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On exponential growth for a certain class of linear systems

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Abstract. We consider a class of stochastic growth models on the integer lattice which includes various interesting examples such as the number of open paths in oriented percolation and the binary contact path process. Under some mild assumptions, we show that the total mass of the process grows exponentially in time whenever it survives. More precisely, we prove that there exists an open path, oriented in time, along which the mass grows exponentially fast.

1. Introduction

1.1. *Overview.* We consider a class of stochastic growth models on the integer lattice $\mathbb{Z}^d$ which includes a time discretization and a special case of the 'linear systems' discussed in Chapter IX of Liggett's book Liggett (2005). One of the simplest examples is the number of distinct open paths on the cluster of contact process studied by Griffeath (1983) and as is discussed there, it can be thought of as a model of population growth with spatial structure. There has recently been some progress on this type of models such as; phase transition for the growth rate of total population (Yoshida (2008)); diffusivity (Nakashima (2009); Nagahata and Yoshida (2009, 2010b)) or localization (Yoshida (2010); Nagahata and Yoshida (2010a)) of the population density. However, the following fundamental question remains: does

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the total population grow exponentially whenever it survives? It is well known that the answer is affirmative for the classical Galton-Watson process (see, e.g., Corollary 1.6 on p.20 in Asmussen and Hering (1983)). In this paper, we show that the same assertion holds for a fairly general growth models with spatial structure. In fact, we show that there exists a single path along which the population grows exponentially.

Remark 1.1. We say that a process survives if there always exists at least one particle, which seems natural. Note, however, that in the theory of linear systems, the term survival often refers to the stronger condition that the total population grows as fast as its expectation, see e.g. Theorem 2.4 on p.433 in Liggett (2005).

1.2. *Setting and main results.* Let us start by describing the definition of the process. Although we have results for both discrete and continuous time processes, we first focus on the discrete time case and discuss the continuous time case in Section 3. We write $\mathbb{N}$ for the set of nonnegative integers and $\mathbb{N}^*$ for $\mathbb{N} \setminus \{0\}$. Let $B = (B_{x,y})_{x,y \in \mathbb{Z}^d}$ be a random matrix of infinite size whose entries take values in $\{0\} \cup [1,\infty)$. We assume that B is translation invariant in the sense that $(B_{x,y})_{x,y\in\mathbb{Z}^d}$ and $(B_{x+z,y+z})_{x,y\in\mathbb{Z}^d}$ has the same law for any $z \in \mathbb{Z}^d$. Using independent copies $\{B_n\}_{n\in\mathbb{N}^*}$ of B, we define a Markov chain $\{M_n\}_{n\in\mathbb{N}} = \{(M_{n,x})_{x\in\mathbb{Z}^d}\}_{n\in\mathbb{N}}$ as follows:

$$M_{0,x} = \delta_{o,x} \text{ and } M_{n,x} = \sum_{y\in\mathbb{Z}^d} M_{n-1,y} B_{n,y,x} \text{ for } n \in \mathbb{N}^*, \tag{1.1}$$

where o denotes the origin of $\mathbb{Z}^d$ and $\delta_{x,y}$ the Kronecker delta:

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \tag{1.2}$$

The resulting process is $[0,\infty]^{\mathbb{Z}^d}$-valued since the sum in (1.1) may diverge. If we regard M_n as a row vector, we can rewrite the above equation as

$$M_n = (\delta_{o,x})_{x\in\mathbb{Z}^d} B_1 \cdots B_n. \tag{1.3}$$

for $n \in \mathbb{N}^*$. We denote the total mass of the process by

$$|M_n| \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} M_{n,x}. \tag{1.4}$$

We call a sequence $\{\Gamma(n)\}_{n=k}^{l} \subset \mathbb{Z}^d$ $(k < l \leq \infty)$ an open path if $B_{n+1,\Gamma(n),\Gamma(n+1)} \geq 1$ for all $k \leq n < l$. For the sake of shorthand, let

$$c_\delta(B) = P(|M_n| \geq 1 \text{ for all } n \in \mathbb{N}) \sup_{x\in\mathbb{Z}^d} P(B_{o,x} \geq 1+\delta) \tag{1.5}$$

for $\delta > 0$. Now we are in position to state our main result.

Theorem 1.2. *Suppose that there exists $\delta > 0$ such that $c_\delta(B_1) > 0$. Then for each $\epsilon > 0$, there exists a random open path $\{\Gamma(n)\}_{n\in\mathbb{N}}$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log M_{n,\Gamma(n)} \geq c_{\delta-\epsilon}(B_1) \log(1+\delta-\epsilon) \tag{1.6}$$

almost surely on $\{|M_n| \geq 1$ for all $n \in \mathbb{N}\}$. In particular, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log |M_n| \geq c_\delta(B_1) \log(1+\delta) \tag{1.7}$$

almost surely on $\{|M_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$ *(since* $c_{\delta-\epsilon}(B_1) \geq c_{\delta(B_1)}$ *and by the continuity of the logarithm).*

The following corollary is more useful in applications than Theorem 1.2 (see Section 3 below). In what follows, we denote the matrix product $B_1 \cdots B_m$ by $\prod_{k=1}^m B_k$.

Corollary 1.3. *Suppose that there exist* $\delta > 0$ *and* $m \in \mathbb{N}^*$ *such that* $c_\delta(\prod_{k=1}^m B_k) > 0$*. Then for each* $\epsilon > 0$*, there exists a random open path* $\{\Gamma(n)\}_{n\in\mathbb{N}}$ *such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log M_{n,\Gamma(n)} \geq \frac{1}{m} c_{\delta-\epsilon}(\textstyle\prod_{k=1}^m B_k) \log(1+\delta-\epsilon) \tag{1.8}$$

almost surely on $\{|M_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$*. In particular, we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log |M_n| \geq \frac{1}{m} c_\delta(\textstyle\prod_{k=1}^m B_k) \log(1+\delta) \tag{1.9}$$

almost surely on $\{|M_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$*. Suppose on the other hand that* $c_\delta(\prod_{k=1}^m B_k) = 0$ *for all* $\delta > 0$ *and* $m \in \mathbb{N}$ *and that there exists* $r_B > 0$ *such that* $B_{x,y} = 0$ *if* $|x-y| \geq r_B$*. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |M_n| \leq 0 \tag{1.10}$$

almost surely with the convention $\log 0 = -\infty$*.*

Remark 1.4. The main point of the above results is its generality. We put no independence assumptions on the elements of $(B_{x,y})_{x,y\in\mathbb{Z}^d}$, nor the finite range assumption except for the last assertion in Corollary 1.3. This for instance allows us to obtain analogous results for certain continuous time models by simply applying the discrete time results (see Subsection 3.2).

Remark 1.5. As a special case of our results, it follows that the number of open paths of length n in supercritical oriented percolation grows exponentially in n (see Subsection 3.1 below). In this special case, the following interesting result has recently obtained by Kesten et al. (2012+), which is valid also in the subcritical phase. Consider the oriented paths of length n which go through a maximal number of open sites. Then, the number of such *maximal* paths grows exponentially in n for all $p > 0$. However, this work seems to have only limited overlap with ours since they mainly focus on the subcritical phase where the number of such paths does not obey the evolution rule (1.1).

2. Proof of Theorem 1.2 and Corollary 1.3

We prove Theorem 1.2 and Corollary 1.3 in this section. Let us briefly explain the strategy to prove Theorem 1.2. We are going to find an infinite open path Γ which goes through *heavy bonds*, i.e. $B_{n+1,\Gamma(n),\Gamma(n+1)} \geq 1+\delta-\epsilon$, many times. To this end, we first construct a path γ which is not necessarily open but it prefers to go through heavy bonds and its construction uses only local information, that is, it does not refer to the future and also not too much to the past. Next, we consider certain good events $\{G_n(\gamma)\}_{n\in\mathbb{N}}$ such that

1. (1) if G_n happens, then $B_{n+1,\gamma(n),\gamma(n+1)} \geq 1+\delta-\epsilon$ and
2. (2) there exists an infinite open path which shares all the bonds where G_n happens with γ.

Then we can prove, essentially due to the locality of γ, the law of large numbers for $\{1_{G_n(\gamma)}\}_{n\in\mathbb{N}}$ and this ensures the existence of the above Γ.

The proof of Corollary 1.3 will be given in the final subsection.

We introduce the notation used in the sequel. For $x, y \in \mathbb{Z}^d$ and $m, n \in \mathbb{N}$, we write

$$(m, x) \rightsquigarrow (m+n, y) \tag{2.1}$$

if there exists an open path $\{x(k)\}_{k=m}^{m+n}$ with $x(m) = x$ and $x(m+n) = y$. We adopt the convention that $(m, x) \rightsquigarrow (m, x)$. We define the process started from $(m, x) \in \mathbb{N} \times \mathbb{Z}^d$ by

$$M_{m,y}^{(m,x)} = \delta_{x,y} \text{ and } M_{n,y}^{(m,x)} = \sum_{z\in\mathbb{Z}^d} M_{n-1,z} B_{n,z,y} \text{ for } n > m. \tag{2.2}$$

We call $(m, x) \in \mathbb{N} \times \mathbb{Z}^d$ a *percolation point* if

$$|M_n^{(m,x)}| \geq 1 \text{ for all } n > m \tag{2.3}$$

and write $\mathcal{P}$ for the set of all percolation points. Finally, we introduce the sigma-fields

$$\mathscr{F}_{m,n} = \sigma\left[B_{k,x,y} : m \leq k \leq n, x, y \in \mathbb{Z}^d\right] \tag{2.4}$$

for $m, n \in \mathbb{N}^*$ with $m \leq n$.

2.1. *Construction of the path.* We assume for simplicity that

$$\text{there exists a site } x \in \mathbb{Z}^d \text{ which maximizes } P(B_{o,x} \geq 1+\delta). \tag{2.5}$$

Otherwise, pick a site for which $P(B_{o,x} \geq 1 + \delta - \epsilon) > 0$ and replace δ by $\delta - \epsilon$ in what follows. Let us fix an enumeration of $\mathbb{Z}^d$ and write Min$A$ for the first element appearing in $A \subset \mathbb{Z}^d$. We define a path $\gamma = \{\gamma(n)\}_{n\in\mathbb{N}}$ according to the following recursive algorithm:

(i) Let $\gamma(0) = o$.

(ii) If $(n, \gamma(n)) \rightsquigarrow (n+1, \gamma(n) + x)$, then let $\gamma(n+1) = \gamma(n) + x$.

(iii) If $(n, \gamma(n)) \not\rightsquigarrow (n+1, \gamma(n) + x)$ and $|M_{n+1}^{(n,\gamma(n))}| \geq 1$, then let

$$\gamma(n+1) = \gamma(n) + \mathrm{Min}\{y \in \mathbb{Z}^d : M_{n+1,\gamma(n)+y}^{(n,\gamma(n))} \geq 1\}.$$

(iv) If $|M_{n+1}^{(n,\gamma(n))}| = 0$ and $\{k \in \mathbb{N} : k \leq n, |M_{n+1}^{(k,\gamma(k))}| \geq 1\} \neq \emptyset$, then let

$$T_n = \max\{k \in \mathbb{N} : k \leq n, |M_{n+1}^{(k,\gamma(k))}| \geq 1\}$$

and

$$\gamma(n+1) = \gamma(T_n) + \mathrm{Min}\{y \in \mathbb{Z}^d : M_{n+1,\gamma(T_n)+y}^{(T_n,\gamma(T_n))} \geq 1\}$$

(v) If $|M_{n+1}^{(n,\gamma(n))}| = 0$ and $\{k \in \mathbb{N} : k \leq n, |M_{n+1}^{(k,\gamma(k))}| \geq 1\} = \emptyset$, then let $\gamma(n+1) = o$.

Also for each $(m, v) \in \mathbb{N} \times \mathbb{Z}^d$, we define a path $\gamma^{(m,v)} = \{\gamma^{(m,v)}(n)\}_{n\geq m}$ in the same way as above but we let $\gamma^{(m,v)}(m) = v$ in (i), restrict the ranges of k to $m \leq k \leq n$ in (iv) and (v), and let $\gamma^{(m,v)}(n+1) = v$ in (v). We denote by $T_n^{(m,v)}$ the corresponding T_n.

The following properties are obvious from the construction:

$$(n,\gamma^{(m,v)}(n)) \rightsquigarrow (n+1,\gamma^{(m,v)}(n)+x) \text{ implies } \gamma^{(m,v)}(n+1)=\gamma^{(m,v)}(n)+x, \tag{2.6}$$

$$(\gamma^{(m,v)}(m),\dots,\gamma^{(m,v)}(n)) \text{ is } \mathscr{F}_{m+1,n}\text{-measurable.} \tag{2.7}$$

We also know that the construction does not go back beyond a percolation point, which will be crucial in the proof of Theorem 1.2:

Proposition 2.1. *Let* $\gamma=\gamma^{(0,o)}$ *be the path constructed above and* $l,m\in\mathbb{N}$*. Then on the event* $|M_{m+l}^{(m,\gamma(m))}|\ge 1$*,*

(a) $\gamma(n)=\gamma^{(m,\gamma(m))}(n)$ *for all* $n=m,m+1,\dots,m+l$*,*
(b) $(m,\gamma(m))\rightsquigarrow(n,\gamma(n))$ *for all* $n=m,m+1,\dots,m+l$*.*

In particular, if $(m,\gamma(m))\in\mathcal{P}$*, then* (a) *and* (b) *hold for all* $n\ge m$*.*

Proof: We prove (a) and (b) simultaneously by induction on $n=m,m+1,\dots,m+l$. They are obviously true for $n=m$. Suppose that the claims hold up to some $n\ge m$. If $|M_{n+1}^{(n,\gamma(n))}|\ge 1$, then both $\gamma(n+1)$ and $\gamma^{(m,\gamma(m))}(n+1)$ are chosen by (ii) or (iii) in the algorithm. Since $\gamma(n)=\gamma^{(m,\gamma(m))}(n)$ by the induction hypothesis, they are chosen in the same manner and thus $\gamma(n+1)=\gamma^{(m,\gamma(m))}(n+1)$. Moreover, we have $(n,\gamma(n))\rightsquigarrow(n+1,\gamma(n+1))$ in this case and hence it follows that

$$(m,\gamma(m))\rightsquigarrow(n,\gamma(n))\rightsquigarrow(n+1,\gamma(n+1)) \tag{2.8}$$

by the induction hypothesis. If, on the other hand, $|M_{n+1}^{(n,\gamma(n))}|=0$, then note that $|M_{n+1}^{(m,\gamma(m))}|\ge 1$ as long as $n+1\le m+l$ by the assumption. In particular, it follows that $\gamma(n+1)$ is chosen by (iv) in the algorithm and $m\le T_n\le n$. Then the induction hypothesis shows that (I) $T_n=T_n^{(m,\gamma(m))}$, (II) $\gamma(T_n)=\gamma^{(m,\gamma(m))}(T_n^{(m,\gamma(m))})$, and (III) $(m,\gamma(m))\rightsquigarrow(T_n,\gamma(T_n))$. Since we have

$$\begin{aligned}\gamma(n+1)&=\mathrm{Min}\{y\in\mathbb{Z}^d: M^{(T_n,\gamma(T_n))}_{n+1,y+\gamma(T_n)}\ge 1\}\\&=\mathrm{Min}\{y\in\mathbb{Z}^d: M^{(T_n^{(m,\gamma(m))},\gamma^{(m,\gamma(m))}(T_n^{(m,\gamma(m))}))}_{n+1,y+\gamma^{(m,\gamma(m))}(T_n^{(m,\gamma(m))})}\ge 1\}\\&=\gamma^{(m,\gamma(m))}(n+1)\end{aligned} \tag{2.9}$$

by (I) and (II) and

$$(m,\gamma(m))\rightsquigarrow(T_n,\gamma(T_n))\rightsquigarrow(n+1,\gamma(n+1)) \tag{2.10}$$

by (III), the proof is complete. □

Finally, we construct an open path Γ on $\{(0,o)\in\mathcal{P}\}$ by connecting percolation points on γ. Assume $(0,o)\in\mathcal{P}$ and let $\tau_1=0$ and

$$\tau_{n+1}=\inf\{k>\tau_n:(k,\gamma(k))\in\mathcal{P}\} \tag{2.11}$$

for $n\ge 1$, so that $(\tau_n,\gamma(\tau_n))$ are nothing but the n-th percolation points on γ. We may assume that $\tau_n<\infty$ for all $n\in\mathbb{N}^*$ since it will be proved in Proposition 2.4 below. (Note that Proposition 2.4 refers to γ only.) Let us first set $\Gamma(\tau_n)=\gamma(\tau_n)$ for all $n\in\mathbb{N}^*$. Next for $\tau_n<k<\tau_{n+1}$ (if any), we define $\Gamma(k)$ recursively as follows: Given $\Gamma(k-1)$, let

$$\Gamma(k)=\mathrm{Min}\{y\in\mathbb{Z}^d:(k-1,\Gamma(k-1))\rightsquigarrow(k,y)\rightsquigarrow(\tau_{n+1},\gamma(\tau_{n+1}))\}. \tag{2.12}$$

Note that $(\tau_n, \gamma(\tau_n)) \rightsquigarrow (\tau_{n+1}, \gamma(\tau_{n+1}))$ by Proposition 2.1-(b). Then, by induction on k, we see that the set on the right-hand side of (2.12) is nonempty and that $(k-1, \Gamma(k-1)) \rightsquigarrow (k, \Gamma(k))$ for all $k \in (\tau_n, \tau_{n+1}]$.

Remark 2.2. A similar construction was used by Kuczek (1989) to show the central limit theorem for the right edge of (1+1)-dimensional supercritical oriented percolation.

2.2. *Law of large numbers for good events.* Throughout this subsection, we keep assuming (2.5) and write γ and Γ for the paths constructed in Subsection 2.1. We consider the following *good* event

$$G_n(\gamma) = \{B_{n+1,\gamma(n),\gamma(n)+x} \geq 1+\delta, (n+1, \gamma(n+1)) \in \mathcal{P}\} \tag{2.13}$$

for each $n \in \mathbb{N}$. Note that on $G_n(\gamma)$, we have $\gamma(n+1) = \gamma(n) + x$ by (2.6) and $(n, \gamma(n)) \in \mathcal{P}$. It follows in particular that

$$\gamma(n) = \Gamma(n) \text{ and } \gamma(n+1) = \Gamma(n+1) \tag{2.14}$$

on $G_n(\gamma) \cap \{(0, o) \in \mathcal{P}\}$. The following proposition shows that $M_{n,\Gamma(n)}$ is multiplied by at least $(1+\delta)$ when $G_n(\gamma)$ occurs and thus explains why this event is good.

Proposition 2.3. *On the event $G_m(\gamma) \cap \{(0, o) \in \mathcal{P}\}$, $M_{n,\Gamma(n)} \geq (1+\delta) M_{m,\Gamma(m)}$ for all $n > m$.*

Proof: We have $M_{m+1,\Gamma(m+1)} \geq B_{m+1,\Gamma(m),\Gamma(m+1)} M_{m,\Gamma(m)} \geq (1+\delta) M_{m,\Gamma(m)}$ by definition. Since Γ is an open path, we have $M_{n,\Gamma(n)} \geq M_{m+1,\Gamma(m+1)}$ for all $n > m+1$ and the claim follows. □

Thanks to this proposition, we have the lower bound

$$M_{n,\Gamma(n)} \geq (1+\delta)^{\sum_{m=0}^{n-1} 1_{G_m(\gamma)}} \tag{2.15}$$

for the process along Γ on the event $\{(0, o) \in \mathcal{P}\}$. Therefore, the proof of Theorem 1.2 is reduced to proving the following law of large numbers for $\{1_{G_n(\gamma)}\}_{n \in \mathbb{N}}$.

Proposition 2.4.

$$\frac{1}{n} \sum_{m=0}^{n-1} 1_{G_m(\gamma)} \to c_\delta(B) \tag{2.16}$$

as $n \to \infty$ P-almost surely, where $c_\delta(B)$ is defined in (1.5).

Proof: By the independence of B_1 and $\{B_n\}_{n \geq 2}$ and translation invariance, we have

$$\begin{aligned} P(G_0(\gamma)) &= P(B_{1,o,x} \geq 1+\delta) P((1,x) \in \mathcal{P}) \\ &= P(B_{1,o,x} \geq 1+\delta) P((0,o) \in \mathcal{P}). \end{aligned} \tag{2.17}$$

Thus (2.16) is indeed the law of large numbers for $\{1_{G_n(\gamma)}\}_{n \in \mathbb{N}}$.

Let us first show that $\{1_{G_n(\gamma)}\}_{n \in \mathbb{N}}$ is a stationary sequence. Fix an increasing sequence of integers $1 \leq m_1 < m_2 < \cdots < m_k$ and consider the probability $P(\bigcap_{i=1}^k G_{m_i}(\gamma))$. We divide the event according to the position of $\gamma(m_1)$ and use Proposition 2.1-(a) to get

$$P\left(\bigcap_{i=1}^k G_{m_i}(\gamma)\right) = \sum_{y \in \mathbb{Z}^d} P\left(\{\gamma(m_1) = y\} \cap \bigcap_{i=1}^k G_{m_i}(\gamma^{(m_1,y)})\right). \tag{2.18}$$

Then by the independence of $\{\gamma(m_1) = y\} \in \mathscr{F}_{1,m_1}$ and $\bigcap_{i=1}^k G_{m_i}(\gamma^{(m_1,y)}) \in \mathscr{F}_{m_1+1,\infty}$ and translation invariance, it follows that the above right hand side equals

$$\sum_{y\in\mathbb{Z}^d} P(\gamma(m_1) = y) P\left(\bigcap_{i=1}^k G_{m_i - m_1}(\gamma)\right) = P\left(\bigcap_{i=1}^k G_{m_i-m_1}(\gamma)\right). \tag{2.19}$$

Applying the same argument to $m_i' = m_i - 1$ $(1 \le i \le k)$, it follows that

$$P\left(\bigcap_{i=1}^k G_{m_i}(\gamma)\right) = P\left(\bigcap_{i=1}^k G_{m_i-m_1}(\gamma)\right) = P\left(\bigcap_{i=1}^k G_{m_i-1}(\gamma)\right),$$

which implies the stationarity of $\{1_{G_n(\gamma)}\}_{n\in\mathbb{N}}$.

Next, we prove that $\{1_{G_n(\gamma)}\}_{n\in\mathbb{N}}$ has the so-called mixing property, which implies the ergodicity and hence the law of large numbers (cf. Durrett (1996), Section 6.4). By Lemma 6.4.4 in Durrett (1996), it suffices to show that for any pair of increasing sequences of integers $0 \le l_1 < l_2 < \cdots < l_j$ and $0 \le m_1 < m_2 < \cdots < m_k$,

$$P\left(\bigcap_{i=1}^j G_{l_i}(\gamma) \cap \bigcap_{i=1}^k G_{m_i+n}(\gamma)\right) \to P\left(\bigcap_{i=1}^j G_{l_i}(\gamma)\right) P\left(\bigcap_{i=1}^k G_{m_i}(\gamma)\right) \tag{2.20}$$

as $n \to \infty$. Note first that for any $m < n$, if $(m+1, \gamma(m+1)) \rightsquigarrow (n, \gamma(n))$ and $(n, \gamma(n)) \in \mathcal{P}$ then $(m+1, \gamma(m+1)) \in \mathcal{P}$. Combined with Proposition 2.1-(b), this yields

$$\begin{aligned} &G_m(\gamma) \cap G_n(\gamma) \\ &\quad = \{B_{m+1,\gamma(m),\gamma(m)+x} \ge 1+\delta, (m+1, \gamma(m+1)) \rightsquigarrow (n, \gamma(n))\} \cap G_n(\gamma). \end{aligned} \tag{2.21}$$

Let us denote the event in braces above by $\tilde{G}_{m,n}(\gamma)$. Then, we can rewrite the event on the left-hand side of (2.20) as

$$\bigcap_{i=1}^j G_{l_i}(\gamma) \cap \bigcap_{i=1}^k G_{m_i+n}(\gamma) = \bigcap_{i=1}^{j-1} \tilde{G}_{l_i,l_{i+1}}(\gamma) \cap \tilde{G}_{l_j,m_1+n}(\gamma) \cap \bigcap_{i=1}^k G_{m_i+n}(\gamma) \tag{2.22}$$

provided $l_j < m_1 + n$. Dividing the event according to the position of $\gamma(m_1+n)$, one can show that

$$\begin{aligned} &P\left(\bigcap_{i=1}^j G_{l_i}(\gamma) \cap \bigcap_{i=1}^k G_{m_i+n}(\gamma)\right) \\ &\quad = P\left(\bigcap_{i=1}^{j-1} \tilde{G}_{l_i,l_{i+1}}(\gamma) \cap \tilde{G}_{l_j,m_1+n}(\gamma)\right) P\left(\bigcap_{i=1}^k G_{m_i+n}(\gamma)\right) \end{aligned} \tag{2.23}$$

exactly in the same way as in the proof of the stationarity. Let us look at

$$\begin{aligned} &\tilde{G}_{l_j,m_1+n}(\gamma) \\ &\quad = \{B_{l_j+1,\gamma(l_j),\gamma(l_j)+x} \ge 1+\delta, (l_j+1, \gamma(l_j+1)) \rightsquigarrow (m_1+n, \gamma(m_1+n))\}, \end{aligned} \tag{2.24}$$

where the dependence on n remains. We recall Proposition 2.1-(b) to see

$$|M_{m_1+n}^{(l_j+1,\gamma(l_j+1))}| \ge 1 \Rightarrow (l_j+1, \gamma(l_j+1)) \rightsquigarrow (m_1+n, \gamma(m_1+n)). \tag{2.25}$$

Since the converse is also valid, we conclude that

$$\tilde{G}_{l_j,m_1+n}(\gamma) = \{B_{l_j+1,\gamma(l_j),\gamma(l_j)+x} \ge 1+\delta, |M_{m_1+n}^{(l_j+1,\gamma(l_j+1))}| \ge 1\} \downarrow G_{l_j}(\gamma) \tag{2.26}$$

as $n \to \infty$. Coming back to (2.23) and using stationarity, we obtain

$$\begin{aligned}\lim_{n\to\infty} P&\left(\bigcap_{i=1}^{j} G_{l_i}(\gamma)\cap\bigcap_{i=1}^{k} G_{m_i+n}(\gamma)\right)\\ &= P\left(\bigcap_{i=1}^{j-1} \tilde{G}_{l_i,l_{i+1}}(\gamma)\cap G_{l_j}(\gamma)\right) P\left(\bigcap_{i=1}^{k} G_{m_i}(\gamma)\right)\\ &= P\left(\bigcap_{i=1}^{j} G_{l_i}(\gamma)\right) P\left(\bigcap_{i=1}^{k} G_{m_i}(\gamma)\right)\end{aligned} \tag{2.27}$$

and we are done. □

2.3. *Proof of Corollary 1.3.*

Proof of Corollary 1.3: For the first half, we apply Theorem 1.2 to $B = \prod_{k=1}^m B_k$ to find an open path $\{\Gamma(n)\}_{n\in\mathbb{N}}$ with respect to $\{\prod_{k=1}^m B_{mn+k}\}_{n\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} \frac{1}{n}\log M_{mn,\Gamma(n)} \geq c_{\delta-\epsilon}(\textstyle\prod_{k=1}^m B_k)\log(1+\delta-\epsilon). \tag{2.28}$$

Since $(\prod_{k=1}^m B_{mn+k})_{m(n+1),\Gamma(n),\Gamma(n+1)} \geq 1$ implies that there exists a sequence $x(0) = \Gamma(n), x(1), \ldots, x(m-1), x(m) = \Gamma(n+1)$ such that

$$B_{mn+k,x(k-1),x(k)} \geq 1 \text{ for all } k = 1,\ldots,m, \tag{2.29}$$

we can construct a path Γ' that is open with respect to $\{B_n\}_{n\in\mathbb{N}^*}$ and $\Gamma'(mn) = \Gamma(n)$ for all $n \in \mathbb{N}^*$. Then the claim follows from the fact $M_{mn+k,\Gamma'(mn+k)} \geq M_{mn,\Gamma'(mn)}$ for all $k = 1,\ldots,m$.

We next prove the second assertion. If $P(|M_n| \geq 1 \text{ for all } n \in \mathbb{N}) = 0$, then (1.10) is obvious. Suppose $\sup_{x\in\mathbb{Z}^d} P((\prod_{k=1}^n B_k)_{o,x} \geq 1+\delta) = 0$ for all $\delta > 0$ and $m \in \mathbb{N}$. This means that $\prod_{k=1}^n B_k$ is a binary matrix for all $n \in \mathbb{N}$ and then it follows $M_{n,x} \leq 1$ for all $(n,x) \in \mathbb{N}\times\mathbb{Z}^d$. Now the finite range assumption implies $\#\{x \in \mathbb{Z}^d : M_{n,x} \geq 1\} \leq (2r_B n)^d$ and hence (1.10). □

3. Applications

3.1. *Linear stochastic evolution and its dual.* The second author has recently introduced a class of stochastic linear systems in Yoshida (2008), called linear stochastic evolutions (LSE for short), which contains various interesting processes such as the number of open paths in site or bond oriented percolation, a time discrete version of the binary contact path process, and the voter model. Using our results, we can completely characterize when the total mass of an LSE grows exponentially.

Let us first recall the definition of an LSE. Let $A = (A_{x,y})_{x,y\in\mathbb{Z}^d}$ be a random matrix satisfying the following:

$$A_{x,y} \geq 0 \text{ for all } x,y \in \mathbb{Z}^d, \tag{3.1}$$

$$\text{the columns } (A_{\cdot,y})_{y\in\mathbb{Z}^d} \text{ are independent}, \tag{3.2}$$

$$A_{x,y} = 0 \text{ if } |x-y| \geq r_A \text{ for some non-random } r_A \in \mathbb{N}, \tag{3.3}$$

$$(A_{x+z,y+z})_{x,y\in\mathbb{Z}^d} \overset{\text{law}}{=} A \text{ for all } z \in \mathbb{Z}^d. \tag{3.4}$$

A few more assumptions were posed in Yoshida (2008) such as square integrability of the matrix elements and a certain aperiodicity (cf. (1.8) in Yoshida (2008)) but we do not need them in this article. On the other hand, we need the following extra assumption

$$A_{x,y} \in \{0\} \cup [1, \infty) \text{ for all } x, y \in \mathbb{Z}^d \tag{3.5}$$

to use the results in Section 1. Let $\{A_n\}_{n \in \mathbb{N}^*}$ be a collection of independent copies of the random matrix A. The LSE generated by $\{A_n\}_{n \in \mathbb{N}^*}$ is the Markov chain $\{N_n\}_{n \in \mathbb{N}}$ with values in $[0, \infty)^{\mathbb{Z}^d}$ defined for given $N_0 \in [0, \infty)^{\mathbb{Z}^d}$ by

$$N_{n+1,y} = \sum_{x \in \mathbb{Z}^d} N_{n,x} A_{n+1,x,y}, \text{ for } n \in \mathbb{N} \text{ and } y \in \mathbb{Z}^d. \tag{3.6}$$

If we consider the dual process of an LSE, then it can be realized in the same way as above but (3.2) is replaced by

$$\text{the rows } (A_{x,\cdot})_{x \in \mathbb{Z}^d} \text{ are independent,} \tag{3.7}$$

see Section 4 in Yoshida (2008) for detail. We call this type of process the dual LSE and write DLSE for short.

Example 1. (Oriented percolation): Let $(\eta_{n,y})_{(n,y) \in \mathbb{N}^* \times \mathbb{Z}^d}$ be $\{0, 1\}$-valued independent and identically distributed random variables with $P(\eta_{n,y} = 1) = p \in [0, 1]$. The LSE $\{N_n\}_{n \in \mathbb{N}}$ generated by

$$A_{n,x,y} = \eta_{n,y} 1_{\{|x-y|=1\}} \tag{3.8}$$

represents the number of open oriented paths up to level n in the oriented site percolation. We call this process *the oriented site percolation* for shorthand. *The bond oriented percolation* can be constructed in a similar way (see p.1036 in Yoshida (2008)).

Example 2. (Binary contact path process): Let $(\eta_{n,y})_{(n,y) \in \mathbb{N}^* \times \mathbb{Z}^d}$ and $(\zeta_{n,y})_{(n,y) \in \mathbb{N}^* \times \mathbb{Z}^d}$ be $\{0, 1\}$-valued independent and identically distributed random variables with

$$P(\eta_{n,y} = 1) = p \in [0, 1] \text{ and } P(\zeta_{n,y} = 1) = q \in [0, 1]. \tag{3.9}$$

We further introduce another family of independent and identically distributed random variables $(e_{n,y})_{(n,y) \in \mathbb{N}^* \times \mathbb{Z}^d}$ which are uniformly distributed on $\{e \in \mathbb{Z}^d : |e| = 1\}$. Then the LSE $\{N_n\}_{n \in \mathbb{N}}$ generated by

$$A_{n,x,y} = \eta_{n,y} 1_{\{e_{n,y} = y - x\}} + \zeta_{n,y} \delta_{x,y} \tag{3.10}$$

gives a time-discrete version of the binary contact path process studied in Griffeath (1983). We simply call this discrete version *the binary contact path process* in this article. We can also define another time-discretization by considering the DLSE generated by

$$A_{n,x,y} = \eta_{n,x} 1_{\{e_{n,x} = y - x\}} + \zeta_{n,x} \delta_{x,y}. \tag{3.11}$$

In this DLSE-version, a site at time n chooses the target of infection whereas in the LSE-version above, a site at time $n + 1$ chooses the source of infection.

The following theorem says that we can characterize when LSE and DLSE grow exponentially in terms of A_1.

Theorem 3.1. *Let $\{N_n\}_{n \in \mathbb{N}}$ be an LSE generated by $\{A_n\}_{n \in \mathbb{N}^*}$ satisfying (3.5) and $|N_0| < \infty$. Then, either of the following holds true:*

(1) *If $c_\delta(A_1) > 0$ for some $\delta > 0$, then*

$$\liminf_{n\to\infty} \frac{1}{n} \log |N_n| \geq c_\delta(A_1) \log(1+\delta) \tag{3.12}$$

P-almost surely on the event $\{|N_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$.

(2) *If $P(|N_n| \geq 1 \text{ for all } n \in \mathbb{N}) > 0$ and $P(A_{1,-x,o} \geq 1 \text{ and } A_{1,-y,o} \geq 1) > 0$ for some distinct $x, y \in \mathbb{Z}^d$, then*

$$\liminf_{n\to\infty} \frac{1}{n} \log |N_n| \geq \frac{1}{2} c_1(A_1A_2) \log 2 > 0 \tag{3.13}$$

P-almost surely on the event $\{|N_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$.

(3) *If both of the assumptions in (1) and (2) fail, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |N_n| \leq 0 \tag{3.14}$$

P-almost surely.

The same assertions hold for DLSE with the second assumption in (2) replaced by

$$P(A_{1,o,x} \geq 1 \text{ and } A_{1,o,y} \geq 1) > 0 \text{ for some distinct } x, y \in \mathbb{Z}^d. \tag{3.15}$$

Proof: We prove (1)–(3) only for the LSE case since the proof for the DLSE case is almost identical. By linearity it suffices to show the claims for the case $N_0 = (\delta_{o,x})_{x\in\mathbb{Z}^d}$. Note first that (1) is a direct consequence of Theorem 1.2. Next, suppose that the assumption in (2) holds. Then it follows from translation invariance and (3.2) that

$$\begin{aligned} &P((A_1A_2)_{o,x+y} \geq 2) \\ &\geq P(A_{1,o,x} \geq 1, A_{1,o,y} \geq 1, A_{2,x,x+y} \geq 1, \text{ and } A_{2,y,x+y} \geq 1) \\ &\geq P(A_{1,-x,o} \geq 1) P(A_{1,-y,o} \geq 1) P(A_{2,-x,o} \geq 1 \text{ and } A_{2,-y,o} \geq 1) \\ &> 0. \end{aligned} \tag{3.16}$$

Therefore Corollary 1.3 shows (3.13). Finally, we prove (3) by checking the assumption for (1.10). If both of the assumptions in (1) and (2) fail, then we have either

$$P(|N_n| \geq 1 \text{ for all } n \in \mathbb{N}) = 0 \tag{3.17}$$

or

$$P\left(A_{1,x,y} \in \{0,1\}\right) = 1 \text{ for all } x, y \in \mathbb{Z}^d \text{ and} \tag{3.18}$$

$$P(A_{1,-x,o} \geq 1 \text{ and } A_{1,-y,o} \geq 1) = 0 \text{ for all distinct } x, y \in \mathbb{Z}^d. \tag{3.19}$$

From (3.18) and (3.19) one can conclude that $(\prod_{k=1}^m A_k)_{x,y} \in \{0,1\}$ for all $m \in \mathbb{N}^*$ and $x, y \in \mathbb{Z}^d$. Thus in both cases, $c_\delta(\prod_{k=1}^m A_k) = 0$ for all $\delta > 0$ and $m \in \mathbb{N}$. $\square$

Theorem 3.1 applies to both site and bond oriented percolations and the binary contact path process. Therefore we see that both the number of open paths in supercritical oriented percolation and the total mass of supercritical binary contact path process grow exponentially on the event of survival.

Remark 3.2. For the oriented site/bond percolation with $d = 1$, it is possible to show by using the rightmost path instead of Γ above that

$$\liminf_{n\to\infty} \frac{1}{n} \log |N_n| \geq c \tag{3.20}$$

with some absolute constant $c > 0$ almost surely on $\{|N_n| \geq 1 \text{ for all } n \in \mathbb{N}\}$. This shows that our lower bound $c_1(A_1 A_2) \log 2$ on the growth rate is not sharp in this case since it is smaller than the percolation probability which decreases to 0 as $p \downarrow p_c$, as was proved in Grimmett and Hiemer (2002).

Finally, we show that surviving LSE and DLSE satisfy the assumption in (1) or (2) in Theorem 3.1 except for a few unimportant examples. Therefore for fairly general LSE and DLSE, we have the coincidence of the events that the process survives and that the total mass grows exponentially. Let us start by defining the exceptional class. We say that an LSE generated by $\{A_n\}_{n \in \mathbb{N}^*}$ is *trivial* if A_n ($n \in \mathbb{N}^*$) are nonrandom, that is, the laws of A_n concentrate on one matrix. A DLSE generated by $\{A_n\}_{n \in \mathbb{N}^*}$ is said to be *trivial* if A_n ($n \in \mathbb{N}^*$) are nonrandom or expressed as

$$A_{n,x,y} = \delta_{x+e_{n,x},y} \tag{3.21}$$

with $\{e_{n,x}\}_{(n,x) \in \mathbb{N}^* \times \mathbb{Z}^d}$ a family of $\mathbb{Z}^d$-valued independent and identically distributed random variables. In the latter case, $\{N_n\}_{n \in \mathbb{N}}$ is nothing but the coalescing random walks for which the questions about survival/extinction and about growth rate are trivial.

Corollary 3.3. *For a nontrivial LSE or DLSE satisfying* (3.5) *and* $|N_0| < \infty$*,*

$$\{|N_n| \geq 1 \text{ for all } n \in \mathbb{N}\} = \left\{ \liminf_{n \to \infty} \frac{1}{n} \log |N_n| > 0 \right\} \tag{3.22}$$

modulo a P-null set. In particular, if

$$\lim_{n \to \infty} \frac{1}{n} E[\log(1 + |N_n|)] = 0, \tag{3.23}$$

then the process dies out almost surely.

Proof: Let $\{N_n\}_{n \in \mathbb{N}}$ be a nontrivial LSE or DLSE generated by $\{A_n\}_{n \in \mathbb{N}^*}$. The second assertion follows from (3.22) and Fatou's lemma. To prove (3.22), it suffices to show that if the assumptions in (1) and (2) in Theorem 3.1 fail, then

$$P(|N_n| \geq 1 \text{ for all } n \in \mathbb{N}) = 0. \tag{3.24}$$

We begin with the LSE case. Suppose that both of the assumptions fail. Then, either (3.24) holds or A_n are binary matrices and for any fixed $y \in \mathbb{Z}^d$, $A_{n,x,y} = 1$ for at most one $x \in \mathbb{Z}^d$. The latter means that our process is stochastically dominated by a finite range version of the nearest neighbor voter model (see p.1037 in Yoshida (2008)). Then, under the non-triviality assumption, one can show that it dies out almost surely by an argument similar to the proof of Lemma 1.3.3 and subsequent Remark in Yoshida (2008). Next we turn to the DLSE case. In this case if both of the assumptions fail, then either (3.24) holds or A_n are binary matrices and for any fixed $x \in \mathbb{Z}^d$, $A_{n,x,y} = 1$ for at most one $y \in \mathbb{Z}^d$. Together with the non-triviality, the latter implies that N_n is a coalescing random walks killed at positive rate, which dies out almost surely. □

Remark 3.4. The last part of above proof gives an alternative proof of Theorem 3.1-(3) for DLSE without the finite range assumption. Since the proofs of (1) and (2) rely only on Theorem 1.2 and the first half of Corollary 1.3, it follows that Theorem 3.1 for DLSE holds without the finite range assumption.

3.2. *Continuous time process.* We discuss the continuous time analogue of our results in this subsection. More precisely, we show results similar to the preceding section for the continuous time version of LSE or DLSE studied in Nagahata and Yoshida (2010a,b).

Let us recall the definition of the process. We introduce a random vector $K = (K_x)_{x\in\mathbb{Z}^d}$ such that each element takes value in $\{0\}\cup[1,\infty)$ and $K_x = 0$ if $|x|$ is larger than some positive constant r_K. Let us further introduce two mutually independent collections of i.i.d. random variables $\{\tau^{z,i}\}_{z\in\mathbb{Z}^d,i\in\mathbb{N}^*}$ and $\{K^{z,i}\}_{z\in\mathbb{Z}^d,i\in\mathbb{N}^*}$ whose distributions are mean-one exponential and the same as that of K, respectively. We suppose that the process $\{Y_t\}_{t\geq 0}$ starts from $Y_0 \in [0,\infty)^{\mathbb{Z}^d}$ and at each time $t = \tau^{z,1} + \tau^{z,2} + \cdots + \tau^{z,i}$ for some $(z, i) \in \mathbb{Z}^d \times \mathbb{N}^*$, the process is updated as follows:

$$Y_{t,x} = \begin{cases} K_0^{z,i} Y_{t-,z} & \text{if } x = z, \\ Y_{t-,x} + K_{x-z}^{z,i} Y_{t-,z} & \text{if } x \neq z. \end{cases} \tag{3.25}$$

We also consider the dual process $Z_t \in [0,\infty)^{\mathbb{Z}^d}$, $t \geq 0$ which evolves in the same way as $\{Y_t\}_{t\geq 0}$ except that (3.25) is replaced by its transpose:

$$Z_{t,x} = \begin{cases} \sum_{y\in\mathbb{Z}^d} K_{y-x}^{z,i} Z_{t-,y} & \text{if } x = z, \\ Z_{t-,x} & \text{if } x \neq z. \end{cases} \tag{3.26}$$

Remark 3.5. Note that the process (3.25) is a continuous-time counterpart of DLSE, while its dual (3.26) is that of LSE.

For processes of above types, we have the following simple characterization of exponential growth in terms of K.

Theorem 3.6. *Let $\{Y_t\}_{t\geq 0}$ be the process defined above satisfying $|Y_0| < \infty$. Then, the following holds:*

(1) *If $P(|Y_t| \geq 1 \text{ for all } t \geq 0) > 0$ and $P(\sum_{x\in\mathbb{Z}^d} K_x \geq 1+\delta) > 0$ for some $\delta > 0$, then*

$$\liminf_{t\to\infty} \frac{1}{t} \log |Y_t| > 0 \tag{3.27}$$

P-almost surely on the event $\{|Y_t| \geq 1 \text{ for all } t \geq 0\}$.

(2) *Otherwise,*

$$\limsup_{t\to\infty} \frac{1}{t} \log |Y_t| \leq 0 \tag{3.28}$$

P-almost surely.

The same assertions hold for the dual process $\{Z_t\}_{t\geq 0}$.

Proof: We prove (1) only for $\{Y_t\}_{t\geq 0}$. We first consider the time discretized process $\{Y_n\}_{n\in\mathbb{N}}$ and apply Theorem 1.2. Then it can be extended to $\{Y_t\}_{t\geq 0}$ as in the proof of Corollary 1.3.

For given $\{\tau^{z,i}\}_{z\in\mathbb{Z}^d,i\in\mathbb{N}^*}$ and $\{K^{z,i}\}_{z\in\mathbb{Z}^d,i\in\mathbb{N}^*}$, let $B_{n+1,x,y}$ be $Y_{n+1,y}^{(n,x)}$, that is, the population at $(n + 1, y)$ starting from one particle at (n, x) (cf. (2.2)). Then it follows that $\{Y_n\}_{n\in\mathbb{N}}$ is nothing but the Markov chain described in Subsection 1.2. Therefore it suffices to check that $P(Y_{1,x} \geq 1 + \delta) > 0$ for some $x \in \mathbb{Z}^d$ when

$Y_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$. Suppose first that $P(K_x \geq 1+\delta) > 0$ for some $x \in \mathbb{Z}^d$. Then,

$$\begin{aligned} &P(Y_{1,x} \geq 1+\delta) \\ &\quad \geq P(\tau^{o,1} < 1, \tau^{x,1} > 1, K_x^{o,1} \geq 1+\delta) \\ &\quad = P(\tau^{o,1} < 1)P(\tau^{x,1} > 1)P(K_x^{o,1} \geq 1+\delta) \\ &\quad > 0. \end{aligned} \tag{3.29}$$

Next, if $K_x \in \{0,1\}$ for all $x \in \mathbb{Z}^d$, then $P(\sum_{x\in\mathbb{Z}^d} K_x \geq 1+\delta) > 0$ implies

$$P(K_x = K_y = 1) > 0 \text{ for some distinct } x, y \in \mathbb{Z}^d. \tag{3.30}$$

Thus we have

$$\begin{aligned} &P(Y_{1,x+y} \geq 2) \\ &\quad \geq P(\tau^{o,1} \in [0,1/2), \tau^{x,1} \in [1/2,1), \tau^{y,1} \in [1/2,1), \tau^{x+y,1} > 1, \\ &\qquad\quad K_x^{o,1} = K_y^{o,1} = K_y^{x,1} = K_x^{y,1} = 1) \\ &\quad > 0. \end{aligned} \tag{3.31}$$

The proof of (2) for $\{Y_t\}_{t\geq 0}$ is immediate since $P(\sum_{x\in\mathbb{Z}^d} K_x = 1) = 1$ implies that $\{Y_t\}_{t\geq 0}$ is a coalescing random walk. To prove (2) for the dual process $\{Z_t\}_{t\geq 0}$, we use the fact that the cardinality of $\mathrm{supp} Z_t = \{x \in \mathbb{Z}^d : Z_{t,x} \geq 1\}$ grows at most polynomially fast, which is proved in Harris (1978). Indeed, $\mathrm{supp} Z_t$ forms an "additive set-valued process" introduced there and (13.10) in Harris (1978) implies that $\#\mathrm{supp} Z_t = O(t^d)$ almost surely. The rest of the proof is very similar to that of Theorem 3.1-(3) and we omit the detail. □

Remark 3.7. It should be pointed out that the generality of Theorem 1.2 is important in this proof. Indeed, the matrix B_n defined above is neither of finite range nor of independent entries.

One can also formulate and prove a continuous version of Corollary 3.3.

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