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Weak weak quenched limits for the path-valued processes of hitting times and positions of a transient, one-dimensional random walk in a random environment

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Abstract. In this article we continue the study of the quenched distributions of transient, one-dimensional random walks in a random environment. In a previous article we showed that while the quenched distributions of the hitting times do not converge to any deterministic distribution, they do have a weak weak limit in the sense that - viewed as random elements of the space of probability measures - they converge in distribution to a certain random probability measure (we refer to this as a weak weak limit because it is a weak limit in the weak topology). Here, we improve this result to the path-valued process of hitting times. As a consequence, we are able to also prove a weak weak quenched limit theorem for the path of the random walk itself.

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1. Introduction and Notation

A random walk in a random environment (RWRE) is a very simple model for random motion in a non-homogeneous random medium. A nearest-neighbor RWRE on $\mathbb{Z}$ may be described as follows. Elements of the set $\Omega = [0,1]^{\mathbb{Z}}$ are called *environments* since they can be used to define the transition probabilities for a Markov chain. That is, for any $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ and any $z \in \mathbb{Z}$, let X_n be a Markov chain with law P_ω^z given by $P_\omega^z(X_0 = z) = 1$ and

$$P_\omega^z(X_{n+1} = y \,|\, X_n = x) = \begin{cases} \omega_x & \text{if } y = x+1 \\ 1-\omega_x & \text{if } y = x-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let Ω be endowed with the natural cylindrical σ-field, and let P be a probability measure on Ω. Then, if ω is a random environment with distribution P, then P_ω^z is a random probability measure and is called the *quenched* law of the RWRE. By averaging over all environments we obtain the *averaged* law of the RWRE

$$\mathbb{P}^z(\cdot) = \int_\Omega P_\omega^z(\cdot) P(d\omega).$$

For ease of notation, the quenched and averaged laws of the RWRE started at $z = 0$ will be denoted by $\mathbb{P}_\omega$ and $\mathbb{P}$, respectively. Expectations with respect to P, P_ω and $\mathbb{P}$ will be denoted by E_P, E_ω and $\mathbb{E}$, respectively.

Throughout this paper we will make the following assumptions on the distribution P on environments.

Assumption 1. The environments are i.i.d. That is, $\{\omega_x\}_{x\in\mathbb{Z}}$ is an i.i.d. sequence of random variables under the measure P.

Assumption 2. The expectation $E_P[\log \rho_0]$ is well defined and $E_P[\log \rho_0] < 0$. Here $\rho_i = \rho_i(\omega) = \frac{1-\omega_i}{\omega_i}$, for all $i \in \mathbb{Z}$.

Assumption 3. The distribution of $\log \rho_0$ is non-lattice under P, and there exists a $\kappa > 0$ such that $E_P[\rho_0^\kappa] = 1$ and $E_P[\rho_0^\kappa \log \rho_0] < \infty$.

From Solomon's seminal paper on RWRE Solomon (1975), it is well known that Assumptions 1 and 2 imply that the RWRE is transient to $+\infty$; that is, $\mathbb{P}(\lim_{n\to\infty} X_n = \infty) = 1$. Moreover, Solomon showed that there exists a law of large numbers in the sense that there exists a constant v_P such that $\lim_{n\to\infty} X_n/n = \mathrm{v}_P$, $\mathbb{P}$-a.s. Solomon also showed that the limiting velocity v_P is non-zero only if $E_P[\rho_0] < 1$, which is equivalent to $\kappa > 1$ when Assumption 3 is in effect as well. Assumption 3 was used by Kesten, Kozlov, and Spitzer in their analysis of the averaged limiting distributions for transient one-dimensional RWRE Kesten et al. (1975). The parameter κ in Assumption 3 determines the magnitude of centering and scaling as well as the type of distribution obtained in the limit. Define the hitting times of the RWRE by

$$T_x = \inf\{n \geq 0 \,:\, X_n = x\}, \quad x \in \mathbb{Z},$$

and for $\kappa \in (0,2)$ define the properly centered and scaled versions of the hitting times and location of the RWRE by

$$\mathfrak{t}_n = \begin{cases} \frac{T_n}{n^{1/\kappa}} & \kappa \in (0,1) \\ \frac{T_n - nD(n)}{n} & \kappa = 1 \\ \frac{T_n - n/\mathrm{v}_P}{n^{1/\kappa}} & \kappa \in (1,2) \end{cases} \quad \text{and} \quad \mathfrak{z}_n = \begin{cases} \frac{X_n}{n^\kappa} & \kappa \in (0,1) \\ \frac{X_n - \delta(n)}{n/(A \log n)^2} & \kappa = 1 \\ \frac{X_n - n\mathrm{v}_P}{\mathrm{v}_P^{1+1/\kappa} n^{1/\kappa}} & \kappa \in (1,2), \end{cases} \tag{1.1}$$

where in the case $\kappa = 1$, $A > 0$ is a certain constant, and $D(n)$ and $\delta(n)$ are certain functions satisfying $D(n) \sim A \log n$ and $\delta(n) \sim n/(A \log n)$, respectively. Also, let $L_{\kappa,b}$ denote the distribution function of a totally skewed to the right stable random variable with index $\kappa \in (0,2)$, scaling parameter $b > 0$, and zero shift; see Samorodnitsky and Taqqu (1994). The following averaged limiting distribution for RWRE was first proved in Kesten et al. (1975).

Theorem 1.1. *Let Assumptions 1 - 3 hold, and let $\kappa \in (0,2)$. Then, there exists a constant $b > 0$ such that for any $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}(\mathfrak{t}_n \le x) = L_{\kappa,b}(x), \;\; x \in \mathbb{R}\,,$$

and

$$\lim_{n\to\infty} \mathbb{P}(\mathfrak{z}_n \le x) = \begin{cases} 1 - L_{\kappa,b}(x^{-1/\kappa}) & \text{for } x > 0 \text{ if } \kappa \in (0,1) \\ 1 - L_{\kappa,b}(-x) & \text{for } x \in \mathbb{R} \text{ if } \kappa \in [1,2). \end{cases}$$

Remark 1.2. The cases $\kappa = 2$ and $\kappa > 2$ were also considered in Kesten et al. (1975), but since our main results are for $\kappa \in (0,2)$ we will limit our focus to these cases. We note, however, that when $\kappa \ge 2$ the averaged limiting distributions for the hitting times and the location of the RWRE are Gaussian.

It is important to note that the limiting distributions in Theorem 1.1 are for the averaged measure $\mathbb{P}$. However, for certain applications the quenched measure P_ω may be more applicable (e.g., for repeated experiments in a fixed non-homogeneous medium), and one naturally wonders if there is a quenched analog of Theorem 1.1. Unfortunately, it was shown in Peterson and Zeitouni (2009) and Peterson (2009) that there is no such strong quenched limiting distribution. That is, for almost every fixed environment ω, there is no centering and scaling (or even environment-dependent centering and scaling) for which the hitting times or location of the RWRE converge in distribution under P_ω.

The negative results of Peterson and Zeitouni (2009) and Peterson (2009) were recently clarified by showing that quenched limiting distributions do exist in a weak sense (Peterson and Samorodnitsky, 2010; Dolgopyat and Goldsheid, 2012; Enriquez et al., 2010). Let $\mathcal{M}_1(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ equipped with the topology of convergence in distribution. Then, since the environment ω is a random variable, the quenched distribution $\mu_{n,\omega} = P_\omega(\mathfrak{t}_n \in \cdot)$ is an $\mathcal{M}_1(\mathbb{R})$-valued function of that random variable. This function can be easily shown to be measurable, hence $\mu_{n,\omega}$ is itself a random variable, namely a $\mathcal{M}_1(\mathbb{R})$-valued random variable. It was shown in Peterson and Samorodnitsky (2010) that there exists a family of $\mathcal{M}_1(\mathbb{R})$-valued random variables $(\pi_{\lambda,\kappa})$ such that $\mu_{n,\omega} \Longrightarrow \pi_{\lambda,\kappa}$ for some $\lambda > 0$, where $\Longrightarrow$ denotes weak convergence of $\mathcal{M}_1(\mathbb{R})$-valued random variables[1].

[1]Throughout the paper, if $(Z_n), Z$ are random variables in some space Ψ, then $Z_n \Longrightarrow Z$ will denote weak convergence (i.e., convergence in distribution) of Ψ-valued random variables.

We will refer to such limits as *weak weak quenched limits* since the quenched distribution converges weakly with respect to the weak topology on $\mathcal{M}_1(\mathbb{R})$. Similar results were obtained independently in Dolgopyat and Goldsheid (2012) and Enriquez et al. (2010).

In Peterson and Samorodnitsky (2010), this weak weak quenched limiting distribution for the hitting times was also used to obtain a result on the quenched distribution of the location of the RWRE. It was shown that

$$P_\omega(\mathfrak{z}_n \le x) \Longrightarrow \begin{cases} \pi_{\lambda,\kappa}[x^{-1/\kappa},\infty) & \text{for } x>0 \text{ if } \kappa\in(0,1) \\ \pi_{\lambda,\kappa}[-x,\infty) & \text{for } x\in\mathbb{R} \text{ if } \kappa\in[1,2), \end{cases} \tag{1.2}$$

and here $\Longrightarrow$ denotes weak convergence of $\mathbb{R}$-valued random variables. Note that this is a weaker statement than the quenched limit that was obtained for the hitting times. Unfortunately, weak convergence of all one-dimensional projections of a random probability measure is not enough to specify the weak limit of the random probability measure. For example, suppose that $\kappa\in(0,1)$. If $\sigma_{\lambda,\kappa}$ is the transformation of the random probability measure $\pi_{\lambda,\kappa}$ defined by letting $\sigma_{\lambda,\kappa}(-\infty,x]$ equal the right hand side of (1.2), one is tempted to guess that $P_\omega(\mathfrak{z}_n\in\cdot)\Longrightarrow\sigma_{\lambda,\kappa}$ in the sense of weak convergence of random probability measures. However, it can be seen from our results below that this is not true (see Corollary 1.8).

1.1. *Main Results.* The original goal of the current paper was to obtain a full weak limit for the random probability measure $P_\omega(\mathfrak{z}_n\in\cdot)$. However, it turned out to be necessary to obtain a weak limit for not just the quenched distribution of the hitting T_n but also for the quenched distribution of the path process of the sequence of hitting times. This result, in turn leads to not only a weak limit for the quenched distribution of X_n but also to the weak limit of the quenched distribution of the entire path of the RWRE, as we will see in the sequel.

To begin, let D_∞ be the space of càdlàg functions (continuous from the right with left limits) on $[0,\infty)$. We will equip D_∞ with the M_1-Skorohod metric $d_\infty^{M_1}$ (instead of the more standard and slightly stronger J_1-Skorohod metric $d_\infty^{J_1}$; the definitions of the Skorohod metrics are given in Section 3). Let $\mathcal{M}_1(D_\infty)$ be the space of probability measures on D_∞ equipped with the topology of weak convergence induced by the M_1-metric $d_\infty^{M_1}$ on D_∞. Since $(D_\infty,d_\infty^{M_1})$ is a Polish space, this topology is equivalent to topology induced by the Prohorov metric ρ^{M_1} (see Section 3 for a precise definition).

For any realization of the random walk and $\varepsilon>0$, let $\mathbb{T}_\varepsilon\in D_\infty$ be defined by

$$\mathbb{T}_\varepsilon(t) = \begin{cases} \varepsilon^{1/\kappa}T_{t/\varepsilon} & \kappa\in(0,1) \\ \varepsilon(T_{t/\varepsilon}-t/\varepsilon D(1/\varepsilon)) & \kappa=1 \\ \varepsilon^{1/\kappa}(T_{t/\varepsilon}-t/(\varepsilon \mathrm{v}_P)) & \kappa\in(1,2) \end{cases} \tag{1.3}$$

(here and in the sequel we define hitting times of non-integer points by $T_x=T_{\lfloor x\rfloor}$.) In the case $\kappa=1$ the function D is the function in (1.1) extended to all $x>0$; we will define it explicitly in Section 4. It is easy to see that, for each environment ω and any $\varepsilon>0$, $\mathbb{T}_\varepsilon$ is a well-defined D_∞-valued random variable; we denote by $m_{\varepsilon,\omega}$ the quenched law of $\mathbb{T}_\varepsilon$ on D_∞. This law is a measurable function of the environment, hence a $\mathcal{M}_1(D_\infty)$-valued random variable. We wish to show that this random variable converges weakly as $\varepsilon\to0$. In order to identify the limit we need to introduce additional notation.

Let $\mathcal{M}_p((0,\infty]\times[0,\infty))$ be the space of Radon point processes on $(0,\infty]\times[0,\infty)$. These are point processes assigning finite mass to $[\varepsilon,\infty]\times[0,T]$ for any $\varepsilon>0$ and $T<\infty$. The topology of vague convergence on this space is metrizable, and converts $\mathcal{M}_p((0,\infty]\times[0,\infty))$ into a complete separable metric space; see Resnick (2008, Proposition 3.17). We denote by $\mathcal{M}_p^f((0,\infty]\times[0,\infty))$ the subset of $\mathcal{M}_p((0,\infty]\times[0,\infty))$ of point processes that do not put any mass on points with infinite first coordinate. Let $\vec{\tau}=\{\tau_i\}_{i\geq 1}$ be a sequence of i.i.d. standard exponential random variables. For a point process $\zeta=\sum_{i\geq 1}\delta_{(x_i,t_i)}\in\mathcal{M}_p^f((0,\infty]\times[0,\infty))$ and $\delta>0$ we define a stochastic process (random path) $W_\delta(\zeta,\vec{\tau})$ with sample paths in D_∞ by

$$W_\delta(\zeta,\vec{\tau})(t)=\sum_{i\geq 1}x_i\tau_i\mathbf{1}_{\{x_i>\delta,t_i\leq t\}}.$$

We also let

$$W(\zeta,\vec{\tau})(t)=\begin{cases}\sum_{i\geq 1}x_i\tau_i\mathbf{1}_{\{t_i\leq t\}} & \text{if the sum is finite}\\ 0 & \text{otherwise.}\end{cases} \tag{1.4}$$

Remark 1.3. The notation $W_\delta(\zeta,\vec{\tau})$ and $W(\zeta,\vec{\tau})$ is somewhat misleading since the actual definitions depend on the (measurable) ordering chosen for the points of ζ. Since $\vec{\tau}$ is an i.i.d. sequence of random variables, the choice of ordering will not affect the laws of $W_\delta(\zeta,\vec{\tau})$ and $W(\zeta,\vec{\tau})$, and we are only concerned with the laws of these processes.

It is clear that $\lim_{\delta\to 0}W_\delta(\zeta,\vec{\tau})=W(\zeta,\vec{\tau})$ in D_∞ for every choice of $\vec{\tau}$ for which $W(\zeta,\vec{\tau})(t)<\infty$ for each $t<\infty$. We will impose assumptions on the point processes ζ such that this holds with probability one.

For any point process ζ such that, with probability 1, $W(\zeta,\vec{\tau})(t)<\infty$ for each $t<\infty$, the definitions of $W_\delta(\zeta,\vec{\tau})$ and $W(\zeta,\vec{\tau})$ induce in natural way probability measures on D_∞. Define functions $\mathcal{H}_\delta,\mathcal{H}:\mathcal{M}_p((0,\infty]\times[0,\infty))\to\mathcal{M}_1(D_\infty)$ by

$$\mathcal{H}_\delta(\zeta)(\cdot)=\mathbf{P}_\tau(W_\delta(\zeta,\vec{\tau})\in\cdot),\quad\text{and}\quad \mathcal{H}(\zeta)(\cdot)=\mathbf{P}_\tau(W(\zeta,\vec{\tau})\in\cdot), \tag{1.5}$$

when $\zeta\in\mathcal{M}_p^f((0,\infty]\times[0,\infty))$ and (in the case of $\mathcal{H}(\zeta)$) when $W(\zeta,\vec{\tau})(t)<\infty$ for each $t<\infty$ with probability 1. Otherwise we define $\mathcal{H}_\delta(\zeta)$ or $\mathcal{H}(\zeta)$, respectively, to be the Dirac point mass at the zero process in D_∞. Here $\mathbf{P}_\tau$ is the distribution of the i.i.d. sequence of the standard exponential random variables $\vec{\tau}=\{\tau_i\}_{i\geq 1}$.

Before stating our theorem we need one last bit of notation. The cases $\kappa\in[1,2)$ require a centering term in the limit. Thus, for any $m\in\mathbb{R}$ let $\ell(m)\in\mathcal{M}_1(D_\infty)$ be the Dirac point mass measure that is concentrated on the linear path $t\mapsto mt$. If X is a D_∞-valued random variable with distribution $\mu\in\mathcal{M}_1(D_\infty)$, then $\mu*\ell(-m)$ is the distribution of the path $\{t\mapsto X(t)-mt\}$.

Theorem 1.4. *Let $m_\varepsilon(\cdot)=m_{\varepsilon,\omega}(\cdot)=P_\omega(\mathbb{T}_\varepsilon\in\cdot)$ be the quenched distribution of the path $\mathbb{T}_\varepsilon$. For $0<\kappa<2$ let $\lambda=C_0\kappa/\bar{\nu}$, where C_0 and $\bar{\nu}$ are given, respectively, by (2.3) and (2.2) below. Let $N_{\lambda,\kappa}$ be a Poisson point process on $(0,\infty]\times[0,\infty)$ whose intensity measure puts no mass on infinite points, and is given by $\lambda x^{-\kappa-1}\,dx\,dt$ on finite points. Then, $m_\varepsilon\Longrightarrow\mu_{\lambda,\kappa}$ as $\varepsilon\to 0$ where*

$$\mu_{\lambda,\kappa}=\begin{cases}\mathcal{H}(N_{\lambda,\kappa}) & \text{if } \kappa\in(0,1)\\ \lim_{\delta\to 0}\mathcal{H}_\delta(N_{\lambda,1})*\ell(-\lambda\log(1/\delta)) & \text{if } \kappa=1\\ \lim_{\delta\to 0}\mathcal{H}_\delta(N_{\lambda,\kappa})*\ell\left(-\lambda\delta^{-\kappa+1}/(\kappa-1)\right) & \text{if } \kappa\in(1,2).\end{cases} \tag{1.6}$$

Remark 1.5. The limits in the definition of $\mu_{\lambda,\kappa}$ in (1.6) when $\kappa \in [1,2)$ are weak limits in $\mathcal{M}_1\big((D_\infty, d_\infty^{M_1})\big)$. In fact, we will see in the sequel that these limits exist even as a.s. limits in $\mathcal{M}_1\big((D_\infty, d_\infty^{M_1})\big)$. Furthermore, the defintion of $\mu_{\lambda,\kappa}$ as $\mathcal{H}(N_{\lambda,\kappa})$ in the case $\kappa \in (0,1)$ is valid by the well-known fact that for each $t < \infty$, $W(N_{\lambda,\kappa}, \vec{\tau})(t) < \infty$ with $\mathbf{P}_\tau$-probability 1 for almost every realization of the Poisson point process $N_{\lambda,\kappa}$.

As mentioned above, we will prove the existence of a weak limit for the quenched distribution of the entire path of the RWRE. To this end, we define a centered and scaled path of the random walk $\chi_\varepsilon \in D_\infty$ by

$$\chi_\varepsilon(t) = \begin{cases} \varepsilon^\kappa X_{\lfloor t/\varepsilon \rfloor} & \kappa \in (0,1) \\ \frac{1}{\varepsilon \delta(1/\varepsilon)^2}\left(X_{\lfloor t/\varepsilon \rfloor} - t\delta(1/\varepsilon)\right) & \kappa = 1 \\ \mathrm{v}_P^{-1-1/\kappa} \varepsilon^{1/\kappa}\left(X_{\lfloor t/\varepsilon \rfloor} - t\mathrm{v}_P/\varepsilon\right) & \kappa \in (1,2), \end{cases} \tag{1.7}$$

where in the case $\kappa = 1$, $\delta(x)$ is a function that satisfies $\delta(x)D(\delta(x)) = x + o(1)$ as $x \to \infty$. Here D is the same function as in (1.3). Note that, since $D(x) \sim A \log x$, this implies that $\delta(x) \sim x/(A\log x)$ so that the scaling factor in the definition of χ_ε when $\kappa = 1$ is asymptotic to $\varepsilon(A \log(1/\varepsilon))^2$ as $\varepsilon \to 0$. Let $p_{\varepsilon,\omega} = P_\omega(\chi_\varepsilon \in \cdot)$ be the quenched law of χ_ε on D_∞. It is a $\mathcal{M}_1(D_\infty)$-valued random variable defined on Ω.

The weak limits of $p_{\varepsilon,\omega}$ will be obtained by comparing the paths of the location of the RWRE χ_ε to appropriately transformed paths of the hitting times $\mathbb{T}_\varepsilon$. To this end, we define two transformations of paths. Let $D_{u,\uparrow}^+ \subset D_\infty$ consist of functions that are (weakly) monotone increasing, with $x(0) \geq 0$ and $\lim_{t\to\infty} x(t) = \infty$. Define the time-space inversion function $\mathfrak{I} : D_{u,\uparrow}^+ \to D_{u,\uparrow}^+$ by

$$\mathfrak{I}x(t) = \sup\{s \geq 0 : x(s) \leq t\},\ t \geq 0,\ x \in D_{u,\uparrow}^+. \tag{1.8}$$

Also, define the spatial reflection function $\mathfrak{R} : D_\infty \to D_\infty$ by $\mathfrak{R}x(t) = -x(t)$, $t \geq 0$, $x \in D_\infty$.

Theorem 1.6. *(***a***) The following coupling results hold.*

(1) *If $\kappa \in (0,1)$, then for any $s < \infty$*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{t \leq s} |\chi_\varepsilon(t) - \mathfrak{I}\mathbb{T}_{\varepsilon^\kappa}(t)| \geq \eta\right) = 0, \quad \forall \eta > 0.$$

(2) *If $\kappa = 1$, then*

$$\lim_{\varepsilon \to 0} \mathbb{P}(d_\infty^{M_1}(\chi_\varepsilon, -\mathbb{T}_{1/\delta(1/\varepsilon)}) \geq \eta) = 0, \quad \forall \eta > 0.$$

(3) *If $\kappa \in (1,2)$, then*

$$\lim_{\varepsilon \to 0} \mathbb{P}(d_\infty^{M_1}(\chi_\varepsilon, -\mathbb{T}_{\varepsilon/\mathrm{v}_P}) \geq \eta) = 0, \quad \forall \eta > 0.$$

*(***b***) Let $p_\varepsilon(\cdot) = p_{\varepsilon,\omega}(\cdot) = P_\omega(\chi_\varepsilon \in \cdot)$ be the quenched distribution of the path χ_ε, and let $\mu_{\lambda,\kappa}$ be the random probability distribution on paths defined in (1.6). Then $p_\varepsilon \Longrightarrow \mu_{\lambda,\kappa} \circ \mathfrak{I}^{-1}$ as $\varepsilon \to 0$ if $\kappa \in (0,1)$ and $p_\varepsilon \Longrightarrow \mu_{\lambda,\kappa} \circ \mathfrak{R}^{-1}$ as $\varepsilon \to 0$ if $\kappa \in [1,2)$, weakly in $\mathcal{M}_1(D_\infty)$.*

Remark 1.7. Note that the nature of conversion from time to space in the limiting random probability measure in $\mathcal{M}_1(D_\infty)$ is very different in the absence of centering term ($\kappa \in (0,1)$) from the case when there is a centering term ($\kappa \in [1,2)$). When $\kappa \in [1,2)$ the conversion is accomplished by multiplying a random path distributed according to the limiting (random) measure by -1. This is, of course, very different from the switching the time and space axes required when $\kappa \in (0,1)$.

Observe that, for any $0 \leq t < \infty$, the map $\Phi_t : \mathcal{M}_1(D_\infty) \to \mathcal{M}_1(\mathbb{R})$ defined by

$$\Phi_t(\mu)(A) = \mu\left(\{x \in D_\infty : x(t) \in A\}\right), \quad \text{for any Borel } A \subset \mathbb{R}, \tag{1.9}$$

is continuous at every $\mu \in \mathcal{M}_1(D_\infty)$ concentrated on paths continuous at t. Since the limiting probability measures on $\mathcal{M}_1(D_\infty)$ obtained in Theorem 1.6 is concentrated on μ with this property, the continuous mapping theorem immediately implies the following weak weak convergence for the distributions of the location of the random walk at fixed times.

Corollary 1.8. *For $0 \leq t < \infty$ let $p_{\varepsilon;t} = p_{\varepsilon,\omega;t} = P_\omega(\chi_\varepsilon(t) \in \cdot) \in \mathcal{M}_1(\mathbb{R})$ be the quenched distribution of $\chi_\varepsilon(t)$, and let $\nu_{\lambda,\kappa}$ be the limiting element of $\mathcal{M}_1(D_\infty)$ given in Theorem 1.6. Then $p_{\varepsilon;t} \Longrightarrow \Phi_t(\nu_{\lambda,\kappa})$ weakly in $\mathcal{M}_1(\mathbb{R})$.*

Theorems 1.4 and 1.6 imply the following corollaries on the convergence of $\mathbb{T}_\varepsilon$ and χ_ε under the averaged measure $\mathbb{P}$.

Corollary 1.9. *For any $\kappa \in (0,2)$, the hitting time paths $\mathbb{T}_\varepsilon$, viewed as random elements of $(D_\infty, d_\infty^{M_1})$, converge weakly under the averaged measure $\mathbb{P}$. Furthermore,*

(1) *if $\kappa \in (0,1)$, the limit is a κ-stable Lévy subordinator;*
(2) *If $\kappa \in [1,2)$, the limit is a κ-stable Lévy process that is totally skewed to the right. Moreover, if $\kappa \in (1,2)$, the limit is a strictly stable Lévy process.*

Since a stable subordinator is a strictly increasing process, its inverse has continuous sample paths. Correspondingly, we can strengthen the topology on the space D_∞ when considering weak convergence of the paths of the location of the RWRE under the average probability measure $\mathbb{P}$ in the case $\kappa \in (0,1)$. To this end, let (D_∞, d_∞^U) denote the space D_∞ equipped with the topology of uniform convergence on compact sets. This space is not separable, but Theorem 6.6 in Billingsley (1999) allows us to conclude weak convergence on the ball-σ-field in that space, the so-called weak$^\circ$ convergence. Moving from the M_1 topology to the J_1 topology, on the other hand, does not cause any difficulties.

Corollary 1.10.

(1) *If $\kappa \in (0,1)$ then the paths χ_ε, viewed as random elements of $(D_\infty, d_\infty^{J_1})$, converge weakly under the averaged measure $\mathbb{P}$ to the inverse of a κ-stable subordinator. Furthermore, χ_ε as random elements of (D_∞, d_∞^U) equipped with the ball-σ-field, we have weak$^\circ$ convergence to the same limit.*
(2) *If $\kappa \in [1,2)$, then the paths χ_ε, viewed as random elements of $(D_\infty, d_\infty^{M_1})$, converge weakly to a κ-stable Lévy process that is totally skewed to the left. Moreover, if $\kappa \in (1,2)$, then the limit is a strictly stable Lévy process.*

Remark 1.11. The statement of Corollary 1.10 in the case $\kappa \in (0,1)$ appeared in Remark 2.5 in Enriquez et al. (2009). To the best of our knowledge the other statements in Corollaries 1.9 and 1.10 are new.

Part (2) of Corollary 1.10 is an immediate consequence of the corresponding part of Corollary 1.9, the coupling results in parts (2) and (3) of Theorem 1.6 and Theorem 3.1 in Billingsley (1999). Further, Whitt (2002, Corollary 13.6.4) says that the operator $\mathfrak{I}$ from the subset $D^+_{u,\uparrow\uparrow} \subset D^+_{u,\uparrow}$ of strictly increasing, nonnegative, unbounded paths endowed with the $d^{M_1}_\infty$ metric to $D^+_{u,\uparrow}$ endowed with the d^U_∞ metric, is continuous. Since, in the case $0 < \kappa < 1$, a κ-stable subordinator is in $D^+_{u,\uparrow\uparrow}$ with probability one, the continuous mapping theorem shows that part (1) of Corollary 1.10 also follows from the corresponding part of Corollary 1.9.

The proof of Corollary 1.9 is also rather straightforward, but, because it introduces certain key ideas and notation used later in the paper, we present the proof here.

Proof of Corollary 1.9: Let $N_{\lambda,\kappa}$ be the Poisson point process on $(0,\infty] \times [0,\infty)$ defined in Theorem 1.4, and let $\vec{\tau} = \{\tau_i\}_{i\geq 1}$ be an i.i.d. sequence of standard exponential random variables; we assume that $N_{\lambda,\kappa}$ and $\vec{\tau}$ are defined on two different probability spaces, with the corresponding probability measures $\mathbf{P}$ and $\mathbf{P}_\tau$. On the product probability space we define

$$Z_{\lambda,\kappa}(t) = \begin{cases} W(N_{\lambda,\kappa},\vec{\tau})(t) & \kappa \in (0,1) \\ \lim_{\delta\to 0} W_\delta(N_{\lambda,\kappa},\vec{\tau})(t) - \lambda t \log(1/\delta) & \kappa = 1 \\ \lim_{\delta\to 0} W_\delta(N_{\lambda,\kappa},\vec{\tau})(t) - \lambda t \delta^{1-\kappa}/(\kappa-1) & \kappa \in (1,2), \end{cases} \tag{1.10}$$

$t \geq 0$. The definition is understood as a.s. convergence in $(D_\infty, d^{M_1}_\infty)$ on the product probability space. This convergence takes place by the proposition in Section 2 of Kallenberg (1974), and it is standard to see that $Z_{\lambda,\kappa}$ is a κ-stable Lévy process with the required properties of Corollary 1.9. In order to show that the averaged distribution of $\mathbb{T}_\varepsilon$ converges to the distribution of $Z_{\lambda,\kappa}$ under the product probability measure $\mathbf{P} \times \mathbf{P}_\tau$, it is enough to show that $\mathbb{P}(\mathbb{T}_\varepsilon \in A) \to \mathbf{P} \times \mathbf{P}_\tau(Z_{\lambda,\kappa} \in A)$ as $\varepsilon \to 0$ for all cylindrical sets $A \subset D_\infty$ such that $\mathbf{P} \times \mathbf{P}_\tau(Z_{\lambda,\kappa} \in \partial A) = 0$. (Recall that the Borel σ-field under all the Skorohod topologies coincides with the cylindrical σ-field; see Theorem 11.5.2 in Whitt (2002).) Let A be such a set. Recall that $\mathbf{P} \times \mathbf{P}_\tau(Z_{\lambda,\kappa} \in A) = \mathbf{E}[\mu_{\lambda,\kappa}(A)]$, where $\mu_{\lambda,\kappa}$ is defined in Theorem 1.4. By Fubini's theorem, $\mu_{\lambda,\kappa}(\partial A) = 0$ almost surely. Also, the evaluation mapping mapping $\mu \mapsto \mu(A)$ on $\mathcal{M}_1(D_\infty)$ is continuous on the set of measures $\{\mu \in \mathcal{M}_1(D_\infty) : \mu(\partial A) = 0\}$. Since the random measure $\mu_{\lambda,\kappa}$ is in this set with probability one, and since Theorem 1.4 implies that $m_{\varepsilon,\omega} \Longrightarrow \mu_{\lambda,\kappa}$, then the mapping theorem implies that $m_{\varepsilon,\omega}(A)$ converges in distribution to $m_{\lambda,\kappa}(A)$. Since these random variables are between 0 and 1, this implies that

$$\lim_{\varepsilon\to 0} \mathbb{P}(\mathbb{T}_\varepsilon \in A) = \lim_{\varepsilon\to 0} E_P[m_{\varepsilon,\omega}(A)] = \mathbf{E}[\mu_{\lambda,\kappa}(A)] = \mathbf{P} \times \mathbf{P}_\tau(Z_{\lambda,\kappa} \in A).$$

□

The limiting random probability measure $\mu_{\lambda,\kappa}$ is a κ-stable random element of $\mathcal{M}_1(D_\infty)$ under convolutions. That is, the convolution of two independent copies of this random probability measure is (after re-scaling and shifting) a random probability measure with the same law. This can be seen in the same way as the stability of the limiting random probability measures on $\mathbb{R}$ was checked in Peterson and Samorodnitsky (2010). Stability of random probability measures on D_∞ does not seen to have been investigated before, but a systematic description of infinitely divisible (in particular, stable) random probability measures on $\mathbb{R}$ was given in Shiga and Tanaka (2006); we recall these notions in Section 7. The latter paper introduced also a notion of $\mathcal{M}_1(\mathbb{R})$-valued Lévy process. If we recall the maps Φ_t, $0 \leq t < \infty$, defined in (1.9), then we can define a (measurable) map

Φ from $\mathcal{M}_1(D_\infty)$ to $D_\infty(\mathcal{M}_1(\mathbb{R}))$ by setting $\Phi(\mu)$ to be the measure-valued path $\{\Phi_t(\mu),\, t \geq 0\}$.

One would expect that a version of Theorem 1.4 would give us a convergence to a $\mathcal{M}_1(\mathbb{R})$-valued Lévy process as well. The following corollary gives such convergence, but only in the sense of convergence of finite dimensional distributions.

Corollary 1.12. *Let $\mu_{\lambda,\kappa}$ and m_ε, be the random probability measures on D_∞ given in Theorem 1.4, $0 < \kappa < 2$ (so that $m_\varepsilon \Longrightarrow \mu_{\lambda,\kappa}$). Then $\Phi(m_\varepsilon)$ converges weakly to $\Phi(\mu_{\lambda,\kappa})$ in the sense of finite dimensional distributions. Moreover, for any $\kappa \in (0,2)$, $\Phi(\mu_{\lambda,\kappa})$ is a stable Lévy process on $\mathcal{M}_1(\mathbb{R})$. It is a strictly stable Lévy process if $\kappa \neq 1$.*

Remark 1.13. One would like to improve the finite dimensional distribution convergence in Corollary 1.12 to a full convergence in distribution of $\mathcal{M}_1(\mathbb{R})$-valued path processes. Such a statement seems would require setting a topology on the space $D_\infty(\mathcal{M}_1(\mathbb{R}))$ of measure-valued path processes. Choosing an appropriate topology seems to be a difficult task as neither the Skorohod J_1-topology nor a natural definition of the Skorohod M_1-topology appear to be sufficient. This is complicated by the fact that the mapping $\Phi : \mathcal{M}_\infty(D_\infty) \to D_\infty(\mathcal{M}_1(\mathbb{R}))$ is not continuous in these topologies (even on the support of the limiting measure $\mu_{\lambda,\kappa}$). These issues are discussed further in Section 7.

2. Random Environment

It will be important for us to identify sections of the environment that contribute the most to the distribution of the hitting times. To this end, we define the ladder locations $\nu_k = \nu_k(\omega)$ of the environment by

$$\nu_0 = 0, \quad \text{and} \quad \nu_k = \inf\left\{ j > \nu_{k-1} : \prod_{i=\nu_{k-1}}^{j-1} \rho_i < 1 \right\} \text{ for } k \geq 1. \tag{2.1}$$

(The ladder locations are those locations where the *potential* of the environment introduced in Sinaĭ (1982) reaches a new minimum to the right of the origin.) Occasionally we will denote ν_1 by ν instead for compactness. Since the environment is i.i.d., the sections of the environment $\{\omega_x : \nu_k \leq x < \nu_{k+1}\}$ between ladder locations are also i.i.d. However, the environment immediately to the left of $\nu_0 = 0$ is different from the environment immediately to the left of ν_k for any $k \geq 1$ since $\prod_{j=i}^{\nu_k-1} \rho_j < 1$ for any $k \geq 1$ and $0 \leq i < \nu_k$ but it can happen that $\prod_{j=i}^{-1} \rho_j \geq 1$ for some $i < 0$. Thus, the environment is not stationary under shifts of the environment by the ladder locations. To resolve this complication we define a new measure Q on environments by

$$Q(\cdot) = P(\cdot\,|\mathcal{R}), \qquad \text{where } \mathcal{R} = \left\{ \omega : \prod_{j=i}^{-1} \rho_j < 1,\, \forall i \leq -1 \right\}.$$

It is important to note that the definition of the measure Q only affects the environment to the left of the origin. Therefore, the blocks of the environment $\{\omega_x : \nu_k \leq x < \nu_{k+1}\}$ between the ladder locations are i.i.d. and have the same distribution under both P and Q. For instance

$$\bar{\nu} := E_P \nu_1 = E_Q \nu_1. \tag{2.2}$$

The measure Q is also stationary under shifts of the environment by the ladder locations in the sense that ω has the same distribution as $\theta^{\nu_k(\omega)}\omega$ under Q and $\nu_1(\theta^{\nu_k(\omega)}\omega) = \nu_{k+1}(\omega) - \nu_k(\omega)$. Therefore, if we let $\beta_i = \beta_i(\omega) = E_\omega[T_{\nu_i} - T_{\nu_{i-1}}]$ for any $i \geq 1$, it follows that $\{\beta_i\}_{i\geq 1}$ is stationary under the measure Q. The following tail asymptotics of the β_i were derived in Peterson and Zeitouni (2009) and will be crucial throughout this paper. There exists a constant $C_0 > 0$ such that

$$Q(\beta_1 > x) = Q(E_\omega T_\nu > x) \sim C_0 x^{-\kappa}, \qquad \text{as } x \to \infty. \tag{2.3}$$

We conclude this section with a simple lemma that will be of use later in the paper.

Lemma 2.1. *Let $\bar{\beta} = E_Q[\beta_1]$. If $\kappa > 1$, then $\bar{\beta} = \bar{\nu}/\mathrm{v}_P$.*

Proof: First, note that the sequence $\{E_\omega[T_i - T_{i-1}]\}_{i\geq 1}$ is ergodic under the measure P (since it represents the shifts of a fixed function of an i.i.d., hence ergodic, sequence). Therefore, Birkhoff's Ergodic Theorem implies that

$$\lim_{n\to\infty} \frac{E_\omega T_n}{n} = E_P[E_\omega T_1] = \mathbb{E}T_1, \quad P\text{-a.s.} \tag{2.4}$$

Since the measure Q is defined by conditioning P on an event of positive probability, we see that this holds Q-a.s. as well. Moreover, if $\kappa > 1$, then the limiting velocity $\mathrm{v}_P = 1/\mathbb{E}T_1 > 0$ (see Solomon (1975) or Zeitouni (2004) for a reference).

Secondly, note that, since the $\{\nu_i - \nu_{i-1}\}_{i\geq 1}$ are i.i.d. under Q , it follows that $\lim_{n\to\infty} \nu_n/n = \bar{\nu}$, Q-a.s. This implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \beta_i = \lim_{n\to\infty} \frac{E_\omega T_{\nu_n}}{n} = \lim_{n\to\infty} \frac{E_\omega T_{\nu_n}}{\nu_n}\frac{\nu_n}{n} = \frac{\bar{\nu}}{\mathrm{v}_P}, \quad Q\text{-a.s.}$$

Finally, since the β_i are stationary under Q, it is a consequence of Birkhoff's Ergodic Theorem that this nonrandom limit of $1/n\sum_{i=1}^n \beta_i$ must coincide with $E_Q[\beta_1]$. □

3. Topological Generalities

3.1. *Skorohod Topologies.* In this section we recall the definitions of the Skorohod J_1 and M_1 metrics on the space D_∞, and the corresponding topologies. We also give certain technical results that will be needed in the sequel. The details that we omit can be found in Billingsley (1999) and Whitt (2002).

For $0 < t < \infty$ the J_1 and M_1 Skorohod metrics on the space D_t of càdlàg functions on $[0,t]$ are defined as follows. Let Λ_t be the set of time-change functions on $[0,t]$ – functions that are strictly increasing and continuous bijections from $[0,t]$ to itself. The Skorohod J_1-metric (on D_t) is defined by

$$d_t^{J_1}(x,y) = \inf_{\lambda\in\Lambda_t} \max\left\{ \sup_{s\leq t} |\lambda(s) - s|, \sup_{s\leq t} |x(\lambda(s)) - y(s)| \right\}.$$

Next, recall that the completed graph of a càdlàg function $x \in D_t$ is the subset $\Gamma_x \subset [0,t]\times\mathbb{R}$ defined by

$$\Gamma_x = \{(u,v) :\, u\in[0,t],\, v = (1-\theta)x(u-) + \theta x(u), \text{ for some } \theta\in[0,1]\}.$$

The natural order $\preceq_{\Gamma_x}$ on the completed graph is given by $(u_1,v_1) \preceq_{\Gamma_x} (u_2,v_2)$ if either $u_1 < u_2$ or $u_1 = u_2$ and $|v_1 - x(u_1-)| \leq |v_2 - x(u_1-)|$. A *parametric representation* of the completed graph Γ_x is a function from $[0,1]$ onto Γ_x that is continuous with respect to the subspace topology on Γ_x and non-decreasing with respect to the order $\preceq_{\Gamma_x}$. Let $\Pi(x)$ be the set of parametric representations of Γ_x, with each parametric representation given by a pair of functions u and v on $[0,1]$ such that $\Gamma_x = \{(u(s),v(s)) : s \in [0,1]\}$. The Skorohod M_1-metric on D_t is defined by

$$d_t^{M_1}(x,y) = \inf_{(u,v)\in\Pi(x),\,(u',v')\in\Pi(y)} \max\left\{\sup_{s\in[0,1]} |u(s)-u'(s)|, \sup_{s\in[0,1]} |v(s)-v'(s)|\right\}.$$

The Skorohod J_1 and M_1-metrics on D_t for all finite t produce corresponding metrics on the space D_∞ by

$$d_\infty^{J_1}(x,y) = \int_0^\infty e^{-t}\left(d_t^{J_1}(x^{(t)},y^{(t)}) \wedge 1\right) dt,$$

and

$$d_\infty^{M_1}(x,y) = \int_0^\infty e^{-t}\left(d_t^{M_1}(x^{(t)},y^{(t)}) \wedge 1\right) dt.$$

Here, for $x \in D_\infty$, the function $x^{(t)} \in D_t$ is the restriction of x to the finite time interval $[0,t]$. Using instead the uniform metric on each D_t produces the metric d_∞^U on the space D_∞.

The following is a list of several useful properties of the Skorohod metrics that we will use throughout the paper; see Whitt (2002).

- $d_\infty^{M_1}(x,y) \le d_\infty^{J_1}(x,y)$.
- $d_\infty^{J_1}(x,y) \le e^{-s} + \sup_{t\le s}|x(t)-y(t)|$ for any $0 < s < \infty$; thus uniform convergence on compact subsets of $[0,\infty)$ implies convergence in the J_1-Skorohod metric.
- $d_t^{M_1}(x,y) \ge |x(t)-y(t)|$ for each $0 < t < \infty$.
- $d_\infty^{J_1}(x_n,x) \to 0$ if and only if $d_t^{J_1}(x_n,x) \to 0$ for all continuity points t of x. An analogous statement is true for the M_1-Skorohod topology.

The J_1 and M_1-metrics generate topologies on the space of càdlàg functions. Even though the two metrics are not complete, each of them has an equivalent metric that is complete. Therefore, the J_1 and M_1 topologies are the topologies of complete separable metric spaces.

The J_1 and M_1-metrics on D_∞ induce in the standard way the corresponding Prohorov's metrics, ρ^{J_1} and ρ^{M_1} on the space of Borel probability measures $\mathcal{M}_1(D_\infty)$. For example, for any $\mu,\pi \in \mathcal{M}_1(D_\infty)$,

$$\rho^{M_1}(\mu,\pi) = \inf\left\{\delta > 0 : \mu(A) \le \pi(A^{\delta,M_1}) + \delta, \text{ for every Borel } A \subset D_\infty\right\}$$

(recall that the J_1 and M_1-metrics generate the same Borel sets on D_∞; these are also the cylindrical sets). Further,

$$A^{\delta,M_1} = \{y : d_\infty^{M_1}(x,y) < \delta, \text{ for some } x \in A\}.$$

Since $(D_\infty, d_\infty^{M_1})$ is a separable metric space, convergence in the Prohorov metric ρ^{M_1} is equivalent to convergence in distribution in $(D_\infty, d_\infty^{M_1})$; see Theorem 3.2.1 in Whitt (2002). Moreover, the space $\left(\mathcal{M}_1(D_\infty), \rho^{M_1}\right)$ is a complete separable metric space (Theorem 6.8 in Billingsley, 1999).

3.2. *Continuity of functionals.* We proceed with two results on the continuity of certain functionals that we will need later. We begin, by recalling the following result from Whitt (2002) on the continuity of the composition map.

Lemma 3.1 (Theorems 13.2.2, 13.2.3 in Whitt, 2002)**.** *The composition map* $\psi : D_\infty \times D_\infty^+ \to D_\infty$ *defined by* $\psi(x,y) = x \circ y$ *is continuous on the set* $D_\infty \times C_{\uparrow\uparrow}^+$*, where* D_∞^+ *is the set of all nonnegative functions in* D_∞*, and* $C_{\uparrow\uparrow}^+$ *is the set of continuous, non-negative, strictly increasing functions on* $[0,\infty)$*. The continuity holds whenever either the* J_1*-topology is used throughout, or the* M_1*-topology is used throughout.*

The composition map ψ induces a map $\Psi : \mathcal{M}_1(D_\infty) \times D_\infty^+ \to \mathcal{M}_1(D_\infty)$ by

$$\Psi(\mu, y)(\{x : x \in A\}) = \mu(\{x : x \circ y \in A\}).$$

Lemma 3.1 leads to the following continuity result for Ψ.

Corollary 3.2. *The map* Ψ *is continuous on the set* $\mathcal{M}_1(D_\infty) \times C_{\uparrow\uparrow}^+$*, if the same topology (either* J_1 *or* M_1*) is used throughout.*

Proof: Suppose that $(\mu_n, y_n) \to (\mu, y) \in \mathcal{M}_1(D_\infty) \times C_{\uparrow\uparrow}^+$. By the Skorohod representation theorem (e.g. Theorem 3.2.2 in Whitt, 2002), there are D_∞-valued random elements $(X_n), X$ defined on a common probability space such that $X_n \sim \mu_n$ for each n, $X \sim \mu$, and $X_n \to X$ a.s. in the corresponding Skorohod metric. By Lemma 3.1 we know that $X_n \circ y_n \to X \circ y$ in the same metric. Since a.s. convergence implies weak convergence, the claim follows. □

3.3. *Deducing weak convergence of random probability measures.* In order to prove weak convergence of a sequence of random probability measures on D_∞ we will often use the coupling technique which we now describe. Suppose that $\mu, \pi \in \mathcal{M}_1(D_\infty)$. Then a *coupling* of μ and π is a probability measure θ on the product space $D_\infty \times D_\infty$ with marginals μ and π, respectively. A coupling of two random probability measures on D_∞ defined on a common probability space is a random element of $\mathcal{M}_1(D_\infty \times D_\infty)$ defined on the same probability space that couples the two measures for every ω. The following simple lemma, which is a path space extension of Lemma 3.1 in Peterson and Samorodnitsky (2010), is the key ingredient in our approach.

Lemma 3.3. *Suppose that* $(\mu_n), (\pi_n)$ *are two sequences of random elements in* $\mathcal{M}_1(D_\infty)$ *defined on a common probability space with probability measure* $\mathbf{P}$ *and expectation* $\mathbf{E}$*. Suppose that one of the following conditions holds.*

(1) $\lim_{n\to\infty} \mathbf{P}(\rho^{M_1}(\mu_n, \pi_n) \geq \eta) = 0$*, for all* $\eta > 0$*.*
(2) *For each* n *there exists a coupling* θ_n *of the random probability measures* μ_n *and* π_n *such that*

$$\lim_{n\to\infty} \mathbf{E}\left[\theta_n(\{(x,y) : d_\infty^{M_1}(x,y) \geq \eta\})\right] = 0, \quad \textit{for all } \eta > 0.$$

(3) *For each* n *there exists a coupling* θ_n *of the random probability measures* μ_n *and* π_n *such that*

$$\lim_{n\to\infty} \mathbf{P}(E_{\theta_n}[d_\infty^{M_1}(x,y)] \geq \eta) = 0, \quad \textit{for all } \eta > 0,$$

where E_{θ_n} *denotes expectations under the measure* θ_n*.*

If $\mu_n \Longrightarrow \mu$ *weakly in* $(\mathcal{M}_1(D_\infty), \rho^{M_1})$*, then* $\pi_n \Longrightarrow \mu$ *weakly in* $(\mathcal{M}_1(D_\infty), \rho^{M_1})$*.*

Proof: Under condition (1) the statement follows from Theorem 3.1 in Billingsley (1999). Next, note that the definition of the Prohorov metric implies that if

$\theta_n(\{(x,y) : d_\infty^{M_1}(x,y) \geq \eta\}) \leq \eta$ then $\rho^{M_1}(\mu_n, \pi_n) \leq \eta$. Therefore,

$$\begin{aligned}\mathbf{P}(\rho^{M_1}(\mu_n,\pi_n) > \eta) &\leq \mathbf{P}(\theta_n(\{(x,y) :\, d_\infty^{M_1}(x,y) \geq \eta\}) > \eta) \\ &\leq \frac{1}{\eta}\mathbf{E}\left[\theta_n(\{(x,y) :\, d_\infty^{M_1}(x,y) \geq \eta\})\right].\end{aligned}$$

Thus, condition (2) implies condition (1). Furthermore, condition (3) implies condition (2) by Chebyshev's inequality. □

The following lemma will allow us to reduce checking condition (3) in Lemma 3.3 to the finite time situation. We note that a similar reduction holds under the metrics d^{J_1} and d^U as well.

Lemma 3.4. *Suppose that $(\mu_n), (\pi_n)$ are two sequences of random elements in $\mathcal{M}_1(D_\infty)$ defined on a common probability space with probability measure $\mathbf{P}$ and expectation $\mathbf{E}$. If for each n there exists a coupling θ_n of the random probability measures μ_n and π_n such that for every $0 < t < \infty$*

$$\lim_{n\to\infty} \mathbf{P}(E_{\theta_n}[d_t^{M_1}(x^{(t)}, y^{(t)})] \geq \eta) = 0, \quad \forall \eta > 0, \tag{3.1}$$

then condition (3) in Lemma 3.3 holds.

Proof: By the bounded convergence theorem, (3.1) implies that

$$\lim_{n\to\infty} \mathbf{E}\left[E_{\theta_n}[d_t^{M_1}(x^{(t)}, y^{(t)})] \wedge 1\right] = 0, \quad \forall t < \infty.$$

By the definition of $d_\infty^{M_1}$, Fubini's Theorem and dominated convergence theorem we immediately see that

$$\begin{aligned}\mathbf{E}\left[E_{\theta_n}[d_\infty^{M_1}(x,y)]\right] &= \mathbf{E}\left[E_{\theta_n}\left[\int_0^\infty e^{-t}\left(d_t^{M_1}(x^{(t)}, y^{(t)}) \wedge 1\right) dt\right]\right] \\ &= \int_0^\infty e^{-t}\mathbf{E}\left[E_{\theta_n}\left[\left(d_t^{M_1}(x^{(t)}, y^{(t)}) \wedge 1\right)\right]\right] dt\end{aligned}$$

vanishes as $n \to \infty$. This implies condition (3) in Lemma 3.3. □

4. Comparison with sums of exponentials

The main goal of this section is to reduce the study of the hitting time process $\mathbb{T}_\varepsilon$ to the study of a process $\mathbb{S}_\varepsilon$ that is defined in terms of sums of exponential random variables. To this end, recall the definition of the ladder locations of the environment in (2.1) and the notation $\beta_i = \beta_i(\omega) = E_\omega[T_{\nu_i} - T_{\nu_{i-1}}]$ for the quenched expectation of the time to cross from ν_{i-1} to ν_i. Also, we expand the measure P_ω to include an i.i.d. sequence of standard exponential random variables (τ_i); it will be used in the coupling procedure below by comparing $T_{\nu_i} - T_{\nu_{i-1}}$ with $\beta_i \tau_i$.

For any realization of the environment we construct random paths $\mathbb{U}_\varepsilon, \mathbb{S}_\varepsilon \in D_\infty$ as follows. For $t \geq 0$,

$$\mathbb{U}_\varepsilon(t) = \begin{cases} \varepsilon^{1/\kappa} T_{\nu_{\lfloor t/\varepsilon \rfloor}} & \kappa \in (0,1) \\ \varepsilon(T_{\nu_{\lfloor t/\varepsilon \rfloor}} - t/\varepsilon D'(1/\varepsilon)) & \kappa = 1 \\ \varepsilon^{1/\kappa}(T_{\nu_{\lfloor t/\varepsilon \rfloor}} - \bar{\beta} t/\varepsilon) & \kappa \in (1,2), \end{cases}$$

and

$$\mathbb{S}_\varepsilon(t) = \begin{cases} \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i \tau_i & \kappa \in (0,1) \\ \varepsilon(\sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i \tau_i - t/\varepsilon D'(1/\varepsilon)) & \kappa = 1 \\ \varepsilon^{1/\kappa}(\sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i \tau_i - \bar{\beta} t/\varepsilon) & \kappa \in (1,2), \end{cases} \tag{4.1}$$

where $D'(x) = E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le x\}}] \sim C_0 \log x$ when $\kappa = 1$ and $\bar{\beta} = E_Q[\beta_1] = E_Q[E_\omega T_\nu]$ when $\kappa \in (1,2)$.

Remark 4.1. The proof below will show that in the case $\kappa = 1$, the function D in definition of $\mathbb{T}_\varepsilon$ can be chosen to be $D(x) = D'(x)/\bar{\nu}$ (recall that $\bar{\nu} = E_Q[\nu_1]$.) In particular, the constant A in Theorem 1.1 satisfies $A = C_0/\bar{\nu}$.

Let $u_\varepsilon = u_{\varepsilon,\omega}, s_\varepsilon = s_{\varepsilon,\omega} \in \mathcal{M}_1(D_\infty)$ be the quenched distributions of $\mathbb{U}_\varepsilon$ and $\mathbb{S}_\varepsilon$, respectively. That is,

$$u_{\varepsilon,\omega} = P_\omega(\mathbb{U}_\varepsilon \in \cdot), \quad \text{and} \quad s_{\varepsilon,\omega} = P_\omega(\mathbb{S}_\varepsilon \in \cdot).$$

We view u_ε and s_ε as random elements in $\mathcal{M}_1(D_\infty)$. The proof of Theorem 1.4 is accomplished via the following two propositions. The first proposition establishes weak convergence of s_ε in $\mathcal{M}_1(D_\infty)$. The notation and the terminology are the same as in Theorem 1.4.

Proposition 4.2. *Let $\lambda = C_0\kappa$, where C_0 is the tail constant in (2.3). The following statements hold under the probability measure Q on the environments.*

(1) *If $\kappa \in (0,1)$, then $s_\varepsilon \Longrightarrow \mathcal{H}(N_{\lambda,\kappa})$.*
(2) *If $\kappa = 1$, then*

$$s_\varepsilon \Longrightarrow \lim_{\delta\to 0} \mathcal{H}_\delta(N_{\lambda,1}) * \ell(-\lambda \log(1/\delta)).$$

(3) *If $\kappa \in (1,2)$, then*

$$s_\varepsilon \Longrightarrow \lim_{\delta\to 0} \mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell(-\lambda\delta^{-\kappa+1}/(\kappa-1)).$$

The second proposition relates a weak limit of s_ε in $\mathcal{M}_1(D_\infty)$ to the corresponding weak limit of m_ε.

Proposition 4.3. *Define $\lambda_0 \in C^+_{\uparrow\uparrow}$ by $\lambda_0(t) = t/\bar{\nu}$. If $s_\varepsilon \Longrightarrow \mu$ weakly in $\mathcal{M}_1(D_\infty)$ under Q, then $m_\varepsilon \Longrightarrow \Psi(\mu, \lambda_0)$ under P.*

Before giving the proofs of Propositions 4.2 and 4.3, we show how they imply Theorem 1.4.

Proof of Theorem 1.4: The proof is essentially the same, whether $\kappa \in (0,1)$, $\kappa = 1$, or $\kappa \in (1,2)$, therefore we only spell out the details in the case $\kappa \in (1,2)$.

First, note that by Propositions 4.2 and 4.3 and Corollary 3.2, under the measure P,

$$m_\varepsilon \Longrightarrow \lim_{\delta\to 0} \Psi\left(\mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell\left(-\frac{\lambda\delta^{-\kappa+1}}{\kappa-1}\right), \lambda_0\right).$$

Therefore, it is enough to show that, for any $\lambda > 0$, with $\lambda' = \lambda/\bar{\nu}$,

$$\Psi\left(\mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell\left(-\frac{\lambda\delta^{-\kappa+1}}{\kappa-1}\right), \lambda_0\right) \stackrel{\text{Law}}{=} \mathcal{H}_\delta(N_{\lambda',\kappa}) * \ell\left(-\frac{\lambda'\delta^{-\kappa+1}}{\kappa-1}\right). \tag{4.2}$$

To see this, note that for any $m > 0$,

$$\Psi\left(\mathcal{H}_\delta\left(\sum_{i\geq 1}\delta_{(x_i,t_i)}\right)*\ell(-m),\lambda_0\right) = \mathcal{H}_\delta\left(\sum_{i\geq 1}\delta_{(x_i,t_i\bar{\nu})}\right)*\ell(-m/\bar{\nu}).$$

If $\sum_{i\geq 1}\delta_{(x_i,t_i)}$ is a Poisson point process with intensity measure $\lambda x^{-\kappa-1}\,dx\,dt$, then $\sum_{i\geq 1}\delta_{(x_i,t_i\bar{\nu})}$ is a Poisson point process with intensity measure $(\lambda/\bar{\nu})x^{-\kappa-1}\,dx\,dt$. This implies (4.2). ☐

It remains to prove Propositions 4.2 and 4.3. Proposition 4.2 will be proved in Section 5, and in the remainder of this section will focus on the proof of Proposition 4.3 which follows immediately from Lemma 3.3 and the following lemmas.

Lemma 4.4. *There exists a coupling of $\mathbb{U}_\varepsilon$ and $\mathbb{S}_\varepsilon$ such that, for any $\eta > 0$,*

$$\lim_{\varepsilon\to 0} Q\left(E_\omega[d_\infty^{J_1}(\mathbb{U}_\varepsilon,\mathbb{S}_\varepsilon)] \geq \eta\right) = 0.$$

Under the assumption of Proposition 4.3, this lemma and part (3) of Lemma 3.3 will imply that $u_\varepsilon \Longrightarrow \mu$ in $\mathcal{M}_1(D_\infty)$ under Q.

Lemma 4.5. *If $u_\varepsilon \Longrightarrow \mu$ in $\mathcal{M}_1(D_\infty)$ under Q, then $m_\varepsilon \Longrightarrow \Psi(\mu,\lambda_0)$ in $\mathcal{M}_1(D_\infty)$ under Q.*

Under the assumption of Proposition 4.3, this lemma will imply that $m_\varepsilon \Longrightarrow \Psi(\mu,\lambda_0)$ in $\mathcal{M}_1(D_\infty)$ under Q.

Lemma 4.6. *There exists a measure $\mathfrak{P}$ on pairs of environments (ω,ω') such that the marginal distributions of ω and ω' are P and Q, respectively, and, for each $\varepsilon > 0$, there exists a coupling $P_\varepsilon = P_{\varepsilon;\omega,\omega'}$ of the random measures $m_{\varepsilon,\omega}$ and $m_{\varepsilon,\omega'}$ such that*

$$\lim_{\varepsilon\to 0}\mathfrak{P}\left(E_\varepsilon[d_\infty^U(x,y)] \geq \eta\right) = 0, \quad \textit{for all } \eta > 0.$$

Under the assumption of Proposition 4.3, this lemma and another appeal to part (3) of Lemma 3.3 will imply the claim of the proposition. We proceed now to prove the three lemmas.

Proof of Lemma 4.6: We use the same construction as in the proof of Lemma 4.2 in Peterson and Samorodnitsky (2010). First let ω and $\tilde{\omega}$ be independent with distributions P and Q respectively. Then, construct ω' by letting

$$\omega'_x = \begin{cases}\tilde{\omega}_x & x \leq -1\\ \omega_x & x \geq 0.\end{cases}$$

Then ω' has distribution Q and is identical to ω on the non-negative integers. Let $\mathfrak{P}$ be the joint law of (ω,ω').

Given a pair of environments (ω,ω'), we construct coupled random walks $\{X_n\}$ and $\{X'_n\}$ so that the marginal laws of $\{X_n\}$ and $\{X'_n\}$ are P_ω and $P_{\omega'}$ respectively. We do that by coordinating all steps of the random walks to the right of 0. That is, since $\omega_x = \omega'_x$ for any $x \geq 0$, we require that on the respective i^{th} visits of the walks X_n and X'_n to site x they both either move to the right or both move to the left. The details of this coupling can be found in Peterson and Samorodnitsky (2010). Let $P_\varepsilon = P_{\varepsilon;\omega,\omega'}$ denote the joint quenched law of the two random walks coupled in this manner; the corresponding expectation is denoted by $E_\varepsilon = E_{\varepsilon;\omega,\omega'}$.

Let T_k, T'_k and $\mathbb{T}_\varepsilon$, $\mathbb{T}'_\varepsilon$ be the hitting times and the path processes of hitting times corresponding to the random walks $\{X_n\}$ and $\{X'_n\}$, respectively. Note that

$$d_\infty^U(\mathbb{T}_\varepsilon, \mathbb{T}'_\varepsilon) \le \sup_{t<\infty} |\mathbb{T}_\varepsilon(t) - \mathbb{T}'_\varepsilon(t)| = \varepsilon^{1/\kappa} \sup_{n\ge 1} |T_n - T'_n|.$$

However, it is easy to see that the coupling of X_n and X'_n is such that $\sup_{n\ge1} |T_n - T'_n| = |L - L'|$, where L and L' are the number of steps that the walks $\{X_n\}$ and $\{X'_n\}$, respectively, spend to the left of 0. It is easy to see (and was shown in the proof of Lemma 4.2 in Peterson and Samorodnitsky, 2010) that $E_{\varepsilon;\omega,\omega'}|L - L'| \le E_\omega L + E_{\omega'} L' < \infty$, $\mathfrak{P}$-a.s. Therefore, for any $\eta > 0$

$$\lim_{\varepsilon\to 0} \mathfrak{P}\left(E_\varepsilon\left[d_\infty^U(\mathbb{T}_\varepsilon, \mathbb{T}'_\varepsilon)\right] \ge \eta\right) \le \lim_{\varepsilon\to0} \mathfrak{P}\left(\varepsilon^{1/\kappa} E_{\omega,\omega'}|L - L'| \ge \eta\right) = 0.$$

□

Proof of Lemma 4.5: We start with a time change in the process $\mathbb{U}_\varepsilon$ to align its jumps with the hitting times of corresponding ladder locations in the process $\mathbb{T}_\varepsilon$. To this end, define $\lambda_\varepsilon \in D_\uparrow^+$, the space of nonnegative non-decreasing functions in D_∞, by

$$\lambda_\varepsilon(t) = \varepsilon \max\{k : \varepsilon\nu_k \le t\}, \ t \ge 0.$$

Then, the renewal theorem implies that $\lim_{\varepsilon\to0} \lambda_\varepsilon(t) = \lambda_0(t)$, Q-a.s, for any fixed $t \ge 0$. Since λ_ε is non-decreasing and λ_0 is continuous and non-decreasing, the convergence is uniform on compact subsets of $[0,\infty)$. Therefore, $\lim_{\varepsilon\to0} d_\infty^{J_1}(\lambda_\varepsilon, \lambda_0) = 0$, Q-a.s. Furthermore, it follows from the functional central limit theorem for renewal sequences with a finite variance that $\varepsilon^{-1/2}(\lambda_\varepsilon - \lambda_0)$ converges weakly in (D_∞, J_1) to a Brownian motion, as $\varepsilon \to 0$. See Theorem 7.4.1 in Whitt (2002).

The assumption $u_\varepsilon \Longrightarrow \mu$ and Corollary 3.2 show that $\Psi(u_\varepsilon, \lambda_\varepsilon) \Longrightarrow \Psi(\mu, \lambda_0)$ under Q, so by Lemmas 3.3 and 3.4, the claim of the present lemma will follow once we show that for every $0 < t < \infty$ and $\eta > 0$,

$$\lim_{\varepsilon\to0} Q\left(E_\omega\left[d_t^{M_1}(\mathbb{U}_\varepsilon \circ \lambda_\varepsilon,\, \mathbb{T}_\varepsilon)\right] \ge \eta\right) = 0. \tag{4.3}$$

To simplify the notation, we omit the superscripts in functions of the type $\mathbb{T}_\varepsilon^{(t)}$. Because of the centering present when $\kappa \in [1,2)$ but not when $\kappa \in (0,1)$, we treat the two cases separately.

Case I: $\kappa \in (0,1)$**.** Note that the definition of λ_ε implies that $\mathbb{U}_\varepsilon(\lambda_\varepsilon(t)) = \varepsilon^{1/\kappa} T_{\nu_j} = \mathbb{T}_\varepsilon(t)$ when $t = \varepsilon\nu_j$. We arrange the respective parametric representations of the completed graphs of the two random functions, $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ and $\mathbb{T}_\varepsilon$, so that at each $s_j = j/(k+1) \in [0,1)$ both parametric representations are equal, to $\left(\varepsilon\nu_j, \varepsilon^{1/\kappa} T_{\nu_j}\right)$. Here k is the largest j so that $\nu_j \le t/\varepsilon$. For $s_j < s < s_{j+1}$ with $j = 0, 1, \ldots, k-1$ we arrange the two parametric representations so that the vertical (v) coordinates always stay the same (see Figure 4.1). Then the distance between the corresponding points on the completed graphs on the interval in that range of s is taken horizontally, and it is, at most, $\varepsilon(\nu_{j+1} - \nu_j)$. This horizontal matching cannot, generally, be performed on the interval $(s_k, 1]$ since the two functions may not be equal at time t. On this interval we keep horizontal (u) coordinates the same. The distance between the corresponding points is now taken vertically, and it is, at most, $\varepsilon^{1/\kappa}(T_{\nu_{k+1}} - T_{\nu_k})$. Therefore,

$$d_t^{M_1}(\mathbb{U}_\varepsilon \circ \lambda_\varepsilon,\, \mathbb{T}_\varepsilon) \le \max\left\{\max_{j<k} \varepsilon(\nu_{j+1} - \nu_j),\, \varepsilon^{1/\kappa}(T_{\nu_{k+1}} - T_{\nu_k})\right\}.$$

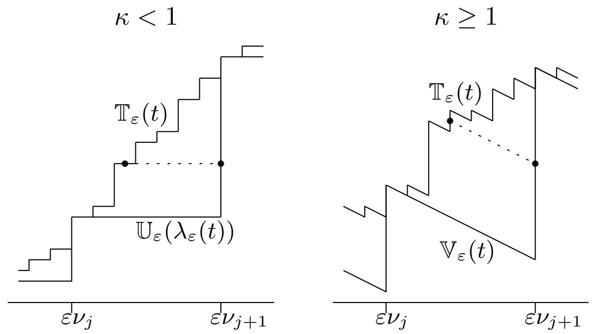

Figure 4.1. A demonstration of the matching of the parameterizations of the completed graphs of $\mathbb{T}_\varepsilon$ with the completed graphs of $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ and $\mathbb{V}_\varepsilon$ when $\kappa \in (0,1)$ and $\kappa \in [1,2)$, respectively.

Since $k \le t/\varepsilon$, we conclude, using stationarity of the sequence $(\nu_{j+1} - \nu_j)$ under Q that for $0 < \varepsilon < 1$ so small that $\varepsilon(\log 1/\varepsilon)^2 \le \eta$,

$$Q\left(E_\omega[d_t^{M_1}(\mathbb{U}_\varepsilon \circ \lambda_\varepsilon,\, \mathbb{T}_\varepsilon)] \ge \eta\right) \le \frac{t}{\varepsilon} Q\left(\nu_1 > \log^2(1/\varepsilon)\right) + Q\left(\varepsilon^{1/\kappa}\beta_{k+1} > \eta \text{ for } t \in [\varepsilon\nu_k, \varepsilon\nu_{k+1})\right).$$

Since ν_1 has some finite exponential moments (see Peterson and Zeitouni, 2009), the first term on the right above vanishes as $\varepsilon \to 0$. For the second term note that $t \in [\varepsilon\nu_k,\, \varepsilon\nu_{k+1})$ is equivalent to $\lambda_\varepsilon(t) = \varepsilon k$, hence

$$\begin{aligned} &Q\left(\varepsilon^{1/\kappa}\beta_{k+1} > \eta \text{ for } t \in [\varepsilon\nu_k, \varepsilon\nu_{k+1})\right) \\ &\le Q\left(|\lambda_\varepsilon(t) - t/\bar{\nu}| > \varepsilon^{1/4}\right) + Q\left(\exists k : |k - t/(\bar{\nu}\varepsilon)| \le \varepsilon^{-3/4},\, \beta_{k+1} > \eta\varepsilon^{-1/\kappa}\right) \\ &\le Q\left(|\lambda_\varepsilon(t) - t/\bar{\nu}| > \varepsilon^{1/4}\right) + 2\varepsilon^{-3/4} Q(\beta_1 > \eta\varepsilon^{-1/\kappa}), \end{aligned}$$

using the stationarity of the (β_k) under Q in the last inequality. The functional central limit theorem for renewal sequences implies that the first probability on the right vanishes as $\varepsilon \to 0$. The second term also vanishes as $\varepsilon \to 0$ by the tail decay (2.3) of β_1. This finishes the proof of (4.3) in the case $\kappa \in (0,1)$.

Case II: $\kappa \in [1,2)$**.** To overcome the difficulty of matching the centering terms of $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ and $\mathbb{T}_\varepsilon$ we define $\mathbb{V}_\varepsilon \in D_\infty$ by

$$\mathbb{V}_\varepsilon(t) = \varepsilon^{1/\kappa} T_{\nu_{\lfloor \lambda_\varepsilon(t)/\varepsilon \rfloor}} - \begin{cases} tD(1/\varepsilon) & \kappa = 1 \\ (t/\mathrm{v}_P)\varepsilon^{-1+1/\kappa} & \kappa \in (1,2). \end{cases}$$

$\mathbb{V}_\varepsilon$ is defined so that the hitting times portion is the same as in $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ while the linear centering is the same as in $\mathbb{T}_\varepsilon$.

Since the only difference between $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ and $\mathbb{V}_\varepsilon$ is in the centering term, we have for any $t < \infty$

$$d_t^{M_1}(\mathbb{U}_\varepsilon \circ \lambda_\varepsilon,\, \mathbb{V}_\varepsilon) \le \sup_{t' \le t} |\lambda_\varepsilon(t') - t'/\bar{\nu}| \begin{cases} D'(1/\varepsilon) & \kappa = 1 \\ \varepsilon^{-1+1/\kappa}\bar{\beta} & \kappa \in (1,2), \end{cases} \tag{4.4}$$

using $D'(1/\varepsilon) = \bar{\nu} D(1/\varepsilon)$ when $\kappa = 1$ and $\bar{\beta} = \bar{\nu}/\mathrm{v}_P$ when $\kappa \in (1,2)$. Recall that the random element of D_t, $t' \mapsto \varepsilon^{-1/2}(\lambda_\varepsilon(t') - t'/\bar{\nu})$, converges weakly in (D_t, J_1) to

Brownian motion, which is a continuous process. Every continuous function in D_t is a continuity point of the mapping $x \mapsto \sup_{t' \le t} |x(t')|$ from D_t to $\mathbb{R}$. Therefore, we can use the continuous mapping theorem to show that the term in the right hand side of (4.4) converges to 0 in Q-probability as $\varepsilon \to 0$, by noticing that both $D'(1/\varepsilon)$ (when $\kappa = 1$) and $\varepsilon^{-1+1/\kappa}$ (when $\kappa \in (1,2)$) are $o(\varepsilon^{-1/2})$. Therefore, in order to prove (4.3) it is enough to show that for every $0 < t < \infty$ and $\eta > 0$,

$$\lim_{\varepsilon \to 0} Q\left(E_\omega \left[d_t^{M_1}(\mathbb{V}_\varepsilon, \mathbb{T}_\varepsilon) \right] > \eta \right) = 0. \tag{4.5}$$

The proof of (4.5) is very similar to the proof of (4.3) when $\kappa \in (0,1)$. Indeed, note that $\mathbb{V}_\varepsilon(t) = \mathbb{T}_\varepsilon(t)$ whenever $t = \varepsilon \nu_j$ for some j. Again, we arrange the respective parametric representations of the completed graphs of the two random functions so that, for k being the largest j so that $\nu_j \le t/\varepsilon$, both parametric representations are equal, to $\left(\varepsilon\nu_j, \varepsilon^{1/\kappa} T_{\nu_j}\right)$ at $s_j = j/(k+1)$, $j = 0, 1, \ldots, k$. For $s_j < s < s_{j+1}$ with $j = 0, 1, \ldots, k-1$ the two parametric representation can be chosen in such a way that the line connecting the two corresponding points is always parallel to the segment, connecting the points $(\varepsilon\nu_j, \mathbb{V}_\varepsilon(\varepsilon\nu_j))$ and $(\varepsilon\nu_{j+1}, \mathbb{V}_\varepsilon(\varepsilon\nu_{j+1}-))$. See Figure 4.1 for a visual representation of this matching. In this case the distance between the two corresponding points does not exceed the length of the above segment, which is shorter than $\varepsilon^{1/2}(\nu_{j+1} - \nu_j)$ for ε small enough. As in the case $\kappa \in (0,1)$, on the interval $(s_k, 1]$ we keep horizontal (u) coordinates of the two parametric representations the same. Overall, we obtain the bound

$$d_t^{M_1}(\mathbb{V}_\varepsilon,\, \mathbb{T}_\varepsilon) \le \max\left\{ \varepsilon^{1/2} \max_{j \le k} (\nu_{j+1} - \nu_j),\, \varepsilon^{1/\kappa}(T_{\nu_{k+1}} - T_{\nu_k}) \right\}.$$

From here we proceed as in the case $\kappa \in (0,1)$ above. □

Proof of Lemma 4.4: By Lemma 3.4 it is enough to show that for each $0 < s < \infty$ and $\eta > 0$,

$$\lim_{\varepsilon \to \infty} Q\left(E_\omega \left[\sup_{t \le s} |\mathbb{U}_\varepsilon(t) - \mathbb{S}_\varepsilon(t)| \right] \ge \eta \right) = 0.$$

Since both $\mathbb{U}_\varepsilon$ and $\mathbb{S}_\varepsilon$ are piecewise linear with the same slope between times $t \in \varepsilon\mathbb{Z}$,

$$\sup_{t \le s} |\mathbb{U}_\varepsilon(t) - \mathbb{S}_\varepsilon(t)| = \varepsilon^{1/\kappa} \max_{k \le \lceil s/\varepsilon \rceil} \left| T_{\nu_k} - \sum_{i=1}^{k} \beta_i \tau_i \right|.$$

Now, it is easy to see that $M_k = T_{\nu_k} - \sum_{i=1}^k \beta_i \tau_i$ is a martingale under P_ω. Therefore, by the Cauchy-Schwartz and L^p-maximum inequalities for martingales,

$$\begin{aligned} E_\omega \left[\sup_{t \le s} |\mathbb{U}_\varepsilon(t) - \mathbb{S}_\varepsilon(t)| \right] &\le \left(E_\omega \left[\sup_{t \le s} |\mathbb{U}_\varepsilon(t) - \mathbb{S}_\varepsilon(t)|^2 \right] \right)^{1/2} \\ &\le \varepsilon^{1/\kappa} \left(4 E_\omega \left[M_{\lceil s/\varepsilon \rceil}^2 \right] \right)^{1/2} \\ &= 2\varepsilon^{1/\kappa} \left(\mathrm{Var}_\omega \left(T_{\nu_{\lceil s/\varepsilon \rceil}} - \sum_{i=1}^{\lceil s/\varepsilon \rceil} \beta_i \tau_i \right) \right)^{1/2}. \end{aligned}$$

Therefore,

$$Q\left(E_\omega\left[\sup_{t\le s}|\mathbb{U}_\varepsilon(t)-\mathbb{S}_\varepsilon(t)|\right]\ge\eta\right)\le Q\left(4\varepsilon^{2/\kappa}\operatorname{Var}_\omega\left(T_{\nu_{\lceil s/\varepsilon\rceil}}-\sum_{i=1}^{\lceil s/\varepsilon\rceil}\beta_i\tau_i\right)\ge\eta^2\right). \tag{4.6}$$

In the proof of Lemma 4.4 in Peterson and Samorodnitsky (2010), a natural coupling of τ_i and $T_{\nu_i}-T_{\nu_{i-1}}$ was constructed so that for any $\eta>0$,

$$\lim_{n\to\infty}Q\left(n^{-2/\kappa}\operatorname{Var}_\omega\left(T_{\nu_n}-\sum_{i=1}^{n}\beta_i\tau_i\right)\ge\eta\right)=0.$$

Applying this to (4.6) completes the proof of the lemma. □

5. Weak weak quenched limits for $\mathbb{S}_\varepsilon$

In this section we prove Proposition 4.2. For any environment ω and $\varepsilon>0$, define a point process by

$$N_\varepsilon=\sum_{i\ge1}\delta_{(\varepsilon^{1/\kappa}\beta_i,\,\varepsilon i)}.$$

We view N_ε as a random element of $\mathcal{M}_p((0,\infty]\times[0,\infty))$. Recalling the definitions of $\mathcal{H}$ in (1.5) and $\mathbb{S}_\varepsilon$ in (4.1), we see that the quenched law of $\mathbb{S}_\varepsilon$ satisfies

$$s_\varepsilon=\begin{cases}\mathcal{H}(N_\varepsilon) & \kappa\in(0,1)\\ \mathcal{H}(N_\varepsilon)*\ell(-D'(1/\varepsilon)) & \kappa=1\\ \mathcal{H}(N_\varepsilon)*\ell(-\bar{\beta}\varepsilon^{-1+1/\kappa}) & \kappa\in(1,2).\end{cases} \tag{5.1}$$

The key to the proof of Proposition 4.2 is the following lemma which shows weak convergence of the point process N_ε.

Lemma 5.1. *Under the measure Q, as $\varepsilon\to0$, the point process N_ε converges weakly in the space $\mathcal{M}_p((0,\infty]\times[0,\infty))$ to a non-homogeneous Poisson point process $N_{\lambda,\kappa}$ with intensity measure $\lambda x^{-\kappa-1}dx\,dt$. Moreover, $\lambda=C_0\kappa$, where C_0 is the tail constant in (2.3).*

Proof: The idea of the proof is similar to that of the proof of Proposition 5.1 in Peterson and Samorodnitsky (2010). It was shown in the above proof that for $0<\varepsilon<1$ there is a stationary under Q sequence of random variables $\left(\beta_i^{(\varepsilon)},\,i=1,2,\ldots\right)$ on Ω such that $\beta_i^{(\varepsilon)}$ and $\beta_j^{(\varepsilon)}$ are independent if $|i-j|>\varepsilon^{-1/2}$, and such that, for some $C,C'>0$,

$$Q\left(\left|\beta_1-\beta_1^{(\varepsilon)}\right|>e^{-\varepsilon^{-1/4}}\right)\le Ce^{-C'\varepsilon^{-1/2}},\ 0<\varepsilon<1\,. \tag{5.2}$$

We define an approximating point process by

$$N_\varepsilon^{(1)}=\sum_{i\ge1}\delta_{(\varepsilon^{1/\kappa}\beta_i^{(\varepsilon)},\,\varepsilon i)},\ 0<\varepsilon<1\,,$$

and proceed by proving the convergence

$$N_\varepsilon^{(1)}\Longrightarrow N_{\lambda,\kappa}\text{ weakly in }\mathcal{M}_p((0,\infty]\times[0,\infty)) \tag{5.3}$$

as $\varepsilon\to0$, under the measure Q.

We start by considering measurable functions $f : (0,\infty] \times [0,\infty) \to \mathbb{R}_+$ of the form

$$f(x,t) = \sum_{i=1}^{k} f_i(x)\mathbf{1}_{[a_{i-1},a_i)}(t),\ (x,t) \in (0,\infty] \times [0,\infty)\,, \tag{5.4}$$

where $k = 1,2,\ldots$, $f_i : (0,\infty] \to \mathbb{R}_+$, $i = 1,\ldots,k$ are continuous functions that vanish for all $0 < x < \delta$ for some $\delta > 0$, and are Lipschitz on the interval (δ,∞), and $0 = a_0 < a_1 < \ldots < a_k < \infty$. We will prove that for such a function,

$$\lim_{\varepsilon\to 0} E_Q\left[e^{-N_\varepsilon^{(1)}(f)}\right] = \exp\left\{-\sum_{i=1}^{k}(a_i - a_{i-1})\int_0^\infty (1-e^{-f_i(x)})\lambda x^{-\kappa-1}\,dx\right\}. \tag{5.5}$$

To this end we define, as in Peterson and Samorodnitsky (2010), for $0 < \tau < 1$,

$$K_\varepsilon(\tau) = \text{card}\big\{i = 1,\ldots,\lfloor a_k/\varepsilon\rfloor :\ \text{both } \beta_i^{(\varepsilon)} > \delta\varepsilon^{-1/\kappa} \text{ and } \beta_j^{(\varepsilon)} > \delta\varepsilon^{-1/\kappa}$$
$$\text{for some } i+1 \le j \le i+\tau/\varepsilon,\ j \le a_k/\varepsilon.\big\};$$

as in the above reference we have

$$\lim_{\tau\to 0}\limsup_{\varepsilon\to 0} Q(K_\varepsilon(\tau) > 0) = 0. \tag{5.6}$$

Define random sets

$$D_\varepsilon^{(j)} = \{a_{j-1}/\varepsilon < i < a_j/\varepsilon :\ \beta_i^{(\varepsilon)} > \delta\varepsilon^{-1/\kappa}\},\ j = 1,\ldots,k,$$

so that

$$\begin{aligned}
E_Q\left[e^{-N_\varepsilon^{(1)}(f)}\right] &= E_Q \exp\left\{-\sum_{j=1}^{k}\sum_{i\in D_\varepsilon^{(j)}} f_j\big(\varepsilon^{1/\kappa}\beta_i^{(\varepsilon)}\big)\right\} \\
&= E_Q\left[\exp\left\{-\sum_{j=1}^{k}\sum_{i\in D_\varepsilon^{(j)}} f_j\big(\varepsilon^{1/\kappa}\beta_i^{(\varepsilon)}\big)\right\}\mathbf{1}\big(K_\varepsilon(\tau)=0\big)\right] \\
&+ E_Q\left[\exp\left\{-\sum_{j=1}^{k}\sum_{i\in D_\varepsilon^{(j)}} f_j\big(\varepsilon^{1/\kappa}\beta_i^{(\varepsilon)}\big)\right\}\mathbf{1}\big(K_\varepsilon(\tau)>0\big)\right] \\
&:= H_\varepsilon^{(1)} + H_\varepsilon^{(2)}.
\end{aligned}$$

It follows from (5.6) that the term $H_\varepsilon^{(2)}$ is negligible as $\varepsilon \to 0$ and then $\tau \to 0$. Furthermore, given the event $\{K_\varepsilon(\tau) = 0\}$, for a fixed $0 < \tau < 1$ and ε small enough, the points in the random set $D_\varepsilon := \cup_j D_\varepsilon^{(j)}$ are separated by more than $\varepsilon^{-1/2}$, so that, given also the set D_ε, the random variables $\beta_i^{(\varepsilon)}$, $i \in D_\varepsilon$ are independent, each one with the conditional distribution of $\beta_1^{(\varepsilon)}$ given $\beta_1^{(\varepsilon)} > \delta\varepsilon^{-1/\kappa}$. Since for every $j = 1,\ldots,k$,

$$E_Q\Big(\exp\big\{-f_j\big(\varepsilon^{1/\kappa}\beta_1^{(\varepsilon)}\big)\big\}\big|\beta_1^{(\varepsilon)} > \delta\varepsilon^{-1/\kappa}\Big) \to \int_1^\infty e^{-f_j(\delta t)}\kappa t^{-(\kappa+1)}\,dt,$$

the claim (5.5) will follow once we check that

$$\begin{aligned}\exp\Big\{-C_0\delta^{-\kappa}\sum_{j=1}^k (a_j-a_{j-1})(1-\alpha_j)\Big\} &\le \lim_{\tau\to 0}\liminf_{\varepsilon\to 0} E_Q\left(\prod_{j=1}^k \alpha_j^{\text{card } D_\varepsilon^{(j)}}\Big| K_\varepsilon(\tau)=0\right)\\ &= \lim_{\tau\to 0}\limsup_{\varepsilon\to 0} E_Q\left(\prod_{j=1}^k \alpha_j^{\text{card } D_\varepsilon^{(j)}}\Big| K_\varepsilon(\tau)=0\right)\\ &\le \exp\Big\{-C_0\delta^{-\kappa}\sum_{j=1}^k (a_j-a_{j-1})(1-\alpha_j)\Big\}\end{aligned}$$

for any $0<\alpha_j<1$, $j=1,\dots,k$. This, however, can be proved in the same way as (48) was proved in Peterson and Samorodnitsky (2010).

In order to prove weak convergence in (5.3), it is enough to prove that for any Lipschitz continuous function $f:\ (0,\infty]\times[0,\infty)\to\mathbb{R}_+$ with support in $[\delta,\infty]\times[0,a]$ for some $0<\delta,a<\infty$,

$$\lim_{\varepsilon\to 0} E_Q\left[e^{-N_\varepsilon^{(1)}(f)}\right] = \exp\left\{-\int_0^\infty\int_0^\infty (1-e^{-f(x,t)})\lambda x^{-\kappa-1}\,dx\,dt\right\}; \tag{5.7}$$

see Resnick (2008) and Remark 5.2 in Peterson and Samorodnitsky (2010). To this end, for $m=1,2,\dots$ we define

$$f_j(x)=f(x,ja/m),\ x\in(0,\infty],\ j=1,\dots,m,$$

and

$$\tilde f(x,t)=\sum_{j=1}^k f_j(x)\mathbf{1}_{[(j-1)a/m,ja/m)}(t),\ \ (x,t)\in(0,\infty]\times[0,\infty)\,.$$

Note that $|f(x,t)-\tilde f(x,t)|\le La/m$ for all finite (x,t), where L is the Lipschitz constant of f. Therefore,

$$\left|E_Q\left[e^{-N_\varepsilon^{(1)}(f)}\right]-E_Q\left[e^{-N_\varepsilon^{(1)}(\tilde f)}\right]\right|\le \frac{La}{m}E_Q\left[N_\varepsilon^{(1)}\big([\delta,\infty]\times[0,a]\big)\right].$$

Notice that, by stationarity,

$$E_Q\left[N_\varepsilon^{(1)}\big([\delta,\infty]\times[0,a]\big)\right]\le a\varepsilon^{-1}Q\big(\beta_1^{(\varepsilon)}>\delta\varepsilon^{-1/\kappa}\big),$$

which, by (2.3) and (5.2), remains bounded as $\varepsilon\to 0$. Since the function $\tilde f$ is of the type (5.4), it follows from (5.5) that

$$\lim_{\varepsilon\to 0} E_Q\left[e^{-N_\varepsilon^{(1)}(\tilde f)}\right] = \exp\left\{-\int_0^\infty\int_0^\infty (1-e^{-\tilde f(x,t)})\lambda x^{-\kappa-1}\,dx\,dt\right\}.$$

This proves (5.7) (and, hence, also (5.3)). It follows by (5.2) and the Lipschitz property that for any function f as in (5.7) we also have

$$\lim_{\varepsilon\to 0} E_Q\left[e^{-N_\varepsilon(f)}\right] = \exp\left\{-\int_0^\infty\int_0^\infty (1-e^{-f(x,t)})\lambda x^{-\kappa-1}\,dx\,dt\right\}.$$

As before, this establishes the weak convergence stated in the lemma. □

We would like to use the representation (5.1) of s_ε, and the fact that $N_\varepsilon\Longrightarrow N_{\lambda,\kappa}$ to obtain a weak limit for s_ε as a random element of $\mathcal{M}_1(D_\infty)$. Unfortunately, the function $\mathcal{H}$ is not continuous and so we need the following lemma which shows that the truncated function $\mathcal{H}_\delta$ is "almost continuous".

Lemma 5.2. *Define subsets* $C_\delta, E \subset \mathcal{M}_p((0,\infty]\times[0,\infty))$ *by*

$$C_\delta = \{\zeta : \zeta(\{\delta,\infty\}\times[0,\infty)) = 0\}$$

and

$$E = \{\zeta : \zeta((0,\infty]\times\{t\}) \le 1, \forall t \in (0,\infty)\} \cap \{\zeta : \zeta((0,\infty]\times\{0\}) = 0\}.$$

Then $\mathcal{H}_\delta$ *is continuous on* $C_\delta \cap E$.

Proof: Suppose that $\zeta_n \to \zeta \in C_\delta \cap E$. We will couple the paths $W_\delta(\zeta_n, \vec{\tau})$ and $W_\delta(\zeta, \vec{\tau})$ by using the same sequence $\vec{\tau}$ of i.i.d. standard exponential random variables. Using this coupling we will show that $\lim_{n\to\infty} W_\delta(\zeta_n, \vec{\tau}) = W_\delta(\zeta, \vec{\tau})$, $\mathbf{P}_\tau$-a.s. Since almost sure convergence implies weak convergence, $H_\delta(\zeta_n) \to \mathcal{H}_\delta(\zeta)$.

To prove that $W_\delta(\zeta_n, \vec{\tau})$ converges a.s. to $W_\delta(\zeta, \vec{\tau})$ it will be enough to show that for every $0 < s < \infty$ such that $W_\delta(\zeta, \vec{\tau})$ is continuous at s, and for every realization $\vec{\tau}$ with finite values,

$$\lim_{n\to\infty} d_s^{J_1}(W_\delta(\zeta_n, \vec{\tau}), W_\delta(\zeta, \vec{\tau})) = 0. \tag{5.8}$$

To this end, take s as above. Then $\zeta([\delta,\infty]\times\{s\}) = 0$. The assumption that $\zeta \in E$ implies that we may order the atoms of ζ in $[\delta,\infty]\times[0,s]$ so that for $M = \zeta([\delta,\infty]\times[0,s])$ we have

$$\zeta\big(\cdot \cap ([\delta,\infty]\times[0,s])\big) = \sum_{i=1}^{M} \delta_{(x_i,t_i)}(\cdot), \quad \text{with } 0 < t_1 < t_2 < \ldots < t_M < s.$$

Similarly, we can order the atoms of ζ_n in $[\delta,\infty]\times[0,s]$ so that for $M_n = \zeta_n([\delta,\infty]\times[0,s])$ we have

$$\zeta_n\big((\cdot \cap ([\delta,\infty]\times[0,s])\big) = \sum_{i=1}^{M_n} \delta_{(x_i^{(n)},t_i^{(n)})}, \quad \text{with } 0 \le t_1^{(n)} \le t_2^{(n)} \le \ldots \le t_{M_n}^{(n)} \le s.$$

The vague convergence of ζ_n to ζ and the fact that ζ has no atoms on the boundary of $[\delta,\infty]\times[0,s]$, imply that for n large enough $M_n = M$ and

$$\lim_{n\to\infty} \max_{i\le M} \left(|x_i^{(n)} - x_i| \vee |t_i^{(n)} - t_i|\right) = 0. \tag{5.9}$$

Therefore, for n sufficiently large, $0 < t_1^{(n)} < t_2^{(n)} < \ldots < t_M^{(n)} < s$. For such n we define a time-change function λ_n^s of the interval $[0,s]$ by $\lambda_n^s(0) = 0$, $\lambda_n^s(s) = s$, $\lambda_n^s(t_i) = t_i^{(n)}$ for all $i \le M$, and extend it everywhere else by linear interpolation. Then,

$$\sup_{t\le s} |\lambda_n^s(t) - t| = \max_{i\le M} |t_i^{(n)} - t_i|,$$

and, since $W_\delta(\zeta_n, \vec{\tau})$ and $W_\delta(\zeta, \vec{\tau})$ are constant between jumps,

$$\sup_{t\le s} |W_\delta(\zeta_n,\vec{\tau})(\lambda_n^s(t)) - W_\delta(\zeta,\vec{\tau})(t)| = \max_{j\le M} \left| \sum_{i=1}^{j} \left(x_i^{(n)} - x_i\right) \tau_i \right| \le \sum_{i=1}^{M} \left|x_i^{(n)} - x_i\right| \tau_i.$$

Therefore, for n sufficiently large,

$$d_s^{J_1}\left(W_\delta(\zeta_n,\vec{\tau}), W_\delta(\zeta,\vec{\tau})\right) \le \max\left\{ \max_{i\le M} |t_i^{(n)} - t_i|, \sum_{i=1}^{M} \left|x_i^{(n)} - x_i\right| \tau_i \right\},$$

which vanishes as $n \to \infty$ by (5.9). This completes the proof of (5.8) and thus of the lemma. □

The relationship between s_ε and N_ε in (5.1) and Lemma 5.2 will allow us now to complete the proof of Proposition 4.2.

Proof of Proposition 4.2: For $\delta > 0$ we define a truncated version of $\mathbb{S}_\varepsilon$ by

$$\mathbb{S}_{\varepsilon,\delta}(t) = \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i \tau_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i > \delta\}} - t\gamma_{\kappa,\varepsilon,\delta},\ t \geq 0\,,$$

with

$$\gamma_{\kappa,\varepsilon,\delta} = \begin{cases} 0 & \kappa \in (0,1) \\ E_Q\left[\beta_1 \mathbf{1}_{\{\varepsilon\beta_1 \in (\delta,1]\}}\right] & \kappa = 1 \\ \varepsilon^{1/\kappa - 1} E_Q\left[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 > \delta\}}\right] & \kappa \in (1,2). \end{cases} \tag{5.10}$$

Then the quenched law of $\mathbb{S}_{\varepsilon,\delta}$ is $s_{\varepsilon,\delta} = \mathcal{H}_\delta(N_\varepsilon) * \ell(-\gamma_{\kappa,\varepsilon,\delta})$.

If $N_{\lambda,\kappa}$ is the Poisson point process as in the statement of Lemma 5.1, then $\mathbf{P}(N_{\lambda,\kappa} \in C_\delta \cap E) = 1$ for any $\delta > 0$. Thus, Lemma 5.1, Lemma 5.2, and the continuous mapping theorem imply that, under the measure Q, $\mathcal{H}_\delta(N_\varepsilon) \Longrightarrow \mathcal{H}_\delta(N_{\lambda,\kappa})$, where $\lambda = C_0\kappa$. Also, by (2.3) and Karamata's theorem,

$$\lim_{\varepsilon\to 0} \gamma_{\kappa,\varepsilon,\delta} = \begin{cases} C_0 \ln(1/\delta) & \kappa = 1 \\ \frac{C_0\kappa}{\kappa-1}\delta^{-\kappa+1} & \kappa \in (1,2). \end{cases}$$

Since the mapping from $\mathcal{M}_1(D_\infty) \times \mathbb{R}$ to $\mathcal{M}_1(D_\infty)$ defined by $(\mu,\gamma) \mapsto \mu * \ell(\gamma)$ is continuous, we conclude that, under the measure Q,

$$s_{\varepsilon,\delta} \Longrightarrow \begin{cases} \mathcal{H}_\delta(N_{\lambda,\kappa}) & \kappa \in (0,1) \\ \mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell(-\lambda \ln(1/\delta)) & \kappa = 1 \\ \mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell(-\lambda\delta^{-\kappa+1}/(\kappa-1)) & \kappa \in (1,2). \end{cases} \tag{5.11}$$

To relate (5.11) to a limit statement about s_ε, we use Billingsley (1999, Theorem 3.2). To this end, it is enough to show that the limit in $\mathcal{M}_1\big((D_\infty, d_\infty^{M_1})\big)$

$$\lim_{\delta\to\infty} \begin{cases} \mathcal{H}_\delta(N_{\lambda,\kappa}) & \kappa \in (0,1) \\ \mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell(-\lambda \ln(1/\delta)) & \kappa = 1 \\ \mathcal{H}_\delta(N_{\lambda,\kappa}) * \ell(-\lambda\delta^{-\kappa+1}/(\kappa-1)) & \kappa \in (1,2), \end{cases} \quad \text{exists } \mathbf{P}_\tau\text{-a.s.} \tag{5.12}$$

and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} Q\left(\rho^{M_1}(s_{\varepsilon,\delta},\, s_\varepsilon) \geq \eta\right) = 0, \quad \forall \eta > 0. \tag{5.13}$$

As in the case of Lemma 3.3, (5.13) will follow from following, stronger, statement: for every $0 < s < \infty$,

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} Q\left(E_\omega\left[\sup_{t\leq s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)|\right] \geq \eta\right) = 0, \quad \forall \eta > 0. \tag{5.14}$$

Therefore, to complete the proof of Proposition 4.2, it remains only to prove (5.12) and (5.14). We divide the proof of these statements into two cases: $\kappa \in (0,1)$ and $\kappa \in [1,2)$.

5.1. *Case I:* $\kappa \in (0,1)$. To prove (5.12) we let $F_1 \subset \mathcal{M}_p((0,\infty] \times [0,\infty))$ be defined by

$$F_1 = \left\{ \zeta = \sum_{i \geq 1} \delta_{(x_i,t_i)} \, : \, \sum_{i\geq 1} x_i \mathbf{1}_{\{t_i \leq t\}} < \infty, \, \forall t < \infty \right\}.$$

(Note that on the set F_1, the sum in the definition of $W(\zeta,\vec{\tau})$ is $\mathbf{P}_\tau$-a.s. finite.) Since $\mathbf{P}(N_{\lambda,\kappa} \in F_1) = 1$ when $\kappa \in (0,1)$, it will be enough to show that $H_\delta(\zeta) \to \mathcal{H}(\zeta)$ as $\delta \to 0$ for any $\zeta \in F_1$. Fix $\zeta = \sum_{i\geq 1} \delta_{(x_i,t_i)} \in F_1$. For $0 < s < \infty$ the obvious coupling of $W(\zeta,\vec{\tau})$ and $W_\delta(\zeta,\vec{\tau})$ gives that

$$\sup_{t \leq s} |W(\zeta,\vec{\tau})(t) - W_\delta(\zeta,\vec{\tau})(t)| = \sup_{t\leq s} \left| \sum_{i\geq 1} x_i \tau_i \mathbf{1}_{\{x_i \leq \delta,\, t_i \leq t\}} \right| = \sum_{i \geq 1} x_i \tau_i \mathbf{1}_{\{x_i \leq \delta,\, t_i \leq s\}}. \tag{5.15}$$

Since $\zeta \in F_1$, finiteness of the mean of an exponential random variable shows that the sum on the right is finite with probability one for any $\delta > 0$. Letting $\delta \to 0$ the dominated convergence theorem shows that $W_\delta(\zeta,\vec{\tau})$ converges almost surely to $W(\zeta,\vec{\tau})$ in the space D_s in the uniform metric, hence also in the M_1-metric, for any $0 < s < \infty$. Therefore, $W_\delta(\zeta,\vec{\tau})$ converges almost surely to $W(\zeta,\vec{\tau})$ in D_∞ as $\delta \to 0$ and, since a.s. convergence implies convergence in distribution, $\mathcal{H}_\delta(\zeta)$ converges to $\mathcal{H}(\zeta)$ in the space $\mathcal{M}_1\big((D_\infty, d_\infty^{M_1})\big)$ as $\delta \to 0$. This proves (5.12). Further, since $W_\delta(N_\varepsilon,\vec{\tau}) = \mathbb{S}_{\varepsilon,\delta}$ and $W(N_\varepsilon,\vec{\tau}) = \mathbb{S}_\varepsilon$, we have by (5.15) with $\zeta = N_\varepsilon$,

$$\begin{aligned} E_\omega\left[\sup_{t\leq s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)| \right] &= E_\omega\left[\sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \varepsilon^{1/\kappa} \beta_i \tau_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \leq \delta\}} \right] \\ &= \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \leq \delta\}}. \end{aligned}$$

By Chebyshev's inequality and stationarity of β_i under Q,

$$\begin{aligned} Q\left(E_\omega\left[\sup_{t\leq s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)| \right] \geq \eta \right) &= Q\left(\varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \leq \delta\}} \geq \eta \right) \\ &\leq \frac{s\varepsilon^{1/\kappa - 1}}{\eta} E_Q\left[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 \leq \delta\}} \right]. \end{aligned}$$

Karamata's theorem and (2.3) imply that $E_Q\left[\beta_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \leq \delta\}}\right] \sim \frac{C_0\kappa}{1-\kappa} \delta^{1-\kappa} \varepsilon^{1-1/\kappa}$ as $\varepsilon \to 0$. Therefore,

$$\lim_{\delta\to 0} \limsup_{\varepsilon \to 0} Q\left(E_\omega\left[\sup_{t\leq s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)| \right] \geq \eta \right) \leq \lim_{\delta \to 0} \frac{sC_0\kappa}{\eta(1-\kappa)} \delta^{1-\kappa} = 0.$$

This proves (5.14).

5.2. *Case II:* $\kappa \in [1,2)$. To prove (5.12), note that the right hand side of (5.12) is the law (with respect to $\mathbf{P}_\tau$) of the random element of D_∞

$$t \mapsto W_\delta(N_{\lambda,\kappa},\vec{\tau})(t) - \begin{cases} \lambda t \log(1/\delta) & \kappa = 1 \\ \frac{\lambda \delta^{1-\kappa} t}{\kappa - 1} & \kappa \in (1,2). \end{cases}$$

It was shown in the proof of Corollary 1.9 that this random element converges almost surely, in the uniform metric, under the joint law $\mathbf{P} \times \mathbf{P}_\tau$ of $(N_{\lambda,\kappa}, \vec{\tau})$. Therefore, convergence takes place in the M_1-metric as well. Fubini's theorem implies that the convergence also holds $\mathbf{P}_\tau$-a.s. for almost every realization of the point process $N_{\lambda,\kappa}$. Once again, a.s. convergence implies convergence in distribution, so (5.12) holds.

To prove (5.14), note that the definition of $\gamma_{\kappa,\varepsilon,\delta}$ in (5.10) implies that

$$\begin{aligned}
&\sup_{t\le s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)| \\
&\quad = \sup_{t\le s} \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i \tau_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} - E_Q[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 \le \delta\}}] \varepsilon^{-1+1/\kappa} t \right| \\
&\quad \le \sup_{t\le s} \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i (\tau_i - 1) \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} \right| \\
&\qquad + \sup_{t\le s} \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \left\{ \beta_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} - E_Q[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 \le \delta\}}] \right\} \right| \\
&\qquad + \varepsilon^{1/\kappa} E_Q[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 \le \delta\}}],
\end{aligned} \tag{5.16}$$

where the last term comes from rounding in the number of terms in the sum. This terms is, clearly, bounded by δ.

For β_i fixed, the sum inside the supremum in the first term in (5.16) is a sum of independent, zero-mean random variables. Thus, by the Cauchy-Schwartz and L^p-maximum inequalities for martingales,

$$\begin{aligned}
&E_\omega \left[\sup_{t\le s} \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \beta_i (\tau_i - 1) \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} \right| \right] \\
&\le \left(4 E_\omega \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i (\tau_i - 1) \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} \right|^2 \right)^{1/2} = 2\varepsilon^{1/\kappa} \left(\sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i^2 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} \right)^{1/2}
\end{aligned}$$

Therefore, for $\eta > 0$ fixed and δ sufficiently small we have

$$\begin{aligned}
&Q \left(E_\omega \left[\sup_{t\le s} |\mathbb{S}_{\varepsilon,\delta}(t) - \mathbb{S}_\varepsilon(t)| \right] \ge \eta \right) \\
&\le Q \left(\varepsilon^{2/\kappa} \sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i^2 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} \ge \eta^2/36 \right) \\
&\quad + Q \left(\sup_{t\le s} \left| \varepsilon^{1/\kappa} \sum_{i=1}^{\lfloor t/\varepsilon \rfloor} \left\{ \beta_i \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i \le \delta\}} - E_Q[\beta_1 \mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_1 \le \delta\}}] \right\} \right| \ge \eta/3 \right).
\end{aligned} \tag{5.17}$$

Notice that

$$\limsup_{\varepsilon\to 0} Q\left(\varepsilon^{2/\kappa}\sum_{i=1}^{\lfloor s/\varepsilon\rfloor}\beta_i^2\mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i\le\delta\}}\ge \eta^2/36\right)$$
$$\le \limsup_{\varepsilon\to 0}\frac{36s\varepsilon^{2/\kappa-1}}{\eta^2}E_Q\left[\beta_i^2\mathbf{1}_{\{\varepsilon^{1/\kappa}\beta_i\le\delta\}}\right]=\frac{36s}{\eta^2}\frac{C_0\kappa}{2-\kappa}\delta^{2-\kappa},$$

where the last equality follows from (2.3) and Karamata's Theorem. This vanishes as $\delta\to 0$ since $\kappa<2$. It remains only to show that the term in (5.17) vanishes as first $\varepsilon\to 0$ and then $\delta\to 0$. A similar statement (without the supremum inside the probability) was shown in Peterson and Samorodnitsky (2010, Lemma 5.5). One can modify the techniques of Peterson and Samorodnitsky (2010) to give a bound on (5.17) that vanishes as first $\varepsilon\to 0$ and then $\delta\to 0$. Since the argument is somewhat technical, we postpone it until Appendix A. □

6. Weak weak quenched limits for the position of the random walk

In this section we prove Theorem 1.6. We start by defining the running maximum version of the scaled path process of the random walk χ_ε in (1.7). For $t\ge 0$, let $X_t^* = \max\{X_k : k\le t\}$ denote the running maximum of the RWRE. The corresponding random element in D_∞ is

$$\chi_\varepsilon^*(t)=\begin{cases}\varepsilon^\kappa X_{t/\varepsilon}^* & \kappa\in(0,1)\\ \frac{1}{\varepsilon\delta(1/\varepsilon)^2}\left(X_{t/\varepsilon}^*-t\delta(1/\varepsilon)\right) & \kappa=1\\ \mathrm{v}_P^{-1-1/\kappa}\varepsilon^{1/\kappa}\left(X_{t/\varepsilon}^*-t\mathrm{v}_P/\varepsilon\right) & \kappa\in(1,2),\end{cases}$$

with the same function δ in the case $\kappa=1$ as in (1.7). The path χ_ε^* is easier to compare to transforms of the hitting times path $\mathbb{T}_\varepsilon$ than the path χ_ε is. The following lemma shows that the quenched distributions of χ_ε and χ_ε^* are asymptotically equivalent, since the distance between χ_ε and χ_ε^* is typically very small.

Lemma 6.1. *For any $s<\infty$ and $\eta>0$,*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\le s}|\chi_\varepsilon(t)-\chi_\varepsilon^*(t)|\ge\eta\right)=0.$$

Proof: The definitions of χ_ε and χ_ε^* imply that for all $\varepsilon>0$ small enough,

$$\sup_{t\le s}|\chi_\varepsilon(t)-\chi_\varepsilon^*(t)| = \max_{k\le s/\varepsilon}(X_k^*-X_k)\begin{cases}\varepsilon^\kappa & \kappa\in(0,1)\\ \frac{1}{\varepsilon\delta(1/\varepsilon)^2} & \kappa=1\\ \varepsilon^{1/\kappa} & \kappa\in(1,2)\end{cases}\le \varepsilon^{\kappa/4}\max_{k\le s/\varepsilon}(X_k^*-X_k).$$

If, for some $0\le k\le s/\varepsilon$, $X_k^*-X_k\ge\eta\varepsilon^{-\kappa/4}$, then, for some location $0\le j\le s/\varepsilon$ the random walk returns to $X_j-\lceil\eta\varepsilon^{-\kappa/4}\rceil$ after visiting location j. Thus, by the stationarity of the environment under the measure $\mathbb{P}$,

$$\mathbb{P}\left(\sup_{t\le s}|\chi_\varepsilon(t)-\chi_\varepsilon^*(t)|\ge\eta\right)\le\mathbb{P}\left(\max_{k\le s/\varepsilon}(X_k^*-X_k)\ge\eta\varepsilon^{-\kappa/4}\right)$$
$$\le(1+s/\varepsilon)\mathbb{P}(T_{-\lceil\eta\varepsilon^{-\kappa/4}\rceil}<\infty).$$

Since $\mathbb{P}(T_{-x}<\infty)$ decays exponentially fast as $x\to\infty$ (see Gantert and Shi (2002, Lemma 3.3)), the term on the right vanishes as $\varepsilon\to 0$. □

We now prove Theorem 1.6. According to Lemma 6.1, we may and will replace χ_ε by χ^*_ε when proving the coupling part. We consider the cases $\kappa \in (0,1)$, $\kappa = 1$, and $\kappa \in (1,2)$ separately.

6.1. *Case I:* $\kappa \in (0,1)$. We wish to compare χ^*_ε with $\mathfrak{I}\mathbb{T}_{\varepsilon^\kappa}$ where $\mathfrak{I}$ is the inversion operator defined in (1.8). To this end, note that

$$X^*_{t/\varepsilon} = \max\{k \in \mathbb{Z} : T_k \le t/\varepsilon\} = \sup\{x \ge 0 : T_{\lfloor x/\varepsilon^\kappa \rfloor} \le t/\varepsilon\}\varepsilon^{-\kappa} - 1.$$

Therefore, for every $t \ge 0$,

$$\chi^*_\varepsilon(t) = \sup\{x \ge 0 : \mathbb{T}_{\varepsilon^\kappa}(x) \le t\} - \varepsilon^\kappa = \mathfrak{I}\mathbb{T}_{\varepsilon^\kappa}(t) - \varepsilon^\kappa,$$

which implies that $\sup_{t<\infty} |\chi^*_\varepsilon(t) - \mathfrak{I}\mathbb{T}_{\varepsilon^\kappa}(t)| = \varepsilon^\kappa$. Hence, we obtain the stated coupling in Theorem 1.6. By Lemma 3.3, it remains only to show that, under the measure P,

$$m_{\varepsilon^\kappa} \circ \mathfrak{I}^{-1} \Longrightarrow \mathcal{H}(N_{\lambda,\kappa}) \circ \mathfrak{I}^{-1}. \tag{6.1}$$

To this end, first note that $\mathfrak{I}$ is continuous on the subset $D^+_{\uparrow\uparrow} \subset D^+_\infty$ of *strictly* increasing functions, when the M_1 topology is used both on the domain and the range (see Whitt (2002, Corollary 13.6.4) for even topologically stronger statement). Therefore, the mapping theorem implies that the function $\mu \mapsto \mu \circ \mathfrak{I}^{-1}$ on $\mathcal{M}_1(D_\infty)$ is continuous on the subset of measures $\{\mu \in \mathcal{M}_1(D_\infty) : \mu(D^+_{\uparrow\uparrow}) = 1\}$. In the notation introduced in (1.10), $\mathcal{H}(N_{\lambda,\kappa}) = \mathbf{P}_\tau(Z_{\lambda,\kappa} \in \cdot)$. Since $Z_{\lambda,\kappa}$ is a κ-stable subordinator under $\mathbf{P} \times \mathbf{P}_\tau$, then $\mathcal{H}(N_{\lambda,\kappa})(D^+_{\uparrow\uparrow}) = \mathbf{P}_\tau(Z_{\lambda,\kappa} \in D^+_{\uparrow\uparrow}) = 1$ for almost every realization of $N_{\lambda,\kappa}$, and so (6.1) follows from Theorem 1.4 and the continuous mapping theorem.

6.2. *Case II:* $\kappa \in (1,2)$. We start by replacing the piecewise constant path of the hitting times in (1.3) by a piecewise linear and continuous version via linear interpolation. Specifically, for $x \in \mathbb{Z}$ and $\theta \in [0,1)$ we let

$$\tilde{T}_{x+\theta} = (1-\theta)T_x + \theta T_{x+1}.$$

Correspondingly, we will define $\tilde{\mathbb{T}}_\varepsilon(t) = \varepsilon^{1/\kappa}(\tilde{T}_{t/\varepsilon} - t/(\varepsilon \mathrm{v}_P))$, $t \ge 0$. The following lemma shows that the M_1-distance between T_ε and $\tilde{\mathbb{T}}_\varepsilon$ is typically small.

Lemma 6.2. *For any $\eta > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}(d^{M_1}_\infty(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \ge \eta) = 0$.*

Proof: As in Lemma 3.4, it is enough to prove that $\mathbb{P}(d^{M_1}_t(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \ge \eta) \to 0$ for every $0 < t < \infty$ and $\eta > 0$. We use a matching of the kind similar to that constructed in the proof of Lemma 4.5. We will describe this matching in the case $\kappa \in (1,2)$, but a similar argument works when $\kappa = 1$ or $\kappa \in (0,1)$. For every $k = 0,1,2,\dots$ such that $\varepsilon k \le t$ we arrange both parametric representations to contain the point $\big(\varepsilon k, \varepsilon^{1/\kappa}(T_k - k/\mathrm{v}_P)\big)$. If $\varepsilon(k+1) \le t$, then between the points $\big(\varepsilon k, \varepsilon^{1/\kappa}(T_k - k/\mathrm{v}_P)\big)$ and $\big(\varepsilon(k+1), \varepsilon^{1/\kappa}(T_{k+1} - (k+1)/\mathrm{v}_P)\big)$ we keep the parametrization of $\tilde{\mathbb{T}}_\varepsilon$ at the former point until the parametrization of $\mathbb{T}_\varepsilon$ reaches the point $\big(\varepsilon(k+1), \varepsilon^{1/\kappa}(T_k - (k+1)/\mathrm{v}_P)\big)$, at which time we complete the two parametrizations in the interval by keeping the slope between the matched points equal to $-\varepsilon^{1/\kappa-1}/\mathrm{v}_P$. Clearly, within this interval the horizontal distance between the two parametrizations is at most ε and the vertical distance is at most $\varepsilon^{1/\kappa}/\mathrm{v}_P$.

If $\varepsilon k < t < \varepsilon(k+1)$, then we use the obvious vertical matching of the parameterizations, with equal horizontal components, and vertical components at most $\varepsilon^{1/\kappa}(T_{\lfloor t/\varepsilon\rfloor+1} - T_{\lfloor t/\varepsilon\rfloor})$ apart. Therefore, for ε small enough

$$d_t^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \le \max\left\{\varepsilon^{1/\kappa}/\mathrm{v}_P,\, \varepsilon^{1/\kappa}(T_{\lfloor t/\varepsilon\rfloor+1} - T_{\lfloor t/\varepsilon\rfloor})\right\}.$$

Since $T_{\lfloor t/\varepsilon\rfloor+1} - T_{\lfloor t/\varepsilon\rfloor}$ has, under the measure $\mathbb{P}$, the same distribution as T_1 we conclude that

$$\limsup_{\varepsilon\to 0} \mathbb{P}\left(d_t^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \ge \eta\right) \le \lim_{\varepsilon\to 0} \mathbb{P}\left(T_{\lfloor t/\varepsilon\rfloor+1} - T_{\lfloor t/\varepsilon\rfloor} \ge \varepsilon^{-1/\kappa}\eta\right) = 0, \qquad (6.2)$$

as required. □

Note that $x \mapsto \tilde{T}_x$ is a strictly increasing and continuous function on $[0,\infty)$. Let $\phi(t)$ be its inverse. Then $\tilde{T}_{\phi(t)} = t$ for all $t \ge 0$. If $T_n \le t < T_{n+1}$ then $X_t^* = n \le \phi(t) < n+1$, so that $\sup_{t\ge 0}|X_t^* - \phi(t)| \le 1$. One consequence of this comparison is that

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{n\to\infty} \frac{X_n^*}{n} = \mathrm{v}_P, \qquad \mathbb{P}\text{-a.s.} \qquad (6.3)$$

Next define $\phi_\varepsilon(t) = \varepsilon\phi(t/\varepsilon)$ for $\varepsilon > 0$ and $\phi_0(t) = \mathrm{v}_P t$. Then, (6.3) implies that ϕ_ε converges pointwise to ϕ_0 as $\varepsilon \to 0$. Moreover, since ϕ_ε and ϕ_0 are monotone increasing and ϕ_0 is continuous, we conclude that ϕ_ε converges uniformly on compact subsets to ϕ_0. In particular, $\lim_{\varepsilon\to 0} d_\infty^U(\phi_\varepsilon, \phi_0) = 0$, $\mathbb{P}$-a.s.

Now, recalling the definition of $\tilde{\mathbb{T}}_\varepsilon$ we obtain that

$$\begin{aligned}
\tilde{\mathbb{T}}_\varepsilon(\phi_\varepsilon(t)) &= \varepsilon^{1/\kappa}(\tilde{\mathbb{T}}_{\phi_\varepsilon(t)/\varepsilon} - \phi_\varepsilon(t)/(\varepsilon \mathrm{v}_P)) \\
&= \varepsilon^{1/\kappa}(\tilde{\mathbb{T}}_{\phi(t/\varepsilon)} - \phi(t/\varepsilon)/\mathrm{v}_P) \\
&= -\mathrm{v}_P^{-1}\varepsilon^{1/\kappa}(\phi(t/\varepsilon) - t\mathrm{v}_P/\varepsilon) \\
&= -\mathrm{v}_P^{-1}\varepsilon^{1/\kappa}(X_{t/\varepsilon}^* - t\mathrm{v}_P/\varepsilon) + \mathrm{v}_P^{-1}\varepsilon^{1/\kappa}(X_{t/\varepsilon}^* - \phi(t/\varepsilon)) \\
&= -\mathrm{v}_P^{1/\kappa}\chi_\varepsilon^*(t) + \mathrm{v}_P^{-1}\varepsilon^{1/\kappa}(X_{t/\varepsilon}^* - \phi(t/\varepsilon)).
\end{aligned}$$

Since $|X_{t/\varepsilon}^* - \phi(t/\varepsilon)| \le 1$ for all t, this implies that

$$d_\infty^U(\chi_\varepsilon^*, -\mathrm{v}_P^{-1/\kappa}\tilde{\mathbb{T}}_\varepsilon \circ \phi_\varepsilon) \le \mathrm{v}_P^{-1-1/\kappa}\varepsilon^{1/\kappa}. \qquad (6.4)$$

Next, we compare $\tilde{\mathbb{T}}_\varepsilon \circ \phi_\varepsilon$ with $\mathbb{T}_\varepsilon \circ \phi_0$. To this end, let $\eta, \eta' > 0$ be fixed. By Corollary 1.9, the laws of $(\mathbb{T}_\varepsilon)$ under $\mathbb{P}$ are tight in $(D_\infty, d_\infty^{M_1})$. Therefore, we can choose a compact subset $K \subset D_\infty$ so that $\mathbb{P}(T_\varepsilon \in K) \ge 1 - \eta'$ for all ε small enough. Further, the composition function $\psi(x,y) = x\circ y$ is continuous at any point $(x, \phi_0) \in D_\infty \times C_{\uparrow\uparrow}^+ \subset D_\infty \times D_\infty$. Therefore, it is uniformly continuous (with the $d_\infty^{M_1}$ metric on each coordinate) at the points of the compact set $K \times \{\phi_0\}$. Choose now $\delta > 0$ such that $d_\infty^{M_1}(x' \circ \phi', x \circ \phi_0) < \eta$ whenever $x \in K$, $d_\infty^{M_1}(x, x') < \delta$, and $d_\infty^{M_1}(\phi_0, \phi') < \delta$. Then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \mathbb{P}(d_\infty^{M_1}(\tilde{\mathbb{T}}_\varepsilon \circ \phi_\varepsilon,\, \mathbb{T}_\varepsilon \circ \phi_0) \ge \eta) \\
&\quad \le \limsup_{\varepsilon\to 0} \mathbb{P}(\mathbb{T}_\varepsilon \notin K) + \mathbb{P}(d_\infty^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \ge \delta) + \mathbb{P}(d_\infty^{M_1}(\phi_\varepsilon, \phi_0) \ge \delta) \\
&\quad \le \eta',
\end{aligned}$$

where the last inequality follows from our choice of the compact set K, Lemma 6.2, and the almost sure convergence of ϕ_ε to ϕ_0. Since $\eta' > 0$ was arbitrary, we see that $\mathbb{P}(d_\infty^{M_1}(\tilde{\mathbb{T}}_\varepsilon \circ \phi_\varepsilon, \mathbb{T}_\varepsilon \circ \phi_0) \geq \eta) \to 0$ for any $\eta > 0$. Combining this with (6.4) we conclude that

$$\lim_{\varepsilon \to 0} \mathbb{P}(d_\infty^{M_1}(\chi_\varepsilon^*, -\mathrm{v}_P^{-1/\kappa}\mathbb{T}_\varepsilon \circ \phi_0) \geq \eta) = 0, \quad \forall \eta > 0. \tag{6.5}$$

Finally, note that the definition of $\mathbb{T}_\varepsilon$ and ϕ_0 imply that

$$\mathrm{v}_P^{-1/\kappa}\mathbb{T}_\varepsilon(\phi_0(t)) = \mathrm{v}_P^{-1/\kappa}\varepsilon^{1/\kappa}(T_{\mathrm{v}_P t/\varepsilon} - t/\varepsilon) = \mathbb{T}_{\varepsilon/\mathrm{v}_P}(t),$$

so that (6.5) proves the coupling part of Theorem 1.6 in the case $\kappa \in (1,2)$. Since $m_{\varepsilon/\mathrm{v}_P} \circ \mathfrak{R}^{-1}$ is the quenched distribution of $-\mathbb{T}_{\varepsilon/\mathrm{v}_P}$, and $\mathfrak{R}$ is a continuous operator, the continuous mapping theorem implies that $m_\varepsilon \circ \mathfrak{R}^{-1} \implies \mu_{\lambda,\kappa} \circ \mathfrak{R}^{-1}$. The coupling now implies that we also have $p_{\varepsilon,\omega} \implies \mu_{\lambda,\kappa} \circ \mathfrak{R}^{-1}$.

6.3. *Case III:* $\kappa = 1$. The proof here is similar to the proof in the case $\kappa \in (1,2)$, so we will omit some of the details. As above, let $\tilde{T}_x$ and $\phi(t)$ be as above, so that $\tilde{T}_{\phi(t)} = t$. We claim that

$$\lim_{t\to\infty} \frac{\phi(t)}{t/\log t} = \frac{1}{A} \qquad \text{in } \mathbb{P}\text{-probability}, \tag{6.6}$$

where A is the positive constant from Theorem 1.1. To see this, first note that by Theorem 1.1, $\lim_{n\to\infty} \frac{T_n}{n \log n} = A$ in $\mathbb{P}$-probability, hence also $\lim_{x\to\infty} \frac{\tilde{T}_x}{x \log x} = A$ in $\mathbb{P}$-probability. Using $x = \phi(t)$ gives us

$$\lim_{t\to\infty} \frac{t}{\phi(t)\log(\phi(t))} = A \qquad \text{in } \mathbb{P}\text{-probability},$$

which proves (6.6).

The function $\delta(x) = \sup\{u > 0 : uD(u) \leq x\}$, $x > 0$, satisfies $\delta(x) \sim x/(A\log x)$ as $x \to \infty$ and $\delta(x)D(\delta(x)) = x + o(\delta(x))$ as $x \to \infty$. We define $\phi_\varepsilon \in D_\infty$ by $\phi_\varepsilon(t) = \phi(t/\varepsilon)/\delta(1/\varepsilon)$. Then the asymptotics of ϕ from (6.6) and the asymptotics of δ imply that for any $t \geq 0$,

$$\lim_{\varepsilon\to 0} \phi_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{\phi(t/\varepsilon)}{\delta(1/\varepsilon)} = t \qquad \text{in } \mathbb{P}\text{-probability}.$$

Once again, since ϕ_ε is non-decreasing, and the identity function is continuous, ϕ_ε converges uniformly on compact subsets (and thus also in the $d_\infty^{J_1}$ metric), in $\mathbb{P}$-probability, to the identity function.

Let $\tilde{\mathbb{T}}_\varepsilon(t) = \varepsilon(\tilde{T}_{t/\varepsilon} - t/\varepsilon D(1/\varepsilon))$, $t \geq 0$. Then

$$\begin{aligned}
\tilde{\mathbb{T}}_{\delta(1/\varepsilon)^{-1}}(\phi_\varepsilon(t)) &= \delta(1/\varepsilon)^{-1}\left(\tilde{T}_{\phi_\varepsilon(t)\delta(1/\varepsilon)} - \phi_\varepsilon(t)\delta(1/\varepsilon)D(\delta(1/\varepsilon))\right) \\
&= \delta(1/\varepsilon)^{-1}\left(\tilde{T}_{\phi(t/\varepsilon)} - \frac{\phi(t/\varepsilon)}{\delta(1/\varepsilon)}\left(\frac{1}{\varepsilon} + o(\delta(1/\varepsilon))\right)\right) \\
&= \delta(1/\varepsilon)^{-1}\left(t/\varepsilon - \frac{X^*_{t/\varepsilon} + \mathcal{O}(1)}{\delta(1/\varepsilon)}\left(\frac{1}{\varepsilon} + o(\delta(1/\varepsilon))\right)\right) \\
&= \frac{1}{\varepsilon\delta(1/\varepsilon)^2}\left(t\delta(1/\varepsilon) - X^*_{t/\varepsilon}\right) + o(1)\frac{X^*_{t/\varepsilon}}{\delta(1/\varepsilon)} + \mathcal{O}\left(\frac{1}{\varepsilon\delta(1/\varepsilon)^2}\right) \\
&= -\chi^*_\varepsilon(t) + o(1)\frac{X^*_{t/\varepsilon}}{\delta(1/\varepsilon)} + \mathcal{O}\left(\frac{1}{\varepsilon\delta(1/\varepsilon)^2}\right),
\end{aligned}$$

where in the third equality we used that $|\phi(t) - X^*_t| \leq 1$ for all t. Since $\varepsilon^{-1}\delta(\frac{1}{\varepsilon})^{-2} \sim A^2\varepsilon\log^2(1/\varepsilon) \to 0$ as $\varepsilon \to 0$, while $X^*_{t/\varepsilon}/\delta(1/\varepsilon)$ converges in probability by Theorem 1.1, this implies that

$$\lim_{\varepsilon\to 0} d^U_\infty(\chi^*_\varepsilon, -\tilde{\mathbb{T}}_{\delta(1/\varepsilon)^{-1}} \circ \phi_\varepsilon) = 0 \qquad \text{in } \mathbb{P}\text{-probability.} \tag{6.7}$$

As in case $\kappa \in (1,2)$ we can use the fact that ϕ_ε converges to the identity function to show that for any $\eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(d^{M_1}_\infty(\tilde{\mathbb{T}}_{\delta(1/\varepsilon)^{-1}} \circ \phi_\varepsilon, \tilde{\mathbb{T}}_{\delta(1/\varepsilon)^{-1}}) \geq \eta\right) = 0. \tag{6.8}$$

Combining (6.7), (6.8) and Lemma 6.1 establishes the coupling part of Theorem 1.6, and the rest is the same as in the case $\kappa \in (1,2)$.

7. $\mathcal{M}_1(\mathbb{R})$-valued Stable Lévy process limits

In this section we discuss Corollary 1.12. We begin with a short proof of the convergence of the finite dimensional distributions of $\Phi(m_\varepsilon)$. Let $m \geq 1$ and $0 \leq t_1 < t_2 < \ldots < t_m$ be given, and define $\Phi_{t_1,\ldots,t_m} : \mathcal{M}_1(D_\infty) \to \mathcal{M}_1(\mathbb{R})^m$ by

$$\Phi_{t_1,t_2,\ldots,t_m}(\mu) = (\Phi_{t_1}(\mu), \Phi_{t_2}(\mu), \ldots, \Phi_{t_m}(\mu)).$$

It is easy to see that $\Phi_{t_1,t_2,\ldots,t_m}$ is continuous at every $\mu \in \mathcal{M}_1(D_\infty)$ concentrated on paths that are continuous at t_i, $i = 1, 2, \ldots, m$; see p. 383 in Whitt (2002). Since $m_\varepsilon \Longrightarrow \mu_{\lambda,\kappa}$ and $\mu_{\lambda,\kappa}$ is, with probability, one concentrated on paths that are continuous at t_i, $i = 1, 2, \ldots, m$, then the continuous mapping theorem implies that $\Phi_{t_1,t_2,\ldots,t_m}(m_\varepsilon) \Longrightarrow \Phi_{t_1,t_2,\ldots,t_m}(\mu_{\lambda,\kappa})$. This proves the convergence of finite dimensional distributions claimed in Corollary 1.12.

We now turn to the stated properties of the random measure-valued path $\Phi(\mu_{\lambda,\kappa})$, namely that $\Phi(\mu_{\lambda,\kappa})$ is a stable Lévy process on $\mathcal{M}_1(\mathbb{R})$. We start by recalling the notions of stable random variables and Lévy processes on $\mathcal{M}_1(\mathbb{R})$; the reader is referred to Shiga and Tanaka (2006) for more details.

Definition 7.1. A $\mathcal{M}_1(\mathbb{R})$-valued random variable μ is a **stable random variable on** $\mathcal{M}_1(\mathbb{R})$ if for any $n \geq 2$ there exist constants $b_n \in \mathbb{R}$ and $c_n > 0$ such that

$$(\mu_1 * \mu_2 * \cdots * \mu_n)(b_n + c_n^{-1}\cdot) \overset{\text{Law}}{=} \mu(\cdot).$$

Here $\mu_1, \mu_2, \ldots \mu_n$ are independent copies of μ. Moreover, if $b_n = 0$ for every $n \geq 2$, then μ is a **strictly stable random variable on** $\mathcal{M}_1(\mathbb{R})$.

Definition 7.2. A $\mathcal{M}_1(\mathbb{R})$-valued stochastic process $\{\Xi(t)\}_{t\geq 0}$ is a **Lévy process on** $\mathcal{M}_1(\mathbb{R})$ if there exists a two parameter family of $\mathcal{M}_1(\mathbb{R})$-valued random variables $\{\Xi_{s,t}\}_{0\leq s<t}$ such that $\Xi(t) = \Xi_{0,t}$ and

(1) $\Xi(0) = \delta_0$ with probability one.
(2) For any $n \geq 2$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, $\{\Xi_{t_{i-1},t_i}\}_{i=1}^n$ are independent and

$$\Xi(t) = \Xi_{t_0,t_1} * \Xi_{t_1,t_2} * \cdots * \Xi_{t_{n-1},t_n}, \quad \text{almost surely.}$$

(3) For any $0 < s < t$, $\Xi(t-s) \stackrel{\text{Law}}{=} \Xi_{s,t}$.
(4) For any fixed $t_0 \geq 0$, the process $\{\Xi(t)\}_{t\geq 0}$ is continuous at t_0 in probability.
(5) There is an event of probability 1 on which every path $\{t \mapsto \Xi(t)\}$ is in $D_\infty(\mathcal{M}_1(\mathbb{R}))$.

Remark 7.3. Part (2) of Definition 7.2 is a version of the independent increments property for stochastic processes with values in $\mathcal{M}_1(\mathbb{R})$ with convolution of measures playing the role of addition. The version of the independent increments property used in the above definition is necessary due to absence of an inverse operation to convolution.

Definition 7.4. A Lévy process $\{\Xi(t)\}_{t\geq 0}$ on $\mathcal{M}_1(\mathbb{R})$ is a **(strictly) stable Lévy process on** $\mathcal{M}_1(\mathbb{R})$ if for every fixed $t \geq 0$, $\Xi(t)$ is a (strictly) stable random variable on $\mathcal{M}_1(\mathbb{R})$.

We are now ready to show that $\Phi(\mu_{\lambda,\kappa})$ is a stable Lévy process on $\mathcal{M}_1(\mathbb{R})$ and is strictly stable when $\kappa \neq 1$. It is easy to check that for any $t \geq 0$, $\Phi_t(\mu_{\lambda,\kappa})$ is a stable random variable on $\mathcal{M}_1(\mathbb{R})$ and is strictly stable if $\kappa \neq 1$ (see the Remark 1.5 and the paragraph following Remark 1.6 in Peterson and Samorodnitsky, 2010), so we will concentrate on showing that $\Phi(\mu_{\lambda,\kappa})$ is a Lévy process on $\mathcal{M}_1(\mathbb{R})$.

We already know that the paths of $\Phi(\mu_{\lambda,\kappa})$ are in $D_\infty(\mathcal{M}_1(\mathbb{R}))$; see the discussion before Theorem 1.12. It is also obvious that $\Phi_0(\mu_{\lambda,\kappa}) = \delta_0$ with probability one since $N_{\lambda,\kappa}((0,\infty] \times \{0\}) = 0$ with probability one. Next, recall the stochastic process $Z_{\lambda,\kappa}$ defined in (1.10). Then $\Phi_t(\mu_{\lambda,\kappa})$ is the distribution of $Z_{\lambda,\kappa}(t)$ under the measure $\mathbf{P}_\tau$, and we define for any $0 \leq s < t$

$$\Phi(\mu_{\lambda,\kappa})_{s,t} = \mathbf{P}_\tau\left(Z_{\lambda,\kappa}(t) - Z_{\lambda,\kappa}(s) \in \cdot\right).$$

Then, the independent increments condition (2) in Definition 7.2 follows from the fact that $\{N_{\lambda,\kappa}(\cdot \cap \left((0,\infty] \times (t_{i-1},t_i]\right)\}_{i=1}^n$ are independent for any $0 = t_0 < t_1 < \cdots < t_n$, and the stationarity condition (3) in Definition 7.2 follows from the shift invariance of the Lebesgue measure governing the time component of the Poisson random measure $N_{\lambda,\kappa}$. Finally, stochastic continuity of $\Phi(\mu_{\lambda,\kappa})$ at fixed points follows from the fact that for each fixed t_0, $N_{\lambda,\kappa}((0,\infty] \times \{t_0\}) = 0$ with probability 1.

7.1. *Topologies on* $D_\infty(\mathcal{M}_1(\mathbb{R}))$. We now give a brief discussion of the difficulties of extending Corollary 1.12 to a full weak convergence $\Phi(m_\varepsilon) \Longrightarrow \Phi(\mu_{\lambda,\kappa})$ of $\mathcal{M}_1(\mathbb{R})$-valued path processes. It is first necessary to decide on a topology for $D_\infty(\mathcal{M}_1(\mathbb{R}))$, the space of càdlàg paths taking values in the space of probability measures on $\mathbb{R}$. Recall that the Prohorov metric ρ on $\mathcal{M}_1(\mathbb{R})$ induces the topology of convergence in distribution and that $(\mathcal{M}_1(\mathbb{R}), \rho)$ is a Polish space. Then, both the uniform and the J_1-topologies have natural extensions to $D_\infty(\mathcal{M}_1(\mathbb{R}))$.

In the proof of Theorem 1.4, it was necessary to equip D_∞ with the M_1-topology to accommodate the fact that the macroscopic jumps of the process of ladder location hitting times were an accumulation of smaller jumps $T_i - T_{i-1}$ for i between consecutive ladder locations. The M_1-topology naturally accomodates such accumulations of jumps while the J_1-topology does not. A similar phenomenon occurs when trying to establish weak convergence $\Phi(m_\varepsilon) \Longrightarrow \Phi(\mu_{\lambda,\kappa})$ in the space of probability measure-valued functions, and thus it is natural to try to equip $D_\infty(\mathcal{M}_1(\mathbb{R}))$ with a Skorohod M_1-topology. This is less standard than defining the Skorohod J_1-topology[2], but, since convex combinations $(1-\theta)\mu+\theta\pi$ of two probability measures form a "line segment" between μ and π in $\mathcal{M}_1(\mathbb{R})$, one can define the M_1-topology and metric on $D_t(\mathcal{M}_1(\mathbb{R}))$ and $D_\infty(\mathcal{M}_1(\mathbb{R}))$ in the natural way. Moreover, the resulting M_1-topology on $D_\infty(\mathcal{M}_1(\mathbb{R}))$ defined in this way is the topology of a complete separable metric space.

Unfortunately, to this point we have been unable to prove weak convergence $\Phi(m_\varepsilon) \Longrightarrow \Phi(\mu_{\lambda,\kappa})$ in the M_1-topology (as defined above) on $D_\infty(\mathcal{M}_1(\mathbb{R}))$. In fact, some preliminary computations suggest that $\{\Phi(m_\varepsilon)\}_{\varepsilon>0}$ is not a tight family of $D_\infty(\mathcal{M}_1(\mathbb{R}))$-valued random variables in this topology, and thus a weaker topology on $D_\infty(\mathcal{M}_1(\mathbb{R}))$ may be needed. We hope to address this in a future paper.

We close this section with an example that demonstrates some of the difficulties establishing weak convergence $\Phi(m_\varepsilon) \Longrightarrow \Phi(\mu_{\lambda,\kappa})$. A natural approach to proving $\Phi(m_\varepsilon) \Longrightarrow \Phi(\mu_{\lambda,\kappa})$ would be to apply Theorem 1.4 and the continuous mapping theorem. Unfortunately, the mapping $\Phi : \mathcal{M}_1(D_\infty) \to D_\infty(\mathcal{M}_1(\mathbb{R}))$ is not continuous. The following example demonstrates this lack of continuity even when in $\mathcal{M}_1(D_\infty)$ we endow the space D_∞ with the strongest of the Skorohod topologies, the J_1-topology, and endow $D_\infty(\mathcal{M}_1(\mathbb{R}))$ with the weakest of the Skorohod topologies, the M_2-topology.

We restrict everything to the interval $[0,1]$ and consider real-valued stochastic processes $X = (X(t),\, 0 \le t \le 1)$ and $X_n = (X_n(t),\, 0 \le t \le 1)$, $n = 1, 2, \ldots$, on the probability space $\big([0,1], \mathcal{B}, \mathrm{Leb}\big)$, defined by

$$X(t;\omega) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 1, & \frac{1}{2} \le t \le 1,\ 0 \le \omega \le \frac{1}{2}, \\ 2, & \frac{1}{2} \le t \le 1,\ \frac{1}{2} < \omega \le 1, \end{cases}$$

and

$$X_n(t;\omega) = \begin{cases} 0, & 0 \le t < \frac{1}{2} - \frac{1}{2^{n+1}}, \\ 1, & \frac{1}{2} - \frac{1}{2^{n+1}} \le t \le 1,\ 0 \le \omega \le \frac{1}{2}, \\ 0, & \frac{1}{2} - \frac{1}{2^{n+1}} \le t < \frac{1}{2} + \frac{1}{2^{n+1}},\ \frac{1}{2} < \omega \le 1, \\ 2, & \frac{1}{2} + \frac{1}{2^{n+1}} \le t \le 1,\ \frac{1}{2} < \omega \le 1, \end{cases}$$

$n = 1, 2, \ldots$. Clearly, each process X and X_n has its sample paths in $D_1 = D[0,1]$. We denote by μ (correspondingly, μ_n) the probability measures these processes generate on the cylindrical sets in $D[0,1]$.

Obviously, for any $\omega \in [0,1]$, $d^{J_1}(X, X_n) \le 2^{-(n+1)}$, so, with probability 1, $X_n \to X$ in $D[0,1]$ equipped with the J_1-topology, so that $\rho^{J_1}(\mu_n, \mu) \to 0$.

Next, in the notation of (1.9), we have

$$\Phi_t(\mu) = \begin{cases} \delta_0, & 0 \le t < \frac{1}{2}, \\ \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, & \frac{1}{2} \le t \le 1 \end{cases}$$

[2]For instance, Whitt defines the Skorohod M_1-topology on $D_t(\Psi)$ if Ψ is a separable Banach space Whitt (2002, p. 382), but $\mathcal{M}_1(\mathbb{R})$ is not a Banach space.

and

$$\Phi_t(\mu_n) = \begin{cases} \delta_0, & 0 \le t < \frac{1}{2} - \frac{1}{2^{n+1}}, \\ \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1, & \frac{1}{2} - \frac{1}{2^{n+1}} \le t < \frac{1}{2} + \frac{1}{2^{n+1}}, \\ \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, & \frac{1}{2} + \frac{1}{2^{n+1}} \le t \le 1, \end{cases}$$

$n = 1, 2, \ldots$, where for $x \in \mathbb{R}$, δ_x is the point mass at x. Note that for any n, there is a point on the completed graph of the element $\Phi(\mu_n)$ of $D_1(\mathcal{M}_1(\mathbb{R}))$ with the second component equal to $(1/2)\delta_0 + (1/2)\delta_1$, and the distance from that point to the completed graph of the $\Phi(\mu)$ has a positive lower bound that does not depend on n. Therefore, $\Phi(\mu_n)$ does not converge to $\Phi(\mu)$ in $D_1(\mathcal{M}_1(\mathbb{R}))$ even if the latter space is endowed with the M_2-topology (see Section 11.5 in Whitt, 2002 for the definition of the M_2-topology).

Appendix A. Estimation of the term in (5.17)

In order to finish the proof of Proposition 4.2 we need to estimate the term in (5.17). In this appendix we achieve that by proving the following lemma.

Lemma A.1. *If $\kappa \in [1, 2)$, then for all $0 < s < \infty$ and $\eta > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} Q\left(\sup_{t \le s} \left| n^{-1/\kappa} \sum_{i=1}^{\lfloor tn \rfloor} \left\{ \beta_i \mathbf{1}_{\{\beta_i \le \delta n^{1/\kappa}\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \delta n^{1/\kappa}\}}] \right\} \right| \ge \eta \right) = 0.$$

Remark A.2. Lemma A.1 is an improvement of Peterson and Samorodnitsky (2010, Lemma 5.5), which stated that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} Q\left(\left| n^{-1/\kappa} \sum_{i=1}^{n} \left\{ \beta_i \mathbf{1}_{\{\beta_i \le \delta n^{1/\kappa}\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \delta n^{1/\kappa}\}}] \right\} \right| \ge \eta \right), \quad \forall \eta > 0.$$

Before giving the proof of Lemma A.1, we introduce new notation. Recall that $\rho_x = (1 - \omega_x)/\omega_x$, and for $i \le j$ let

$$\Pi_{i,j} = \prod_{k=i}^{j} \rho_k, \quad R_{i,j} = \sum_{k=i}^{j} \Pi_{i,k}, \quad W_{i,j} = \sum_{k=i}^{j} \Pi_{k,j}, \quad W_j = \sum_{k=-\infty}^{j} \Pi_{k,j}. \tag{A.1}$$

This notation is often useful for writing certain quenched expectations or probabilities in compact form. For instance, it is easy to show that $E_\omega^i[T_{i+1}] = 1 + 2W_i$ (see Zeitouni, 2004 for a reference). In particular,

$$\beta_i = E_\omega[T_{\nu_i} - T_{\nu_{i-1}}] = \sum_{j=\nu_{i-1}}^{\nu_i - 1} E_\omega^j[T_{j+1}] = \nu_i - \nu_{i-1} + 2 \sum_{j=\nu_{i-1}}^{\nu_i - 1} W_j. \tag{A.2}$$

It will be important for us to be able to control the tails, under the measure Q, of the random variables of the type $W_{\nu_m - 1}$. Since under Q the environment is stationary under shifts by the ladder locations, these random variables all have the same distribution under Q as W_{-1}. Further, it was shown in Peterson and Zeitouni (2009, Lemma 2.2) that W_{-1} has, under Q, exponential tails. That is, there exist constants $C_1, C_2 > 0$ such that for any $x > 0$,

$$Q(W_{-1} > x) \le C_1 e^{-C_2 x}. \tag{A.3}$$

We now proceed to prove Lemma A.1.

Proof of Lemma A.1: First, note that $\beta_i \mathbf{1}_{\{\beta_i \le \delta n^{1/\kappa}\}} = \beta_i \wedge \delta n^{1/\kappa} - \delta n^{1/\kappa} \mathbf{1}_{\{\beta_i > \delta n^{1/\kappa}\}}$. Thus,

$$
\begin{aligned}
&Q\left(\sup_{t \le s} \left| n^{-1/\kappa} \sum_{i=1}^{\lfloor tn \rfloor} \left\{ \beta_i \mathbf{1}_{\{\beta_i \le \delta n^{1/\kappa}\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \delta n^{1/\kappa}\}}] \right\} \right| \ge \eta \right) \\
&\le Q\left(\sup_{t \le s} \left| n^{-1/\kappa} \sum_{i=1}^{\lfloor tn \rfloor} \left\{ \beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_i \wedge \delta n^{1/\kappa}] \right\} \right| \ge \eta/2 \right) \qquad \text{(A.4)} \\
&\quad + Q\left(\delta \sup_{t \le s} \left| \sum_{i=1}^{\lfloor tn \rfloor} \mathbf{1}_{\{\beta_i > \delta n^{1/\kappa}\}} - \lfloor tn \rfloor Q(\beta_i > \delta n^{1/\kappa}) \right| \ge \eta/2 \right). \qquad \text{(A.5)}
\end{aligned}
$$

Note that (2.3) implies that $\lfloor tn \rfloor Q(\beta_i > \delta n^{1/\kappa}) \to tC_0 \delta^{-\kappa}$ and, moreover, that the convergence is uniform in $t \in [0, s]$. Therefore, to bound the term in (A.5) it is enough to show that for all $0 < s < \infty$ and $\eta > 0$,

$$
\lim_{\delta \to 0} \limsup_{n \to \infty} Q\left(\delta \sup_{t \le s} \left| \sum_{i=1}^{\lfloor tn \rfloor} \mathbf{1}_{\{\beta_i > \delta n^{1/\kappa}\}} - tC_0 \delta^{-\kappa} \right| \ge \eta \right) = 0. \qquad \text{(A.6)}
$$

Now, for any $\delta > 0$ let $G_\delta : \mathcal{M}_p((0, \infty] \times [0, \infty)) \to D_\infty^+$ (we equip the latter space with the J_1 topology) be defined by

$$
G_\delta(\zeta)(t) = \zeta((\delta, \infty] \times [0, t]), \ t \ge 0.
$$

Then $\sum_{i=1}^{\lfloor tn \rfloor} \mathbf{1}_{\{\beta_i > \delta n^{1/\kappa}\}} = G_\delta(N_{1/n})(t)$. It is easy to see that G_δ is continuous on the set of point processes with no atoms on the line $\{\delta\} \times [0, \infty)$. Since $N_{\lambda,\kappa}$ belongs to this set with probability 1, and $N_{1/n} \overset{Q}{\Longrightarrow} N_{\lambda,\kappa}$, the continuous mapping theorem implies that $G_\delta(N_{1/n}) \overset{Q}{\Longrightarrow} G_\delta(N_{\lambda,\kappa})$. Furthermore, the supremum over a compact interval is a continuous mapping from D_∞^+ equipped with the J_1 topology to the real line. Therefore,

$$
\begin{aligned}
&\limsup_{n \to \infty} Q\left(\delta \sup_{t \le s} \left| \sum_{i=1}^{tn} \mathbf{1}_{\{\beta_i > \delta n^{1/\kappa}\}} - tC_0 \delta^{-\kappa} \right| \ge \eta \right) \\
&\le Q\left(\delta \sup_{t \le s} \left| G_\delta(N_{\lambda,\kappa})(t) - tC_0 \delta^{-\kappa} \right| \ge \eta \right).
\end{aligned}
$$

Note that $G_\delta(N_{\lambda,\kappa})$ is a homogeneous one-dimensional Poisson process with rate $\lambda/\kappa \delta^{-\kappa} = C_0 \delta^{-\kappa}$. Therefore, using once again the L^p-maximum inequality for martingales, we have

$$
Q\left(\delta \sup_{t \le s} \left| G_\delta(N_{\lambda,\kappa})(t) - tC_0 \delta^{-\kappa} \right| \ge \eta \right) \le \frac{\delta^2}{\eta^2} \operatorname{Var}_Q \left(G_\delta(N_{\lambda,\kappa})(s) \right) = \frac{\lambda s \delta^{2-\kappa}}{\eta^2 \kappa}.
$$

Since $\kappa < 2$ this last term vanishes as $\delta \to 0$ for any $\eta > 0$ and $s < \infty$. This completes the proof of (A.6) and, therefore, it only remains to estimate the term in (A.4).

To this end, we assume for (notational) simplicity that $s = 1$, in which case our task reduces to showing that for any $\eta > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} Q\left(\max_{k\le n}\left|n^{-1/\kappa}\sum_{i=1}^{k}\left\{\beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_1\wedge \delta n^{1/\kappa}]\right\}\right| > \eta\right) = 0. \quad \text{(A.7)}$$

For a fixed n and $\delta \in (0,1]$ denote

$$S_j = n^{-1/\kappa}\sum_{i=1}^{j}\left\{\beta_i\wedge\delta n^{1/\kappa} - E_Q[\beta_1\wedge\delta n^{1/\kappa}]\right\}, \quad j = 1,\ldots,n. \quad \text{(A.8)}$$

For $\eta > 0$ let $A_m = \{\max_{j<m}|S_j| \le \eta < |S_m|\}$. Then,

$$\begin{aligned} Q&\left(\max_{k\le n}\left|n^{-1/\kappa}\sum_{i=1}^{k}\left\{\beta_i\wedge\delta n^{1/\kappa} - E_Q[\beta_1\wedge\delta n^{1/\kappa}]\right\}\right| > \eta\right) = \sum_{m=1}^{n} Q(A_m)\\ &\le Q\left(|S_n|\ge \eta/2\right) + \sum_{m=1}^{n-1} Q\left(A_m\cap\{|S_n|\le\eta/2\}\right)\\ &\le Q\left(|S_n|\ge \eta/2\right) + \sum_{m=1}^{n-1} Q\left(A_m\cap\{|S_n - S_m| > \eta/2\}\right). \end{aligned} \quad \text{(A.9)}$$

It was shown in the proof of Lemma 5.5 in Peterson and Samorodnitsky (2010) that for some constant C,

$$\mathrm{Var}_Q\left(\sum_{i=1}^{n}\beta_i\wedge\delta n^{1/\kappa}\right) \le C\delta^{2-\kappa}n^{2/\kappa}. \quad \text{(A.10)}$$

By Markov's inequality this shows that the term $Q\left(|S_n|\ge\eta/2\right)$ does not contribute to the limit in (A.7). Therefore, it remains only to bound the sum on the right in (A.9). If the β_i were independent, then the general term in this sum would be equal to $Q(A_m)Q(|S_n - S_m| > \eta/2)$ and the sum could be handled in the same way as the term $Q\left(|S_n|\ge\eta/2\right)$ above. While the β_i are not independent under Q, they have good mixing properties and the following lemma gives an upper bound on the general term in the sum, not far off from what it would be if the β_i were independent.

Lemma A.3. *There are constants $C, C' > 0$ such that for any $n = 1,2,\ldots$, $\delta\in(0,1]$, $m = 1,\ldots,n$ and $\eta > 0$,*

$$Q(A_m\cap\{|S_n - S_m| > \eta\}) \le Ce^{-C'\eta n^{1/\kappa}/\log n} + \frac{1}{\eta^2}C\delta^{2-\kappa}\left(Q(A_m) + 1/n\right).$$

Assuming the statement of Lemma A.3, the proof of Lemma A.1 can be completed by writing (changing the constants as necessary)

$$\begin{aligned} &\sum_{m=1}^{n-1} Q(A_m\cap\{|S_n - S_m| > \eta/2\})\\ &\qquad\le \sum_{m=1}^{n-1}\left\{Ce^{-C'\eta n^{1/\kappa}/\log n} + \frac{1}{\eta^2}C\delta^{2-\kappa}\left(Q(A_m)+1/n\right)\right\}\\ &\qquad\le Cne^{-C'\eta n^{1/\kappa}/\log n} + \frac{C\delta^{\kappa-2}}{\eta^2}. \end{aligned}$$

Both terms vanish under the limits in (A.7), so we only need to prove Lemma A.3. □

Proof of Lemma A.3: Define a (discrete time) filtration on $\Omega = [0,1]^{\mathbb{Z}}$ by $\mathcal{G}_n = \sigma(\omega_i : i \le n)$, $n = 0,1,2,\ldots$. Then for each $m = 0,1,2,\ldots$, $\nu_m - 1$ is a stopping time with respect to that filtration, and we denote $\mathcal{F}_m = \mathcal{G}_{\nu_m - 1}$, $m = 1,2,\ldots$. Since each β_j with $j \le m$ is $\mathcal{F}_m$-measurable, so is each S_j with $j \le m$. Therefore,

$$Q\left(A_m \cap \{|S_n - S_m| > \eta\}\right) = E_Q\left[\mathbf{1}_{\{A_m\}} Q\left(|S_n - S_m| > \eta \,\middle|\, \mathcal{F}_m\right)\right]. \tag{A.11}$$

Conditioned on $\mathcal{F}_m$, the difference $S_n - S_m$ no longer has zero mean, but we will show that the conditional mean is typically small. We begin by comparing the conditional and unconditional means of $\beta_j \wedge \delta n^{1/\kappa}$. To this end we make explicit the dependence of β_j on $\mathcal{F}_m$. Recall the definitions of $\Pi_{i,j}$, $W_{i,j}$ and $R_{i,j}$ in (A.1) and note that $W_i = W_{k,i} + \Pi_{k,i} W_{k-1}$ for any $k \le i$. Therefore, for any $1 \le m < j$ we can rewrite(A.2) as

$$\begin{aligned}
\beta_j &= \nu_j - \nu_{j-1} + 2 \sum_{i=\nu_{j-1}}^{\nu_j - 1} \left(W_{\nu_m,i} + W_{\nu_m - 1}\Pi_{\nu_m,i}\right) \\
&= \nu_j - \nu_{j-1} + 2 \sum_{i=\nu_{j-1}}^{\nu_j - 1} W_{\nu_m,i} + 2W_{\nu_m - 1}\Pi_{\nu_m,\nu_{j-1}-1} R_{\nu_{j-1},\nu_j - 1} \\
&=: \beta_{m,j} + 2W_{\nu_m - 1}\Pi_{\nu_m,\nu_{j-1}-1} R_{\nu_{j-1},\nu_j - 1}.
\end{aligned}$$

Note that $\beta_{m,j}$ is independent of $\mathcal{F}_m$. We enlarge, if necessary, the probability space to define a random variable $\tilde{W}$ with the same distribution as $W_{\nu_m - 1}$ and independent of all (ω_x); in particular, $\tilde{W}$ is independent of $\mathcal{F}_m$. Denote $\tilde{\beta}_j = \beta_{m,j} + 2\tilde{W}\Pi_{\nu_m,\nu_{j-1}-1} R_{\nu_{j-1},\nu_j - 1}$, so that

$$E_Q\left[\beta_j \wedge \delta n^{1/\kappa} \,\middle|\, \mathcal{F}_m\right] - E_Q[\beta_j \wedge \delta n^{1/\kappa}] = E_Q\left[\beta_j \wedge \delta n^{1/\kappa} - \tilde{\beta}_j \wedge \delta n^{1/\kappa} \,\middle|\, \mathcal{F}_m\right].$$

Observe that $R_{\nu_{j-1},\nu_j - 1} \le \beta_{m,j} \le \min(\beta_j, \tilde{\beta}_j)$. Thus, if $R_{\nu_{j-1},\nu_j - 1} \ge \delta n^{1/\kappa}$, then both β_j and $\tilde{\beta}_j$ are larger than $\delta n^{1/\kappa}$ as well. This implies that

$$\begin{aligned}
&\left| E_Q\left[\beta_j \wedge \delta n^{1/\kappa} \,\middle|\, \mathcal{F}_m\right] - E_Q[\beta_j \wedge \delta n^{1/\kappa}] \right| \\
&\quad \le E_Q\left[|\beta_j - \tilde{\beta}_j| \mathbf{1}_{\{R_{\nu_{j-1},\nu_j - 1} \le \delta n^{1/\kappa}\}} \,\middle|\, \mathcal{F}_m\right] \\
&\quad = E_Q\left[2\Pi_{\nu_m,\nu_{j-1}-1} R_{\nu_{j-1},\nu_j - 1} |W_{\nu_m - 1} - \tilde{W}| \mathbf{1}_{\{R_{\nu_{j-1},\nu_j - 1} \le \delta n^{1/\kappa}\}} \,\middle|\, \mathcal{F}_m\right] \\
&\quad \le 2\big(E_Q[\Pi_{0,\nu-1}]\big)^{j-m-1} E_Q[R_{0,\nu-1}\mathbf{1}_{\{R_{0,\nu-1} \le \delta n^{1/\kappa}\}}] \left(E_Q[\tilde{W}] + W_{\nu_m - 1}\right),
\end{aligned} \tag{A.12}$$

where in the last inequality we used the fact that the blocks of environment between ladder locations are i.i.d. under the measure Q. Since $R_{0,\nu-1} \le \beta_1$, there exists a constant C so that $Q(R_{0,\nu-1} > x) \le C x^{-\kappa}$. This implies that

$$E_Q[R_{0,\nu-1}\mathbf{1}_{\{R_{0,\nu-1} \le \delta n^{1/\kappa}\}}] \le E_Q[R_{0,\nu-1}] < \infty, \qquad \text{when } \kappa > 1$$

and $E_Q[R_{0,\nu-1}\mathbf{1}_{\{R_{0,\nu-1} \le \delta n^{1/\kappa}\}}] \le C \log n$ when $\kappa = 1$ for some other C. Thus, we can always bound this expectation by $C \log n$ for some $C > 0$. Also, the definition of

ν implies that $r := E_Q[\Pi_{0,\nu-1}] < 1$ and (A.3) implies that $E_Q[\tilde{W}] = E_Q[W_{-1}] < \infty$. Thus, there exists a constant C so that

$$\left| E_Q\left[\beta_j \wedge \delta n^{1/\kappa} \,\middle|\, \mathcal{F}_m\right] - E_Q[\beta_j \wedge \delta n^{1/\kappa}]\right| \le C \log n\, r^{j-m-1}\left(1 + W_{\nu_m - 1}\right),$$

implying that

$$\begin{aligned} |E_Q[S_n - S_m \,|\, \mathcal{F}_m]| &\le n^{-1/\kappa} \sum_{j=m+1}^{n} \left| E_Q\left[\beta_j \wedge \delta n^{1/\kappa} \,\middle|\, \mathcal{F}_m\right] - E_Q[\beta_j \wedge \delta n^{1/\kappa}]\right| \\ &\le C n^{-1/\kappa} \log n \left(1 + W_{\nu_m - 1}\right). \end{aligned}$$

Applying Chebyshev's inequality conditionally, we obtain

$$\begin{aligned} &Q\left(|S_n - S_m| > \eta \,|\, \mathcal{F}_m\right) \\ &\quad \le \mathbf{1}\{|E_Q[S_n - S_m \,|\, \mathcal{F}_m]| > \eta/2\} + Q\left(|S_n - S_m - E_Q[S_n - S_m \,|\, \mathcal{F}_m]| > \eta/2\right) \\ &\quad \le \mathbf{1}\{1 + W_{\nu_m - 1} > n^{1/\kappa}\eta/(2C\log n)\} + \frac{4}{\eta^2}\operatorname{Var}_Q\left(S_n - S_m \,|\, \mathcal{F}_m\right). \end{aligned} \tag{A.13}$$

To handle the conditional variance in (A.13) we write

$$\begin{aligned} \operatorname{Var}_Q\left(S_n - S_m \,|\, \mathcal{F}_m\right) = n^{-2/\kappa} &\sum_{j=m+1}^{n} \operatorname{Var}_Q\left(\beta_j \wedge \delta n^{1/\kappa} \,|\, \mathcal{F}_m\right) \\ &+ 2n^{-2/\kappa} \sum_{m<j<k\le n} \operatorname{Cov}_Q\left(\beta_j \wedge \delta n^{1/\kappa},\, \beta_k \wedge \delta n^{1/\kappa} \,|\, \mathcal{F}_m\right). \end{aligned} \tag{A.14}$$

Upper bounds on the conditional variance and conditional covariance terms above can be obtained in a similar manner to the proof of (49) in Peterson and Samorodnitsky (2010). One adapts this approach to take into account the conditioning on $\mathcal{F}_m$, by replacing β_j by $\beta_{m,j}$, and then controlling the difference between the two similarly to what was done above when bounding $E[S_n - S_m \,|\, \mathcal{F}_m]$. Doing this we obtain that there exist constants $C > 0$ and $r \in (0,1)$ such that

$$\operatorname{Var}_Q\left(\beta_j \wedge \delta n^{1/\kappa} \,|\, \mathcal{F}_m\right) \le C\delta^{2-\kappa} n^{2/\kappa - 1}\left(1 + r^{j-m-1} W_{\nu_m - 1}^2\right)$$

and

$$\operatorname{Cov}_Q\left(\beta_j \wedge \delta n^{1/\kappa},\, \beta_k \wedge \delta n^{1/\kappa} \,|\, \mathcal{F}_m\right) \le C\delta^{2-\kappa} n^{2/\kappa - 1}\left(1 + r^{j-m-1} W_{\nu_m - 1}^2\right)\sqrt{r^{k-j-1}}.$$

Using these bounds in (A.14), we see that for some $C > 0$,

$$\operatorname{Var}_Q(S_n - S_m \,|\, \mathcal{F}_m) \le C\delta^{2-\kappa}\left(1 + \frac{W_{\nu_m - 1}^2}{n}\right). \tag{A.15}$$

Combining (A.11), (A.13), and (A.15) we obtain

$$\begin{aligned} &Q(A_m \cap \{|S_n - S_m| > \eta\}) \\ &\quad \le Q(C'(1 + W_{\nu_m - 1}) > \eta n^{1/\kappa}/\log n) + \frac{1}{\eta^2} C\delta^{2-\kappa} E_Q\left[\mathbf{1}_{\{A_m\}}\left(1 + W_{\nu_m - 1}^2/n\right)\right] \\ &\quad \le C e^{-C'\eta n^{1/\kappa}/\log n} + \frac{1}{\eta^2} C\delta^{2-\kappa}\left(Q(A_m) + 1/n\right) \end{aligned}$$

where the constants C, C' may change from line to line (in the last inequality we used (A.3) and the fact that $W_{\nu_m - 1}$ has the same distribution as W_{-1} under Q). This gives us the statement of the lemma. □

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