^{1-a} N_{c_n^a r}^{(x)}, \tag{3.21}$$

since, given $S^{(n)}$, $Z^{(n)}$ is identically distributed with $\Xi(S^{(n)}, \{T_i,\, i \in \mathbb{N}^*\})$ – see Remark 1.

In order to establish (3.20), we will make a precise choice of c_n and switch to another version of τ, which properly rescaled converges strongly, rather than weakly. We follow Fontes et al. (2002), Section 3. Let

$$c_n = \left(\inf\{t \geq 0 : \mathbb{P}(\tau_1 > t) \leq n^{-1}\}\right)^{-1}, \tag{3.22}$$

$$\tau_x^{(n)} := c_n^{-1}\, g_n(U(x) - U(x - 1/n)),\; x \in (0,1] \cap \tfrac{1}{n}\mathbb{Z} \tag{3.23}$$

$$g_n(y) = c_n\, G^{-1}(n^{1/\alpha} y),\; y \geq 0, \tag{3.24}$$

where G^{-1} is the inverse of the function G defined by the following condition.

$$\mathbb{P}(U(1) > G(x)) = \mathbb{P}(\tau_1 > x),\; x \geq 0 \tag{3.25}$$

We then have that $\tau^{(n)} := \{\tau_x^{(n)},\, x \geq 1\}$ is equally distributed with τ for every $n \geq 1$.

For $x \in (0,1] \cap \frac{1}{n}\mathbb{Z}$, let now

$$\gamma_x^{(n)} = c_n \tau_x^{(n)}, \tag{3.26}$$

and define

$$\tilde{S}_r^{(n)} := \sum_{x=1}^{n} (\gamma_{x/n}^{(n)})^{1-a} \tilde{N}_r^{(n,x)}, \tag{3.27}$$

where, given γ,

$$\tilde{\mathcal{N}}^{(n)} = \{\tilde{N}^{(n,x)} := (\tilde{N}_r^{(n,x)})_{r \geq 0},\; x \in \mathbb{N}^*\} \tag{3.28}$$

is a family of independent Poisson counting processes such that the rate of $N^{(n,x)}$ is $(\gamma_x^{(n)})^a$.

One now readily checks, using the identity in distribution of $\tau^{(n)}$ and τ for every $n \geq 1$, together with the above definitions, that $\tilde{S}^{(n)} := (\tilde{S}_r^{(n)})_{r \geq 0}$ has the same distribution (induced by $(\gamma, \tilde{\mathcal{N}}^{(n)})$) as $S^{(n)}$ under P' for every $n \geq 1$. So it is enough to show that

$$\tilde{S}^{(n)} \xrightarrow{J_1, P_2} S, \tag{3.29}$$

where P_2 is the probability measure induced on (D, J_1) by the joint distribution of γ and $\tilde{\mathcal{N}}^{(n)}$.

Now since, given γ, $\tilde{S}^{(n)}$ is a subordinator for each $n \geq 1$, it is enough to show the convergence of the Laplace exponents of $\tilde{S}^{(n)}$, $n \geq 1$, as $n \to \infty$, for almost every realization of γ, to the Laplace exponent of S given γ, which is itself a subordinator. (See Corollary 3.6 page 374 in Jacod and Shiryaev (1987).)

A straightforward computation yields

$$\tilde{\varphi}_n(\lambda) := \sum_{x \in (0,1] \cap \frac{1}{n}\mathbb{Z}} (\gamma_x^{(n)})^a \left(1 - e^{-\lambda (\gamma_x^{(n)})^{1-a}}\right) \tag{3.30}$$

as the Laplace exponent of $\tilde{S}^{(n)}$, $n \geq 1$.

Now let

$$\mathfrak{T}_\delta = \{x \in [0,1] :\, \gamma(x) > \delta\} = \{x_1 < \ldots < x_K\}, \tag{3.31}$$

and

$$\mathfrak{T}_\delta^{(n)}=\left\{x_1^{(n)}=\frac{1}{n}\lceil nx_1\rceil<\ldots<x_K^{(n)}=\frac{1}{n}\lceil nx_K\rceil\right\}, \tag{3.32}$$

where the strict inequalities in (3.32) hold provided n is large enough (for each fixed δ).

Lemma 3.1 in Fontes et al. (2002) implies that for every $\delta>0$

$$\sum_{x\in\mathfrak{T}_\delta^{(n)}}(\gamma_x^{(n)})^a(1-e^{-\lambda(\gamma_x^{(n)})^{1-a}})\to\sum_{x\in\mathfrak{T}_\delta}\gamma_x^a(1-e^{-\lambda\gamma_x^{1-a}}) \tag{3.33}$$

almost surely as $n\to\infty$. One also readily checks that

$$\sum_{x\in(0,1]\cap\frac{1}{n}\mathbb{Z}\setminus\mathfrak{T}_\delta^{(n)}}(\gamma_x^{(n)})^a(1-e^{-\lambda(\gamma_x^{(n)})^{1-a}})\le\lambda\sum_{x\in(0,1]\cap\frac{1}{n}\mathbb{Z}\setminus\mathfrak{T}_\delta^{(n)}}\gamma_x^{(n)}. \tag{3.34}$$

Since, as argued in paragraphs of (3.25-3.28) in Fontes et al. (2002), we have that the $\lim_{\delta\to0}\limsup_{n\to\infty}$ of the sum in the right hand side of (3.34) vanishes almost surely, we may conclude that

$$\tilde{\varphi}_n(\lambda)\to\varphi(\lambda):=\sum_{x\in[0,1]}\gamma_x^a(1-e^{-\lambda\gamma_x^{1-a}}),\ \lambda\ge0, \tag{3.35}$$

almost surely. This convergence holds in principle for each $\lambda\ge0$, but it may be argued to hold simultaneously for every $\lambda\ge0$ from the monotonicity of $\tilde{\varphi}_n$ for every $n\ge1$, and the continuity of φ. The right hand side of (3.35) is the Laplace exponent of S given γ, so the proof is complete. ■

3.2. *Scaling limit of Z at small times.* In this subsection, we assume $0\le a<\alpha$. Let

$$Z_t^{(\varepsilon)}=\varepsilon^{-1}Z_{\varepsilon t}. \tag{3.36}$$

Before stating a convergence result for $Z^{(\varepsilon)}$, let us describe the limit process. Let $(\hat{S}_t)_{t\ge0}$ be an $\hat{\alpha}$-stable subordinator, where

$$\hat{\alpha}=\frac{\alpha-a}{1-a}, \tag{3.37}$$

and whose Laplace exponent is given by $\hat{\varphi}(\lambda)=\hat{c}\lambda^{\hat{\alpha}}$, where $\hat{c}$ is a constant to be determined below.

We may then write $\hat{S}$ as a partial sum of its increments as follows.

$$\hat{S}_r=\sum_{x\in[0,r]}\hat{\gamma}_x, \tag{3.38}$$

where $\{\hat{\gamma}_x,\,x\in\mathbb{N}^*\}$ are the increments of $\hat{S}$.

Let now

$$\hat{\Gamma}_r=\sum_{x\in[0,r]}\hat{\gamma}_xT_x, \tag{3.39}$$

where

$$\mathcal{T}':=\{T_x,\,x\in[0,\infty)\} \tag{3.40}$$

is an iid family of mean 1 exponential random variables, independent of $\hat{S}$.

Remark 9. One may readily check that $\hat{\Gamma}$ is also an $\hat{\alpha}$-stable subordinator (under the joint distribution of $\hat{S}$ and $\{T_x,\,x\in[0,\infty)\}$).

Now define

$$\hat{Z}_t = \begin{cases} \hat{\gamma}_x, & \text{if } t \in [\hat{\Gamma}_{x-}, \hat{\Gamma}_x) \text{ for some } x \in [0,\infty) \\ 0, & \text{for all other } t \geq 0, \text{ if any.} \end{cases} \tag{3.41}$$

Remark 10. $\hat{Z}$ may be seen as a process in random environment, where $\hat{S}$ is the environment. Indeed, given $\hat{S}$, $\hat{Z}$ is Markovian. And the distribution of $\hat{Z}$ (integrated over the environment) makes it a self similar process of index 1, that is, $(\hat{Z}_t)_{t\geq 0} = (c^{-1}\hat{Z}_{ct})_{t\geq 0}$ in distribution for every constant $c > 0$. This latter property explains the aging behavior of Z in its small time scaling regime, as established below.

Remark 11. Given $\hat{S}$, $\hat{Z}$ may be identified in distribution to $\Xi(\hat{S}, \{T_i,\, i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 1.

Before we state this subsection's result, let, for γ fixed, $P_3 = P_3^\gamma$ denote the the probability measure induced on (D, J_1) by the joint distribution of $\hat{\mathcal{N}}$ and $\hat{\mathcal{T}}$ – given above in respectively (3.12) and (3.13).

Theorem 3.2. *If* $0 \leq a < \alpha$ *then for almost every* γ

$$(Z_t^{(\varepsilon)})_{t\geq 0} \xrightarrow{J_1, P_3} (\hat{Z}_t)_{t\geq 0}. \tag{3.42}$$

as $\varepsilon \to 0$.

Remark 12. Perhaps more precisely, Theorem 3.2 states that for almost every γ, the distribution of $(Z_t^{(\varepsilon)})$ under P_3 converges to that of $(\hat{Z}_t)$ under P_4, the probability measure induced on (D, J_1) by the joint distribution of γ and $\mathcal{T}'$.

Corollary 13. If $0 \leq a < \alpha$ then

$$(Z_t^{(\varepsilon)})_{t\geq 0} \xrightarrow{J_1, P_5} (\hat{Z}_t)_{t\geq 0} \tag{3.43}$$

as $\varepsilon \to 0$, where P_5 denotes the probability measure induced on (D, J_1) by the joint distribution of γ, $\hat{\mathcal{N}}$ and $\hat{\mathcal{T}}$.

Remark 14. The above corollary follows immediately from the preceding theorem, since P_5 is obtained by integrating P_3 over the distribution of γ. Below we will nevertheless give a direct (sketchy) argument for the corollary, much simpler than the one for the theorem next.

Proof of Theorem 3.2

Let

$$\hat{S}_r^{(\varepsilon)} = \varepsilon^{-1} \sum_{x\in[0,1]} \gamma_x^{1-a} \hat{N}^x_{\varepsilon^{\hat{\alpha}} r},\ r \geq 0 \tag{3.44}$$

where $\hat{\alpha}$ was introduced in (3.37) above. Then, given γ and $\varepsilon > 0$, $(\hat{S}_t^{(\varepsilon)},\, t \geq 0)$ is a subordinator, and its Laplace exponent equals

$$\hat{\varphi}^{(\varepsilon)}(\lambda) = \varepsilon^{\hat{\alpha}} \sum_{x\in[0,1]} \gamma_x^a (1 - e^{-\lambda\varepsilon^{-1}\gamma_x^{1-a}}),\ \lambda \geq 0. \tag{3.45}$$

By Lemma 2.1, and recalling the discussion at the end of Section 2, to get the result, it is enough to establish the limit

$$\hat{S}^{(\varepsilon)} \xrightarrow{J_1, P_5} \hat{S} \tag{3.46}$$

as $\varepsilon \to 0$ for a.e. γ. Since we are dealing with subordinators, it suffices to show that for almost every γ

$$\hat{\varphi}^{(\varepsilon)}(\lambda) \to \hat{c}\lambda^{\hat{\alpha}}, \ \lambda \geq 0, \tag{3.47}$$

as $\varepsilon \to 0$, for some positive finite constant $\hat{c}$. This is obvious for $\lambda = 0$, so let us fix $\lambda > 0$, and write

$$\lambda^{-\hat{\alpha}}\hat{\varphi}^{(\varepsilon)}(\lambda) = R^{-\alpha} \sum_{x\in[0,1]} (R\gamma_x)^a (1 - e^{-(R\gamma_x)^{1-a}}) \tag{3.48}$$

with $R = (\varepsilon^{-1}\lambda)^{\frac{1}{1-a}}$, and then argue in the sequel that the left hand side converges to a constant as $R \to \infty$ for a.e. γ.

We start by considering

$$W := R^{-\alpha} \sum_{x\in[0,1]} \sum_{i=1}^{R\delta^{-1}} (R\gamma_x)^a (1 - e^{-(R\gamma_x)^{1-a}}) \mathbb{I}_{\{\gamma_x \in [\frac{\delta}{R}(i-1), \frac{\delta}{R} i]\}}. \tag{3.49}$$

Since the difference between W and the left hand side of (3.48) is bounded above by

$$R^{-(\alpha-a)} \sum_{x\in[0,1]} \gamma_x^a \, \mathbb{I}_{\{\gamma_x > 1\}}, \tag{3.50}$$

which vanishes as $R \to \infty$ for a.e. γ, it is enough to establish the convergence result for W. We estimate it as follows.

$$W - X_1 \quad \leq \quad R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^+ := R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} (\delta i)^a (1 - e^{-(\delta i)^{1-a}}) M_i \tag{3.51}$$

$$W \quad \geq \quad R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^- := R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} (\delta(i-1))^a (1 - e^{-(\delta(i-1))^{1-a}}) M_i, \tag{3.52}$$

where $X_1 = R^{-\alpha} \sum_{x\in[0,1]} (R\gamma_x)^a (1 - e^{-(R\gamma_x)^{1-a}}) \mathbb{I}_{\{\gamma_x \in [0, \frac{\delta}{R}]\}}$ and M_i is the number of points of $\mathcal{P}$ in the region $[0,1] \times [\frac{\delta}{R}(i-1), \frac{\delta}{R} i]$ (recall paragraph of (3.11) above).

X_1 can be bounded above by $R^{-\alpha} \sum_{x\in[0,1]} (R\gamma_x) \, \mathbb{I}_{\{\gamma_x \in [0, \frac{\delta}{R}]\}}$, and this has the same distribution as $R^{-\alpha} \sum_{x\in[0,R^\alpha]} \gamma_x \, \mathbb{I}_{\{\gamma_x \in [0,\delta]\}}$ for every $R > 0$, by the scale invariance of γ. We can use standard large deviation estimates for the latter expression to conclude that X_1 can be ignored in the limits as $R \to \infty$ and then $\delta \to 0$ (here we may use the existence of a positive exponential moment for $\sum_{x\in[0,1]} \gamma_x \, \mathbb{I}_{\{\gamma_x \in [0,\delta]\}}$ for any δ, a result that follows as an application of Campbell Theorem – see Kingman (1993)). We concentrate on the right hand sides of (3.51, 3.52).

We start with (3.51). By the exponential Markov inequality, we get, for given $\theta, \xi > 0$,

$$\mathbb{P}\left(R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^+ \geq R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E} X_i^+ + \xi \right) \leq \frac{A}{B} \tag{3.53}$$

where $A = \mathbb{E} e^{\theta \sum_{i=1}^{R\delta^{-1}} X_i^+}$ and $B = e^{\theta \sum_{i=1}^{R\delta^{-1}} \mathbb{E} X_i^+ + R^\alpha \xi}$.

Since M_i, $i \geq 2$, are independent Poisson random variables, we obtain

$$\frac{A}{B} = e^{-R^\alpha \xi \theta + \sum_{i=2}^{R\delta^{-1}} (e^{c_i \theta} - 1 - c_i \theta) \, \mathbb{E} M_i}, \tag{3.54}$$

where $c_i = (\delta i)^a(1 - e^{-(\delta i)^{1-a}})$.

We choose $\theta = R^{-b}$ with $a < b < \alpha < 2b$. Then, using the estimate

$$\mathbb{E}M_i = \int_{\frac{\delta}{R}(i-1)}^{\frac{\delta}{R}i} \frac{\alpha}{x^{1+\alpha}}dx \le \frac{R^\alpha}{\delta^\alpha(i-1)^{1+\alpha}}, \tag{3.55}$$

we find that the sum in the exponent in (3.54) is bounded above by

$$\sum_{i=2}^{R\delta^{-1}} \frac{R^\alpha}{\delta^\alpha(i-1)^{1+\alpha}}(c_iR^{-b})^2 \le 2\frac{R^{\alpha-2b}}{\delta^{\alpha-2a}}\sum_{i=1}^{R\delta^{-1}} i^{-(1+\alpha-2a)} \tag{3.56}$$

Since the sum on the right of (3.56) is bounded by constant times $R^{2a-\alpha} \vee \log R$, and using the above estimates, we find that the exponent in (3.54) is bounded above by

$$-R^{\alpha-b}\xi + \text{ const } R^{-c'}, \tag{3.57}$$

for some constant $c' > 0$. We can then apply Borel-Cantelli and conclude that for a.e. γ, given $\xi > 0$

$$R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} X_i^+ \le R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ + \xi \tag{3.58}$$

for all large enough R.

Conversely, we can conclude that given $\xi > 0$, for a.e. γ and all R large enough

$$R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} X_i^- \ge R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^- - \xi. \tag{3.59}$$

(3.58) and (3.59) then imply that

$$\liminf_{R\to\infty} R^{-\alpha}\sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^- \le \liminf_{R\to\infty} W \le \limsup_{R\to\infty} W \le \limsup_{R\to\infty} R^{-\alpha}\sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^+. \tag{3.60}$$

To conclude, it is enough to verify that

$$\liminf_{\delta\to 0}\liminf_{R\to\infty} R^{-\alpha}\sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^- = \limsup_{\delta\to 0}\limsup_{R\to\infty} R^{-\alpha}\sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^+ \tag{3.61}$$

is a (positive finite) constant $\hat{c}$.

We begin with the following estimate.

$$\mathbb{E}X_i^+ = (\delta i)^a(1 - e^{-(\delta i)^{1-a}})\int_{\frac{\delta}{R}(i-1)}^{\frac{\delta}{R}i} \frac{\alpha}{x^{1+\alpha}}dx \le (\delta i)^a(1 - e^{-(\delta i)^{1-a}})\frac{\delta}{R}\frac{\alpha}{(\frac{\delta}{R}(i-1))^{1+\alpha}} \tag{3.62}$$

Summing up:

$$\begin{aligned} R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ &\le R^{-\alpha}\sum_{i=2}^{R\delta^{-1}} (\delta i)^a(1 - e^{-(\delta i)^{1-a}})\frac{\delta}{R}\frac{\alpha}{(\frac{\delta}{R}(i-1))^{1+\alpha}} \\ &= \alpha\sum_{i=2}^{R\delta^{-1}} \delta^{a-\alpha}\frac{1 - e^{-(\delta i)^{1-a}}}{i^{1+\alpha-a}}\left(\frac{i}{i-1}\right)^{1+\alpha} \end{aligned} \tag{3.63}$$

Now as $R \to \infty$, the latter sum converges to a series, which is readily seen to be an approximation to an integral. We find that

$$\limsup_{\delta\to 0}\limsup_{R\to\infty} R^{-\alpha}\sum_{i=2}^{R\delta^{-1}}\mathbb{E}X_i^+ \leq \alpha\int_0^\infty \frac{1-e^{-x^{1-a}}}{x^{1+\alpha-a}}dx. \tag{3.64}$$

We similarly find the latter expression as a lower bound for

$$\liminf_{\delta\to 0}\liminf_{R\to\infty} R^{-\alpha}\sum_{i=2}^{R\delta^{-1}}\mathbb{E}X_i^-$$

and (3.61) follows, with the right hand side of (3.64) as the constant $\hat{c}$. □

(Direct) Proof of Corollary 13 (sketchy)

Under P_5 we may use a different, more suitable version of γ. In view of the right hand side of (3.48), we replace γ_x by $R^{-1}\gamma_{R^\alpha x}$, $x \in [0,1]$, with R as in the above proof. The Laplace exponent of the corresponding version of $\hat{S}^{(\varepsilon)}$ is then readily seen to equal

$$\lambda^{\hat{\alpha}}\, R^{-\alpha}\sum_{x\in[0,R^\alpha]}\gamma_x^a(1-e^{-\gamma_x^{1-a}}). \tag{3.65}$$

Since $\sum_{x\in[0,1]}\gamma_x^a(1-e^{-\gamma_x^{1-a}})$ is integrable, with mean $\hat{c}$, as can be checked by an application of Campbell Theorem, the Law of Large Numbers yields the almost sure convergence of (3.65) to $\hat{c}\lambda^{\hat{\alpha}}$ (simultaneously for all $\lambda \geq 0$, once one uses monotonicity and continuity of the functions involved, as previously argued – see the end of the proof of Theorem 3.1 above). Since that is the Laplace exponent of $(\hat{Z}_t)$, we conclude that the version of $(Z_t^{(\varepsilon)})$ with γ replaced by $\gamma^R := \{R^{-1}\gamma_{R^\alpha x}, x \in [0,1]\}$ converges in $P_3^{\gamma^R}$-distribution to $(\hat{Z}_t)$ for almost every γ. Upon integrating over the distribution of γ, we get the convergence in P_5-distribution. □

Remark 15. A few words about the cases where $\alpha \leq a \leq 1$. When $a > \alpha$, we have that Z is a jump process in $\mathbb{N}^*$ (see Remark 5) with $Z(0) = \gamma_x$ with probability proportional to γ_x^a, $x \in [0,1]$. It follows then that $Z^{(\varepsilon)} \to \infty$ identically almost surely as $\varepsilon \to 0$.

The case $a = \alpha$ demands more delicate analysis. We have that $\hat{\varphi}^{(\varepsilon)}(\lambda)$ (see 3.45), when scaled with a factor of $|\log\varepsilon|^{-1}$ (instead of $\varepsilon^{\hat{\alpha}} = 1$ in this case), converges to a number r independent of $\lambda > 0$ as $n \to \infty$ in probability, and this is the Laplace exponent of a subordinator which equals 0 for an exponentially distributed amount of time of rate r, and then jumps to ∞, where it stays. One may then argue from this that $Z^{(\varepsilon)} \to \infty$ identically as $\varepsilon \to 0$ in probability.

3.3. *Aging in the K process.* Theorem 3.2 may be viewed as an aging result for Z, since $\hat{Z}$ is nontrivial and self similar with index 1. Corresponding aging results for two-time correlation functions follow.

Below we consider three examples of correlation functions related to aging, and derive scaling limit/aging results for them as a consequence of Theorem 3.2 (as well as of other results derived above). Other correlation functions can be similarly treated.

Example 1. We start with the time correlation function introduced in Bouchaud (1992), which is the one that is usually studied in connection with his model. Let

$$\bar{\Pi}(t,s;\gamma)=\mathbb{P}(\text{no jump of } Z \text{ on } [t,t+s]|\gamma) \tag{3.66}$$

(see Remark 16 below).

Let $\Phi\in D$, let $\mathcal{D}(\Phi)$ denote the set of discontinuities of Φ, that is, $\mathcal{D}(\Phi)=\{t\geq 0:\Phi(t)\neq\Phi(t-)\}$, and consider $F:D\times(0,\infty)\times(0,\infty)\to\{0,1\}$ such that

$$F(\Phi;t,s)=1\{[t,t+s]\cap\mathcal{D}(\Phi)=\emptyset\}. \tag{3.67}$$

Then we have that

$$\bar{\Pi}(\varepsilon t,\varepsilon s;\gamma)=\mathbb{E}[F(Z^{(\varepsilon)};t,s)|\gamma]. \tag{3.68}$$

Let also

$$\hat{\Pi}(t,s)=\mathbb{P}(\text{no jump of } \hat{Z} \text{ on } [t,t+s]). \tag{3.69}$$

Since deterministic single times are almost surely continuity points of $\hat{Z}$, we have that $F(\cdot;t,s)$ is almost surely continuous under the distribution of $\hat{Z}$. We thus conclude from Theorem 3.2 that if $0\leq a<\alpha$, then for almost every γ

$$\lim_{\varepsilon\to 0}\bar{\Pi}(\varepsilon t,\varepsilon s;\gamma)=\mathbb{E}[F(\hat{Z};t,s)]=\hat{\Pi}(t,s). \tag{3.70}$$

The aging phenomenon, namely $\hat{\Pi}(\cdot,\cdot)$ being a (nontrivial) function of the ratio of its arguments, then follows from the self similarity with index 1 (and nontriviality) of $\bar{Z}$, but in this case there is an explicit expression for $\hat{\Pi}$, obtained as follows. One readily checks that the right hand side of (3.69) equals $\mathbb{P}([t,t+s]\cap\mathcal{R}(\hat{\Gamma})=\emptyset)$, where $\mathcal{R}(\Phi)$ is the range of $\Phi\in D$. Since $\hat{\Gamma}$ is an $\hat{\alpha}$-stable subordinator (see Remark 9), an application of the Dynkin and Lamperti arcsine law theorem for that probability yields

$$\hat{\Pi}(t,s)=\frac{\sin(\pi\hat{\alpha})}{\pi}\int_{s/(t+s)}^{1}\theta^{-\hat{\alpha}}(1-\theta)^{\hat{\alpha}-1}\,d\theta. \tag{3.71}$$

The limit in (3.70) was first obtained in Bouchaud (1992) (as the expression in (3.71)) for the case where $a=0$. The general case $0\leq a\leq 1$ was first studied in Gayrard (2012) (see Theorem 3.3 for the case $a<\alpha$, and Theorem 3.4 for the case $a>\alpha$; the particular limit (3.70) is (7.5) in that reference).

In case $a\geq\alpha$, the discussion in Remark 15 indicates that the limit in (3.70) is identically 1, and that aging is thus interrupted.

For the next examples, we restrict a to $[0,\alpha)$.

Example 2. Let

$$\bar{R}(t,s;\gamma)=\mathbb{P}(Z(t)=Z(t+s)|\gamma). \tag{3.72}$$

Then, the difference between $\bar{R}(\varepsilon t,\varepsilon s;\gamma)=\mathbb{P}(\hat{Z}_t^{(\varepsilon)}=\hat{Z}_{t+s}^{(\varepsilon)}|\gamma)$ and $\bar{\Pi}(\varepsilon t,\varepsilon s;\gamma)$ is given by

$$\mathbb{P}(\hat{Z}_t^{(\varepsilon)}=\hat{Z}_{t+s}^{(\varepsilon)};\,\hat{Z}_t^{(\varepsilon)}\neq\hat{Z}_{t+r}^{(\varepsilon)} \text{ for some } r\in[0,s]|\gamma). \tag{3.73}$$

Let $\hat{\mathcal{P}}^{(\varepsilon)}$ and $\hat{\mathcal{P}}$ denote the point processes in $(0,\infty)\times(0,\infty)$ associated to $\hat{S}^{(\varepsilon)}$ and $\hat{S}$, respectively, i.e.,

$$\begin{aligned}\hat{\mathcal{P}}^{(\varepsilon)}&=\left\{\left(t,\hat{S}_t^{(\varepsilon)}-\hat{S}_{t-}^{(\varepsilon)}\right):\ t>0,\ \hat{S}_t^{(\varepsilon)}-\hat{S}_{t-}^{(\varepsilon)}>0\right\},\\ \hat{\mathcal{P}}&=\left\{\left(t,\hat{S}_t-\hat{S}_{t-}\right):\ t>0,\ \hat{S}_t-\hat{S}_{t-}>0\right\}.\end{aligned} \tag{3.74}$$

The convergence in distribution $\hat{S}^{(\varepsilon)} \to \hat{S}$ argued in the proof of Theorem 3.2 implies that

$$\hat{\mathcal{P}}^{(\varepsilon)} \to \hat{\mathcal{P}} \tag{3.75}$$

as $\varepsilon \to 0$ in distribution (in the point process sense; for almost every γ).

Let also $\hat{\Gamma}^{(\varepsilon)}(t) = \varepsilon^{-1}\Gamma(\varepsilon^{\hat{\alpha}}t)$, $t \geq 0$ (see paragraph of (3.14) above). We have that

$$\hat{\Gamma}^{(\varepsilon)} \to \hat{\Gamma} \tag{3.76}$$

in distribution for almost every γ (see (3.39) above). This claim may be argued as follows. Since $(\hat{\Gamma}^{(\varepsilon)}_t)$ is a subordinator, an entirely similar reasoning to the one employed in the proof of Theorem 3.2 may also be employed to establish this result. It also follows from a continuity property of $(\hat{\Gamma}^{(\varepsilon)}_t)$ as a function of $(\hat{S}^{(\varepsilon)}_t)$ and $\mathcal{T}$ similar to the one established in Lemma 2.1, and similarly proven. We leave the details for the interested reader.

For arbitrary $\delta, T > 0$, consider now the event

$$A^{(\varepsilon)}_{\delta,T,t,s} = \{\hat{Z}^{(\varepsilon)}_t > \delta,\, \hat{Z}^{(\varepsilon)}_{t+s} > \delta,\, \hat{\Gamma}^{(\varepsilon)}(T) > t+s\}, \tag{3.77}$$

and let $B^{(\varepsilon)}_{\delta,T}$ be the event that there exist two points in $\hat{\mathcal{P}}^{(\varepsilon)} \cap \{(0,2T)\times(\delta/2,\infty)\}$ with the same second coordinate. Now one readily gets from the above convergence results that

$$\lim_{\varepsilon\to 0} \mathbb{P}(A^{(\varepsilon)}_{\delta,T,t,s}|\gamma) = \mathbb{P}(\hat{Z}_t > \delta,\, \hat{Z}_{t+s} > \delta,\, \hat{\Gamma}(T) > t+s), \tag{3.78}$$

and this can be made arbitrarily close to 1 by choosing δ and T appropriately. We also have that

$$\lim_{\varepsilon\to 0} \mathbb{P}(B^{(\varepsilon)}_{\delta,T}|\gamma) = \mathbb{P}(B_{\delta,T}), \tag{3.79}$$

where $B_{\delta,T}$ is the event corresponding to $B^{(\varepsilon)}_{\delta,T}$ upon replacing $\hat{\mathcal{P}}^{(\varepsilon)}$ by $\hat{\mathcal{P}}$. The latter probability clearly vanishes. Now, since the intersection of the event in the probability in (3.73) and $A^{(\varepsilon)}_{\delta,T,t,s}$ is contained in $B^{(\varepsilon)}_{\delta,T}$, we conclude from the above that

$$\lim_{\varepsilon\to 0} \bar{R}(\varepsilon t, \varepsilon s; \gamma) = \lim_{\varepsilon\to 0} \bar{\Pi}(\varepsilon t, \varepsilon s; \gamma) = \hat{\Pi}(t,s) \tag{3.80}$$

for almost every γ.

Remark 16. The *aging* correlation functions

$$\begin{aligned} \Pi(t,s;\gamma) &= \mathbb{P}(\text{no jump of } Y \text{ on } [t,t+s]|\gamma), & (3.81)\\ R(t,s;\gamma) &= \mathbb{P}(Y(t) = Y(t+s)|\gamma) & (3.82) \end{aligned}$$

are more widely considered in the literature than their barred versions (3.66) and (3.72) above. In the present case there is almost surely no difference, since a.s. $\gamma_x \neq \gamma_y$ provided $x \neq y$ and $\gamma_x > 0$.

The above examples could be done either by considering the clock processes $\hat{\Gamma}^{(\varepsilon)}$ and $\hat{\Gamma}$ on their own, together with (3.76), in the case of Example 1, or, in the case of Example 2, we used, besides Theorem 3.2, convergence results for S and Γ (in the appropriate scale), and in both examples the limit is a correlation function of the limiting clock process $\hat{\Gamma}$. Our last example is natural from the aging point of view, requires Theorem 3.2 alone, and the limit is not a function of $\hat{\Gamma}$ alone.

Example 3. Let

$$Q(t,s;\gamma) = \mathbb{P}\left(\sup_{r\in[0,t]} Z(r) < \sup_{r\in[0,t+s]} Z(r)|\gamma\right). \tag{3.83}$$

This function was suggested in Fontes et al. (2002) as a "measure of the prospects for novelty in the system". $\hat{Z}$ is almost surely continuous in single deterministic times, so we have that

$$\lim_{\varepsilon\to 0} Q(\varepsilon t, \varepsilon s;\gamma) = \mathbb{P}\left(\sup_{r\in[0,t]} \hat{Z}(r) < \sup_{r\in[0,t+s]} \hat{Z}(r)\right) =: \hat{Q}(t,s), \tag{3.84}$$

since the function $1\{\sup_{r\in[0,t]} \Phi(r) < \sup_{r\in[0,t+s]} \Phi(r)\}$ is continuous in $\Phi \in D$ for almost every Φ under the distribution of $\hat{Z}$. We note that $\hat{Q}(t,s)$ is a function of the ratio t/s only, by the self similarity of $\hat{Z}$, but an explicit expression is not available, as far as we know, as it is for $\hat{\Pi}(t,s)$.

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