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Conditions for exchangeable coalescents to come down from infinity

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Abstract. An improved condition to come down from infinity for exchangeable coalescent processes with simultaneous multiple collisions of ancestral lineages is provided. For non-critical coalescents this leads to an improved necessary and sufficient condition for the coalescent to come down from infinity. An analog conjecture for the full class of exchangeable coalescents is presented. New examples of critical coalescents are studied in detail. The results extend those obtained by J. Schweinsberg in Section 5.5 of 'Coalescents with simultaneous multiple collisions', Electron. J. Probab. 5, 1–50.

1. Introduction

Exchangeable coalescents (with simultaneous multiple collisions) are Markov processes $\Pi := (\Pi_t)_{t\geq 0}$ taking values in $\mathcal{P}$, the set of partitions of $\mathbb{N} := \{1, 2, \ldots\}$. Schweinsberg (2000a) characterizes Π in terms of a finite measure Ξ on the infinite simplex $\Delta := \{x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, |x| := \sum_{i=1}^{\infty} x_i \leq 1\}$. Throughout the paper, we use for $x = (x_1, x_2, \ldots) \in \Delta$ the notation $(x, x) := \sum_{i=1}^{\infty} x_i^2$. We furthermore decompose the measure $\Xi = a\delta_0 + \Xi_0$, where $a := \Xi(\{0\})$, δ_0 denotes the Dirac measure in $0 := (0, 0, \ldots) \in \Delta$ and Ξ_0 having no mass at 0. Additionally, set $\nu(dx) := \Xi_0(dx)/(x, x)$. Note that ν is a measure on Δ having no mass at zero.

For $n \in \mathbb{N}$, the function $\varrho_n : \mathcal{P} \to \mathcal{P}_n$ denotes the restriction to the set $\mathcal{P}_n$ of partitions of $\{1, \ldots, n\}$. For $\xi \in \mathcal{P}$ and as well for $\xi \in \mathcal{P}_n$, the notation $|\xi|$ is used for the number of blocks (equivalence classes) of ξ. With $N := (N_t)_{t\geq 0} := (|\Pi_t|)_{t\geq 0}$ we denote the so-called block counting process of Π.

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For $n \in \mathbb{N}$ let $\Pi^{(n)} = (\Pi_t^{(n)})_{t\geq 0} := (\varrho_n \circ \Pi_t)_{t\geq 0}$ denote the coalescent process restricted to $\mathcal{P}_n$. Furthermore, let $N^{(n)} := (N_t^{(n)})_{t\geq 0} := (|\Pi_t^{(n)}|)_{t\geq 0}$ denote the corresponding block counting process of $\Pi^{(n)}$. It is known (see, for example, Möhle, 2010, p. 2162) that the block counting process $N^{(n)}$ moves from a state $m \in \{2,\ldots,n\}$ to a state $k \in \{1,\ldots,m-1\}$ at the rate

$$\begin{aligned} g_{mk} &:= \lim_{h\searrow 0} \frac{P(N_{t+h}^{(n)} = k \,|\, N_t^{(n)} = m)}{h} \\ &= a\binom{m}{2} 1_{\{k=m-1\}} + \int_\Delta P(Y(m,x)=k)\,\nu(dx), \end{aligned} \tag{1.1}$$

where $Y(m,x) := X_0(m,x) + \sum_{i=1}^\infty 1_{\{X_i(m,x)\geq 1\}}$ and $(X_0(m,x), X_1(m,x),\ldots)$ has an infinite multinomial distribution with parameters $m \in \mathbb{N}$ and $(1-|x|, x_1, x_2, \ldots)$. Note that $Y(m,x)$, $m \in \mathbb{N}$, $x \in \Delta$, has mean

$$\mathrm{E}(Y(m,x)) = m(1-|x|) + \sum_{i=1}^\infty (1-(1-x_i)^m). \tag{1.2}$$

Expressions for the probabilities $P(Y(m,x)=k)$, $k \in \{1,\ldots,m\}$, which occur below the integral in (1.1), are available. For details on the distribution of $Y(m,x)$ we refer the reader to the appendix. Similarly as in (1.1), the total rates $g_m := \sum_{k=1}^{m-1} g_{mk}$, $m \in \mathbb{N}$, can be expressed in terms of the measure Ξ. For some information on related infinite urn schemes we refer the reader to the seminal work of Karlin (1967) and to the survey of Gnedin et al. (2007). We recall the definition of coming down from infinity and staying infinite (Pitman, 1999, p. 1886 and Schweinsberg, 2000a, p. 38). A coalescent $\Pi = (\Pi_t)_{t\geq 0}$ is called *standard* if Π_0 is the partition of $\mathbb{N}$ into singletons.

Definition 1.1. (coming down from infinity, staying infinite) Let Ξ be a finite measure on Δ, let $\Pi = (\Pi_t)_{t\geq 0}$ be a standard Ξ-coalescent and let $N := (N_t)_{t\geq 0} := (|\Pi_t|)_{t\geq 0}$ denote the corresponding block counting process. We say that Π comes down from infinity if $P(N_t < \infty) = 1$ for all $t > 0$. We say that Π stays infinite if $P(N_t = \infty) = 1$ for all $t > 0$.

Schweinsberg (2000b) provides a necessary and sufficient condition for the Λ-coalescent to come down from infinity. In Schweinsberg (2000a) he extents his methods used in Schweinsberg (2000b) and derives a similar sufficient condition and (under additional constraints) a similar necessary condition for Ξ-coalescents. We recall Schweinsberg's results in the following Theorems 1.2 and 1.3. Theorem 1.2 below essentially summarizes the results around Lemma 31 in Schweinsberg (2000a). In order to state the theorem it is convenient to introduce the set $\Delta_f := \{x \in \Delta : x_1 + \cdots + x_n = 1 \text{ for some } n \in \mathbb{N}\}$ and to define the absorption times $T_\infty := \inf\{t > 0 : N_t = 1\}$ and $T_n := \inf\{t > 0 : N_t^{(n)} = 1\}$ of the block counting process $N^{(n)}$, $n \in \mathbb{N}$. In the biological context T_n is called the time back to the most recent common ancestor (MRCA) of a sample of size n. Note that $0 \leq T_1 \leq T_2 \leq \cdots$ and that $\lim_{n\to\infty} T_n(\omega) = T_\infty(\omega)$ $(\in [0,\infty])$ for all $\omega \in \Omega$. By monotone convergence, $\mathrm{E}(T_\infty) = \lim_{n\to\infty} \mathrm{E}(T_n) \in [0,\infty]$. In Schweinsberg (2000a) it is explained how the problem of coming down from infinity for a coalescent satisfying $\Xi(\Delta_f) > 0$ can be reduced to the same problem for a coalescent which satisfies $\Xi(\Delta_f) = 0$. In the following we therefore focus without loss of generality mostly on coalescents satisfying $\Xi(\Delta_f) = 0$. It is also assumed without loss of generality that Ξ is not the zero measure.

Theorem 1.2. *(Schweinsberg, 2000a) Let Ξ be a finite measure on Δ satisfying $\Xi(\Delta_f) = 0$ and let Π be a standard Ξ-coalescent. Then either Π comes down from infinity or Π stays infinite. Moreover, Π comes down from infinity if and only if $\mathrm{E}(T_\infty) < \infty$ or, equivalently, if and only if the sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ is bounded.*

The sequence $(T_n)_{n\in\mathbb{N}}$ satisfies (see, for example, Freund and Möhle, 2009, Eq. (1.2)) the distributional recursion $T_1 = 0$ and $T_n \stackrel{d}{=} \tau_n + T_{I_n}$, where I_n is independent of $T_1, \ldots, T_{n-1}$ with distribution $p_{nk} := P(I_n = k) = g_{nk}/g_n$, $k \in \{1, \ldots, n-1\}$, and τ_n is independent of T_{I_n} and exponentially distributed with parameter g_n, $n \in \mathbb{N}$. Note that for Λ-coalescents this recursive structure was already found by Pitman (1999, p. 1886) and the same recursion holds for Ξ-coalescents due to the strong Markov property of $(\Pi_t^{(n)})_{t\geq 0}$. In particular, the expectation $\mathrm{E}(T_n)$ follows the recursion $\mathrm{E}(T_1) = 0$ and

$$\mathrm{E}(T_n) \;=\; \mathrm{E}(\tau_n) + \mathrm{E}(T_{I_n}) \;=\; \frac{1}{g_n} + \sum_{k=1}^{n-1} p_{nk}\mathrm{E}(T_k), \qquad n \in \{2, 3, \ldots\}. \tag{1.3}$$

Note that $\mathrm{E}(T_2) = 1/g_2 = 1/\Xi(\Delta)$. Thus, under the natural restriction that $\Xi(\Delta_f) = 0$, Theorem 1.2 together with the recursion (1.3) provides a necessary and sufficient condition for the standard Ξ-coalescent Π to come down from infinity in terms of the infinitesimal rates g_{nk}, $n, k \in \mathbb{N}$ with $k < n$, of its associated block counting process. However, the analysis of the recursion (1.3) is not as simple as it seems to be at a first glance. In general it is not straightforward to decide from (1.3) directly for which given triangular array of infinitesimal rates g_{nk}, $n, k \in \mathbb{N}$, with $k < n$, the sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ is bounded. The study of functionals of $\Pi^{(n)}$, such as the absorption time T_n, has recently become one of the main research interests in coalescent theory (see, for example, Gnedin et al., 2011 and Gnedin et al., 2012). For example, for many beta coalescents, the limiting distribution as $n \to \infty$ of T_n, properly centered and scaled, is known (Gnedin et al., 2012, Table 2). However, for arbitrary Ξ-coalescents, no general results are available, not even for the asymptotics of the expectation $\mathrm{E}(T_n)$ as $n \to \infty$.

So far, Schweinsberg (2000a) provided the most general criterion for $\mathrm{E}(T_\infty) < \infty$. In order to state his result, put $\Delta^\varepsilon := \{x \in \Delta : |x| \leq 1 - \varepsilon\}$ for $\varepsilon \in (0, 1)$ and define the function $\gamma : [0, \infty) \to \mathbb{R}$ by

$$\gamma(q) \;:=\; \gamma_\Xi(q) \;:=\; a\binom{q}{2} + \int_\Delta \sum_{i=1}^{\infty} ((1 - x_i)^q - 1 + qx_i)\,\nu(dx), \qquad q \geq 0. \tag{1.4}$$

Note that, by (1.1) and (1.2), $\gamma(n) = \sum_{k=1}^{n-1}(n - k)g_{nk}$ for all $n \in \mathbb{N}$. The following theorem summarizes Propositions 32 and 33 of Schweinsberg (2000a).

Theorem 1.3. *(Schweinsberg, 2000a) Let Ξ be a finite measure on Δ satisfying $\Xi(\Delta_f) = 0$ and let Π be a standard Ξ-coalescent.*
a) *If $\sum_{n=2}^{\infty} 1/\gamma(n) < \infty$, then $\mathrm{E}(T_\infty) < \infty$, so Π comes down from infinity.*
b) *If $\sum_{n=2}^{\infty} 1/\gamma(n) = \infty$ and $\nu(\Delta \setminus \Delta^\varepsilon) < \infty$ for some $\varepsilon \in (0, 1)$, then $\mathrm{E}(T_\infty) = \infty$, so Π stays infinite.*

However, there do exist Ξ-coalescents with $\Xi(\Delta_f) = 0$, $\sum_{n=2}^{\infty} 1/\gamma(n) = \infty$ and yet $\mathrm{E}(T_\infty) < \infty$. The most prominent example of such a coalescent is provided by

Schweinsberg (2000a, p. 42, Example 34). We refer the reader to Example 6.1 b) in Section 6, where Schweinsberg's example is re-explored and generalized. There are many other examples, in which Schweinsberg's criterion,

$$\sum_{n=2}^{\infty} \frac{1}{\gamma(n)} \ < \ \infty, \tag{1.5}$$

does not work. All of them necessarily violate the additional assumption in Theorem 1.3 b). We believe that such coalescents are worth studying and therefore give them a special name.

Definition 1.4. Let Ξ be a finite measure on Δ and let Π be a standard Ξ-coalescent. If $\nu(\Delta \setminus \Delta^{\varepsilon}) = \infty$ for all $\varepsilon \in (0,1)$, then Π is called critical.

Several examples of critical coalescents are introduced and studied in Section 6 in detail.

All other approaches concerning the problem of coming down from infinity for Ξ-coalescents Berestycki et al. (2010); Foucart (2011, 2012); Limic (2010); Schweinsberg (2000a,b) are also based on the function γ in (1.4) or on the closely related and slightly larger function $\psi : [0,\infty) \to [0,\infty)$, defined via

$$\psi(q) \ := \ \psi_{\Xi}(q) \ := \ a\frac{q^2}{2} + \int_{\Delta} \sum_{i=1}^{\infty} (e^{-qx_i} - 1 + qx_i)\, \nu(dx), \qquad q \geq 0. \tag{1.6}$$

None of them extends Schweinsberg's criterion. Note that $\gamma(q) \sim \psi(q)$ as $q \to \infty$ (Lemma 3.3) and therefore $\sum_{n=2}^{\infty} 1/\gamma(n) < \infty$ if and only if $\sum_{n=2}^{\infty} 1/\psi(n) < \infty$. Furthermore, ψ is the characteristic exponent of a spectrally positive Lévy process which is as well a martingale (Proposition 3.4).

The speed of coming down from infinity for Λ-coalescents is studied in Berestycki et al. (2010). Furthermore, Limic (2010, 2012) studies the speed of coming down from infinity for so-called *regular coalescents* satisfying the regularity condition $\int_{\Delta} |x|^2\, \nu(dx) < \infty$. Note that all Λ-coalescents are regular and all regular coalescents are non-critical (Proposition 2.1).

There exists a Markov process $Y = (Y_t)_{t \geq 0}$ with state space $[0,1]$, which is dual to the block counting process $N = (N_t)_{t \geq 0}$.

It is known (see, for example, Birkner et al., 2009, Remark 5.3) that, under the restriction that $\Xi(\Delta_f) = 0$, the Ξ-coalescent comes down from infinity if and only if Y hits its boundary $\{0,1\}$ in finite time almost surely. The coming down from infinity property is therefore essentially equivalent to the almost sure finiteness of the boundary hitting time $T := \inf\{t > 0 : Y_t \in \{0,1\}\}$ of Y, a problem of its own complexity, since Y has jumps. Relations of this form between the coming down from infinity property of the Ξ-coalescent and the finiteness of the boundary hitting times of certain jump processes further emphasize the importance of the coming down from infinity problem.

The paper is organized as follows. Our main results are presented in Section 2. Section 3 deals with the analysis of the functions γ and ψ defined in (1.4) and (1.6). Section 4 analyzes the functions δ and ϕ defined in (2.1) and (2.2). We consider this analysis to be important for further studies concerning the problems presented in Section 2. The proofs of the main results are provided in Section 5. The proofs make heavily use of the representation of the rates (1.1) in terms of an infinite multinomial distribution. Several new examples of critical coalescents are defined and analyzed in Section 6.

2. Results

We start with a proposition on properties of critical coalescents. For $k \in \mathbb{N}$ let Δ_k denote the set of all points $x = (x_1, x_2, \ldots) \in \Delta$ satisfying $x_i = 0$ for all $i > k$.

Proposition 2.1. *Let Ξ be a finite measure on Δ and let Π be a critical standard Ξ-coalescent. Then $\int_\Delta |x|^k\, \nu(dx) = \infty$ for all $k \in \mathbb{N}$. Moreover, $\Xi(\Delta \setminus \Delta_k) > 0$ for all $k \in \mathbb{N}$, i.e. Π allows with positive probability for an arbitrary large number of multiple collisions at the same time.*

Remark 2.2. There exist non-critical Ξ-coalescents satisfying $\int_\Delta |x|^k \nu(dx) = \infty$ for all $k \in \mathbb{N}$, for example, if Ξ assigns for each $m \in \mathbb{N}$ mass 2^{-m} to the point $x^{(m)} \in \Delta$ whose first 2^{m-1} coordinates are all equal to 2^{-m} and all other coordinates are equal to 0.

Our second result sheds some light on the asymptotics of $\mathrm{E}(T_n)$ as $n \to \infty$. It also provides an upper bound of general order $\log n$ for the expected absorption time.

Theorem 2.3. *Let Π be a standard Ξ-coalescent with $\Xi(\{0\}) = 0$. Then*

$$\lim_{n\to\infty} \frac{\log n}{\mathrm{E}(T_n)} \;=\; \int_\Delta -\log(1-|x|)\, \nu(dx) \;\in\; (0,\infty].$$

In particular, for any standard Ξ-coalescent, there exists a constant $C = C(\Xi) \in (0,\infty)$ such that $\mathrm{E}(T_n) \le C \log n$ for all $n \in \mathbb{N}$. Clearly, there exist coalescents for which $\mathrm{E}(T_n)$ grows much slower than $\log n$. The sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ can even be bounded, as for example for all coalescents with $\Xi(\{0\}) > 0$. However, in general the order $\log n$ of the upper bound cannot be improved since there obviously exist measures Ξ with $\int_\Delta -\log(1-|x|)\,\nu(dx) < \infty$. Note that critical coalescents satisfy $\int_\Delta -\log(1-|x|)\,\nu(dx) = \sum_{n=1}^\infty (1/n) \int_\Delta |x|^n\,\nu(dx) = \infty$. Theorem 2.3 may also be interpreted in the sense that the height of n-coalescent trees grows at most logarithmically in n. Note that trees of (at most) logarithmic height are particulary studied in graph theory.

We now turn to the problem of coming down from infinity. Our main idea is to replace the function γ in (1.4) by a properly modified larger function. Instead of using γ and ψ we work with the modified functions $\delta : (0,\infty) \to \mathbb{R}$ defined via

$$\delta(q) \;:=\; \delta_\Xi(q) \;:=\; a\binom{q}{2} - q\int_\Delta \log\left(1 - \frac{1}{q}\sum_{i=1}^\infty ((1-x_i)^q - 1 + qx_i)\right)\nu(dx), \qquad q > 0, \tag{2.1}$$

and its slightly larger modification $\phi : (0,\infty) \to [0,\infty)$, defined via

$$\phi(q) \;:=\; \phi_\Xi(q) \;:=\; a\frac{q^2}{2} - q\int_\Delta \log\left(1 - \frac{1}{q}\sum_{i=1}^\infty (e^{-qx_i} - 1 + qx_i)\right)\nu(dx), \qquad q > 0. \tag{2.2}$$

Note that δ and ϕ are well defined (Lemma 4.1) and differentiable (Lemma 4.4). We have not been able to show that $\delta(q) \sim \phi(q)$, $q \to \infty$. Under the constraint that $\int_\Delta 1/(1-|x|)\,\Xi(dx) < \infty$, the asymptotics $\delta(q) \sim \phi(q)$ as $q \to \infty$ holds (Proposition 4.5). The functions δ and ϕ are more involved than γ and ψ. However, these functions turn out to be helpful to decide whether a coalescent comes down from infinity or not. Note that, by (1.2), for $n \in \mathbb{N}$, $\delta(n) = a\binom{n}{2} - n\int_\Delta \log \mathrm{E}(Y(n,x)/n)\,\nu(dx)$. This relation between δ and $Y(n,x)$ is crucial for our analysis.

Theorem 2.4. *Let* Π *be a standard* Ξ*-coalescent. If*

$$\int_2^\infty \frac{dq}{\delta(q)} \;<\; \infty, \tag{2.3}$$

then $\mathrm{E}(T_\infty) < \infty$*, so the* Ξ*-coalescent comes down from infinity.*

The proof of Theorem 2.4 is closely related to the proof of the previous Theorem 2.3 and exploits the fact that the map $q \mapsto \delta(q)/q$ is non-decreasing on $[1,\infty)$ (see Corollary 4.2). By definition it is readily checked that $\gamma(q) \le \delta(q)$ for all $q > 0$. From this fact and from the monotonicity it follows that

$$\int_2^\infty \frac{dq}{\delta(q)} \;\le\; \sum_{n=2}^\infty \frac{1}{\gamma(n)}. \tag{2.4}$$

Therefore (2.3) is at least as good as Schweinsberg's criterion. However, it turns out that (2.3) is even stronger than (1.5). For example, in Schweinsberg's example mentioned in Section 1 we have $\int_2^\infty dq/\delta(q) < \infty = \sum_{n=2}^\infty 1/\gamma(n)$. For a whole class of critical coalescents, in which (2.3) is applicable and (1.5) is not, we refer the reader to Example 6.1 b) in Section 6. Schweinsberg's example will be covered there as well. The proof of Theorem 2.4 gives a bit more information. It shows that, for any Ξ-coalescent (no matter whether it comes down from infinity or not),

$$\mathrm{E}(T_n) \;\le\; \sum_{k=2}^n \frac{1}{\delta(k)}\frac{k}{k-1} \;\le\; 2\sum_{k=2}^n \frac{1}{\delta(k)}, \qquad n \in \{2,3,\ldots\}.$$

The first inequality implies that $\limsup_{n\to\infty} \mathrm{E}(T_n)/\sum_{k=2}^n 1/\delta(k) \le 1$, provided that $\sum_{k=2}^\infty 1/\delta(k) = \infty$. Let us now consider the converse implication, ensuring that the coalescent stays infinite. By (2.4) the following proposition is a direct consequence of Theorem 1.3 b) and Theorem 2.4.

Proposition 2.5. *Let* Π *be a non-critical standard* Ξ*-coalescent with* $\Xi(\Delta_f) = 0$*. Then* Π *comes down from infinity if and only if (2.3) holds.*

This all leads to the following problem that we present the coalescent community. So far the authors have not been able to solve this problem.

Problem 2.6. *Show that Proposition 2.5 holds without the restriction that the coalescent is non-critical. If this is not possible, find weaker/other conditions, under which Proposition 2.5 still holds.*

Remark 2.7. If the coalescent Π has a Kingman part ($a = \Xi(\{0\}) > 0$), then (see, for example, Limic, 2010, Remark 2) Π comes down from infinity and $\delta(k) \ge ak(k-1)/2$, $k \in \mathbb{N}$, which means that (2.3) is satisfied. Proposition 2.5 thus holds for $a > 0$, whether the coalescent is critical or not.

At the end of this section we present an interesting variant of (2.3). For $n \in \mathbb{N}$ define

$$\tilde{\delta}(n) \;:=\; n\sum_{k=1}^{n-1}(\log n - \log k)g_{nk} \;=\; n\lim_{t\searrow 0}\frac{1}{t}\mathrm{E}\Big(-\log\frac{N_t^{(n)}}{n}\Big), \tag{2.5}$$

where the last equality follows from (1.1). Note that $\tilde{\delta}(1) = 0$ and that $\tilde{\delta}(2) = 2(\log 2)\Xi(\Delta) > 0$, and, by (1.1),

$$\frac{\tilde{\delta}(n)}{n} = \left(-\log\frac{n-1}{n}\right) a\binom{n}{2} + \int_\Delta \mathrm{E}\left(-\log\frac{Y(n,x)}{n}\right)\nu(dx), \qquad n \in \{2,3,\ldots\}. \tag{2.6}$$

We believe that, for arbitrary but fixed $x \in \Delta$, the map

$$n \mapsto \mathrm{E}\left(-\log\frac{Y(n,x)}{n}\right) = \sum_{k=1}^\infty \frac{1}{k}\,\mathrm{E}\left(\left(1-\frac{Y(n,x)}{n}\right)^k\right) \tag{2.7}$$

is non-decreasing in $n \in \mathbb{N}$, which would imply that the map $n \mapsto \tilde{\delta}(n)/n$ is non-decreasing in $n \in \mathbb{N}$. However, we have not been able to verify this monotonicity property rigorously. In the following result, which is much in the spirit of Theorem 2.4 and verified in the same manner (see end of Section 5), the monotonicity of the map $n \mapsto \tilde{\delta}(n)/n$ is therefore assumed.

Proposition 2.8. *Let Π be a standard Ξ-coalescent and suppose that the map $n \mapsto \tilde{\delta}(n)/n$ is non-decreasing in $n \in \mathbb{N}$. If $\sum_{n=2}^\infty 1/\tilde{\delta}(n) < \infty$, then Π comes down from infinity.*

Since Jensen's inequality implies $\tilde{\delta}(n) \geq \delta(n)$ for all $n \in \mathbb{N}$, we conclude that $\sum_{n=2}^\infty 1/\tilde{\delta}(n) \leq \sum_{n=2}^\infty 1/\delta(n)$. Therefore, if the map in (2.7) is non-decreasing, Proposition 2.8 would be at least as good as Theorem 2.4.

Problem 2.9. *Show that (2.7) is non-decreasing in $n \in \mathbb{N}$.*

3. Analysis of the functions γ and ψ

The study of the functions γ and ψ, defined in (1.4) and (1.6) respectively, was at the core of previous works on coming down from infinity. Most of the results in this section are already known from the literature (see, for example, the works of Foucart (2011, 2012) and Limic (2010, 2012)), however, we provide them for completeness and add a few more details. We start with an analysis of the functions below the integrals in (1.4) and (1.6).

Lemma 3.1. *The functions $g : (0,\infty)\times\Delta \to \mathbb{R}$ and $h : (0,\infty)\times\Delta \to \mathbb{R}$, defined via*

$$g(q,x) := \frac{1}{q}\sum_{i=1}^\infty((1-x_i)^q - 1 + qx_i) = \sum_{i=1}^\infty\left(\frac{(1-x_i)^q}{q} - \frac{1}{q} + x_i\right)$$

and

$$h(q,x) := \frac{1}{q}\sum_{i=1}^\infty(e^{-qx_i} - 1 + qx_i) = \sum_{i=1}^\infty\left(\frac{e^{-qx_i}}{q} - \frac{1}{q} + x_i\right),$$

are both twice differentiable with respect to q, non-decreasing in $q \in (0,\infty)$ and concave. Furthermore, $\lim_{q\to\infty} g(q,x) = |x|$ and $\lim_{q\to\infty} h(q,x) = |x|$, where $|x| := \sum_{i=1}^\infty x_i$. Moreover, $\lim_{q\to 0} h(q,x) = 0$, whereas $\lim_{q\to 0} g(q,x) = \sum_{i=1}^\infty(x_i + \log(1-x_i)) \leq 0$ $(:= -\infty$ if $x = (1,0,0,\ldots))$.

Remark 3.2. The functions g and h are even infinitely often differentiable with respect to $q \in (0,\infty)$. However, we do not need this property in our further considerations.

Proof: For $x = (1,0,0,\ldots) \in \Delta$, $g(q,x) = 1 - 1/q$ is obviously twice differentiable, non-decreasing in $q \in (0,\infty)$ and concave. Assume now that $x \neq (1,0,0,\ldots)$. Then, $x_i < 1$ for all $i \in \mathbb{N}$, so we can take the logarithm of $1 - x_i$. Note that $\sum_{i=1}^\infty (-\log(1-x_i)) < \infty$, since, $-\log(1-t) = \sum_{k=1}^\infty t^k/k \le t\sum_{k=0}^\infty t^k = t/(1-t) \le 2t$ for all $t \in [0,\frac{1}{2}]$. For $k \in \mathbb{N}$ define $g_k : (0,\infty) \times \Delta \to \mathbb{R}$ via

$$g_k(q,x) \;:=\; \sum_{i=1}^k \left(\frac{(1-x_i)^q}{q} - \frac{1}{q} + x_i\right), \qquad q \in (0,\infty), x = (x_1,x_2,\ldots) \in \Delta.$$

Applying the inequality $t \le -\log(1-t)$, $t \in [0,1)$, with $t := 1 - (1-x_i)^q \in [0,1)$ it follows that $(1-(1-x_i)^q)/q \le -\log(1-x_i)$, $i \in \mathbb{N}$, $q \in (0,\infty)$. Therefore, for all $q \in (0,\infty)$,

$$\begin{aligned} |g(q,x) - g_k(q,x)| &= \left|\sum_{i=k+1}^\infty \frac{(1-x_i)^q - 1}{q} + x_i\right| \\ &\le \sum_{i=k+1}^\infty \frac{1-(1-x_i)^q}{q} + \sum_{i=k+1}^\infty x_i \\ &\le \sum_{i=k+1}^\infty (-\log(1-x_i)) + \sum_{i=k+1}^\infty x_i \;\to\; 0 \end{aligned}$$

as $k \to \infty$, so $g(.,x) \to g_k(.,x)$ uniformly on $(0,\infty)$. Furthermore, each g_k is differentiable with respect to q with continuous derivative $g_k'(q,x) = q^{-2}\sum_{i=1}^k (1-(1-x_i)^q + (1-x_i)^q\log((1-x_i)^q))$. We have pointwise convergence $g_k'(q,x) \to q^{-2}\sum_{i=1}^\infty (1-(1-x_i)^q+(1-x_i)^q\log((1-x_i)^q))$ as $k \to \infty$. Applying the inequalities $0 \le (1 - t^q + t^q\log t^q)/q^2 \le (\log t)^2/2$, $t \in (0,1]$, $q \in (0,\infty)$, with $t := 1 - x_i$ it is readily checked that this pointwise convergence holds even uniformly on $(0,\infty)$. Therefore, $g(.,x)$ is differentiable with respect to q with derivative $g'(q,x) = \lim_{k\to\infty} g_k'(q,x) = q^{-2}\sum_{i=1}^\infty (1-(1-x_i)^q + (1-x_i)^q\log((1-x_i)^q))$, which is non-negative, since $1 - t + t\log t \ge 0$ for all $t \in (0,1]$. Analogously, h is differentiable with respect to q with derivative $h'(q,x) = q^{-2}\sum_{i=1}^\infty (1 - e^{-qx_i} - qx_ie^{-qx_i})$, which is non-negative, since $1 - e^{-y} - ye^{-y} \ge 0$ for all $y \in [0,\infty)$.

Similarly it follows that g has second derivative $g''(q,x) = q^{-3}\sum_{i=1}^\infty \big(2(1-x_i)^q + (1-x_i)^q(\log((1-x_i)^q))^2 - 2 - 2(1-x_i)^q\log((1-x_i)^q)\big)$, which is non-positive, since $2t + t\log^2 t - 2 - 2t\log t \le 0$ for $t \in (0,1]$. Thus $g(.,x)$ is concave. Analogously, h has second derivative $h''(q,x) = q^{-3}\sum_{i=1}^\infty (2e^{-qx_i} + (qx_i)^2e^{-qx_i} - 2 + 2qx_ie^{-qx_i})$, which is non-positive, since $2e^{-y} + y^2e^{-y} - 2 + 2ye^{-y} \le 0$ for all $y \in [0,\infty)$. Thus $h(.,x)$ is concave.

Fix $x = (x_1,x_2,\ldots) \in \Delta$. For $q \in (0,\infty)$ and $i \in \mathbb{N}$ define $f_q(i) := ((1-x_i)^q - 1 + qx_i)/q$ and $f(i) := x_i$. Clearly, for arbitrary but fixed $i \in \mathbb{N}$, $f_q(i) \to f(i)$ as $q \to \infty$. Moreover, $0 \le f_q(i) \le f(i)$ for all $i \in \mathbb{N}$ and all $q \ge 1$. Note that f is integrable with respect to the counting measure $\varepsilon_{\mathbb{N}}$ on $\mathbb{N}$, since $\int f\,d\varepsilon_{\mathbb{N}} = |x| \le 1$. By dominated convergence, $g(q,x) := \int f_q\,d\varepsilon_{\mathbb{N}} \to \int f\,d\varepsilon_{\mathbb{N}} = |x|$ as $q \to \infty$. The same arguments work with $f_q(i)$ replaced by $\tilde{f}_q(i) := (e^{-qx_i} - 1 + qx_i)/q$ showing that $h(q,x) \to |x|$ as $q \to \infty$. The limits as $q \to 0$ are obtained similarly. $\square$

The next lemma in particular shows that the functions γ and ψ in (1.4) and (1.6) are well defined in the sense that they cannot take the value ∞. Moreover, the equivalence of γ and ψ is established. Previous works Foucart (2011, 2012);

Limic (2010, 2012) on coming down from infinity for Ξ-coalescents used inequalities of the form $c\gamma \le \psi \le C\gamma$ for some constants $c, C \in (0,\infty)$.

Lemma 3.3. *For all $q \ge 0$, $0 \le \psi(q) \le (q^2/2)\Xi(\Delta)$ and $\gamma(q) \le \psi(q)$. Moreover, $\psi(q) - \gamma(q) \le (q/2)\Xi(\Delta)$ for all $q \ge 1$, $\psi(q) \le 2\gamma(q)$ for all $q \ge 2$ and $\psi(q) \sim \gamma(q)$ as $q \to \infty$. In particular, the series $\sum_{n=2}^\infty 1/\gamma(n)$ converges if and only if the series $\sum_{n=2}^\infty 1/\psi(n)$ converges.*

Proof: Note that $0 \le e^{-t} - 1 + t \le t^2/2$ for $t \in [0,\infty)$. Hence, $0 \le \psi(q) \le (q^2/2)\Xi(\Delta)$, $q \ge 0$. Clearly $(1-x)^q \le e^{-qx}$ for $x \in [0,1]$ and $q \ge 0$, which implies that $\gamma(q) \le \psi(q)$ for all $q \ge 0$. Assume now that $q \ge 1$. By the mean value theorem, $(e^{-x})^q - (1-x)^q \le q(e^{-x} - (1-x)) \le qx^2/2$ for $x \in [0,1]$ and $q \in [1,\infty)$ and, therefore,

$$\begin{aligned}\psi(q) - \gamma(q) &= a\frac{q}{2} + \int_\Delta \sum_{i=1}^\infty (e^{-qx_i} - (1-x_i)^q)\,\nu(dx)\\ &\le a\frac{q}{2} + \frac{q}{2}\int_\Delta \sum_{i=1}^\infty x_i^2\,\nu(dx) \;=\; \frac{q}{2}\Xi(\Delta), \qquad q \ge 1.\end{aligned}$$

We now verify that $\psi(q) \le 2\gamma(q)$ for $q \ge 2$. For $\Xi = 0$ this is obvious. Assume now that Ξ is not the zero measure. By Corollary 4.2 the map $q \mapsto \gamma(q)/q$ is non-decreasing on $[1,\infty)$. For $q \ge 2$ it follows that $\gamma(q)/q \ge \gamma(2)/2 = \Xi(\Delta)/2$, or, equivalently, $q \le D\gamma(q)$ for $q \ge 2$ with $0 < D := 2/\Xi(\Delta) < \infty$. For $q \ge 2$ it follows that

$$\psi(q) \;=\; \psi(q) - \gamma(q) + \gamma(q) \;\le\; \frac{q}{2}\Xi(\Delta) + \gamma(q) \;\le\; \frac{D\gamma(q)}{2}\Xi(\Delta) + \gamma(q) \;=\; 2\gamma(q).$$

It remains to verify that $\psi(q) \sim \gamma(q)$ as $q \to \infty$. If $a = 0$ and $\int_\Delta |x|\,\nu(dx) < \infty$, then, see (4.1), $\lim_{q\to\infty} \gamma(q)/q = \lim_{q\to\infty} \psi(q)/q = \int_\Delta |x|\,\nu(dx) < \infty$ and, therefore, $\lim_{q\to\infty} \psi(q)/\gamma(q) = 1$.

Suppose now that $a > 0$ or $\int_\Delta |x|\,\nu(dx) = \infty$. If $a > 0$ then obviously $\gamma(q)/q \ge a(q-1)/2 \to \infty$. Otherwise we have $a = 0$ and, see (4.1), $\gamma(q)/q \to \int_\Delta |x|\,\nu(dx) = \infty$. Therefore, $q/\gamma(q) \to 0$ as $q \to \infty$, so for each $\varepsilon > 0$ there exists $q_0 = q_0(\varepsilon) > 1$ such that $q/\gamma(q) < 2\varepsilon/\Xi(\Delta)$ for all $q > q_0$. For all $q > q_0$ it follows that $\gamma(q) \le \psi(q) = \psi(q) - \gamma(q) + \gamma(q) \le (q/2)\Xi(\Delta) + \gamma(q) < \varepsilon\gamma(q) + \gamma(q)$, or, equivalently, $1 \le \psi(q)/\gamma(q) < 1 + \varepsilon$ for all $q > q_0$. Thus, $\psi(q) \sim \gamma(q)$ as $q \to \infty$. □

For monotonicity properties of the maps $q \mapsto \gamma(q)/q$ and $q \mapsto \psi(q)/q$, $q \ge 0$, we refer the reader to Corollary 4.2 and the remark thereafter.

The remaining part of this section concerns the relation of the function ψ with Lévy processes in the spirit of Bertoin and Le Gall (2006). We do not use these relations in our further considerations. However, we think that relations to Lévy processes are worth to mention, since they could turn out to be important for future work.

Proposition 3.4. *The function ψ in (1.6) is the characteristic exponent of a Lévy process $X = (X_t)_{t \ge 0}$, i.e. $\mathrm{E}(e^{-qX_t}) = e^{t\psi(q)}$, $q, t \in [0,\infty)$. Moreover, X is spectrally positive and a martingale.*

Remark 3.5. The map $q \mapsto \psi(q)/q$ is (see also Foucart, 2012, Lemma 9) the Laplace exponent of a subordinator.

Proof: Define the measure ϱ on $((0,1],\mathcal{B}\cap(0,1])$ via $\varrho(B):=\int_\Delta\sum_{i=1}^\infty 1_B(x_i)\,\nu(dx)$ for all Borel sets $B\subseteq(0,1]$. Let us verify that

$$\int_{(0,1]} f(u)\varrho(du) \;=\; \int_\Delta \sum_{i=1}^\infty f(x_i)\,\nu(dx) \tag{3.1}$$

for all measurable non-negative or integrable functions $f:[0,1]\to\mathbb{R}$ satisfying $f(0)=0$. For indicator functions $f=1_B$ with B any Borel set in $(0,1]$, (3.1) holds by the definition of ϱ. It is then straightforward to extend (3.1) stepwise to elementary functions, non-negative functions and finally to integrable functions. Choosing $f(u):=u^2$ is follows that $\int_{(0,1]}u^2\,\varrho(du)=\int(x,x)\,\nu(dx)=\Xi_0(\Delta)<\infty$. For arbitrary but fixed $q\in[0,\infty)$ we can also choose $f(u):=e^{-qu}-1+qu$, and it follows that $\psi(q)=(a/2)q^2+\int_{(0,1]}(e^{-qu}-1+qu)\,\varrho(du)$ for all $q\geq 0$. By the Lévy-Khintchine representation (see, for example, Bertoin, 1992, p. 307), ψ is the characteristic exponent of a spectrally positive Lévy process $X=(X_t)_{t\geq0}$, i.e. X has independent homogeneous increments, no negative jumps and satisfies $\mathrm{E}(e^{-qX_t})=e^{t\psi(q)}$ for all $q,t\geq 0$. Note that the associated Lévy measure ϱ is concentrated on $(0,1]$. It is readily checked that $\psi'(q)=aq+\int_{(0,1]}u(1-e^{-qu})\,\varrho(du)$ for all $q>0$, from which it follows that $\psi'(0+)=0$ and $\mathrm{E}(X_t)=-t\psi'(0+)=0$ for all $t\geq 0$. It remains to note that a Lévy process $X=(X_t)_{t\geq0}$ satisfying $\mathrm{E}(X_t)=0$ for all $t\geq0$ is a martingale. $\Box$

Remark 3.6. Note that $\psi'(q)=aq+\int_{(0,1]}u(1-e^{-qu})\,\varrho(du)$ and that $\psi''(q)=a+\int_{(0,1]}u^2e^{-qu}\,\varrho(du)$ as well as $\psi^{(k)}(q)=(-1)^k\int_{(0,1]}u^ke^{-qu}\,\varrho(du)$, $k\geq3$, $q\in(0,\infty)$. In particular, $\psi'(0+)=0$, $\psi''(0+)=a+\int_{(0,1]}u^2\,\varrho(du)=\Xi(\Delta)<\infty$ and $\psi^{(k)}(0+)=(-1)^k\int_{(0,1]}u^k\,\varrho(du)<\infty$, $k\geq3$. These derivatives are useful to compute the moments of X_t. For example, $\mathrm{Var}(X_t)=\mathrm{E}(X_t^2)=t\psi''(0+)=t\Xi(\Delta)$ and $\mathrm{E}(X_t^3)=-t\psi'''(0+)=t\int_{(0,1]}u^3\,\varrho(du)=\int_\Delta\sum_{i=1}^\infty x_i^3\,\nu(dx)$.

Example 3.7. 1. Assume that $\nu=\delta_x$ for some $x=(x_1,x_2,\ldots)\in\Delta\setminus\{0\}$. Then $\psi(q)=\sum_{i=1}^\infty(e^{-qx_i}-1+qx_i)$, $q\geq0$. The Lévy measure ϱ is the counting measure on $\{x_i:i\in\mathbb{N},x_i>0\}$ which assigns mass 1 to each $x_i>0$. The Lévy process $(X_t)_{t\geq0}$ has the form $X_t=\sum_{i=1}^\infty x_i(N_i(t)-t)$, $t\geq0$, where $N_i=(N_i(t))_{t\geq0}$, $i\in\mathbb{N}$, are i.i.d. homogeneous Poisson processes all with parameter 1.

2. Assume that Π does not have proper frequencies, or, equivalently that $\Xi(\{0\})=0$ and $\int_\Delta|x|\,\nu(dx)<\infty$ (see Schweinsberg, 2000a, Proposition 30). Then $X_t=\int_\Delta\sum_{i=1}^\infty x_i(N_{i,x}(t)-t)\,\nu(dx)$, $t\geq0$, where $N_{i,x}=(N_{i,x}(t))_{t\geq0}$, $i\in\mathbb{N}$, $x\in\Delta$, are i.i.d. homogeneous Poisson processes with parameter 1.

Problem 3.8. *The representations of the Lévy process X in the last examples lead to the conjecture that X can be constructed as well pathwise directly from (the Poisson process construction of) the coalescent. How does this pathwise construction work?*

4. Analysis of the functions δ and ϕ

We now turn to the functions δ and ϕ defined in (2.1) and (2.2). Recall that δ is at the core of this article (see, for example, Theorem 2.4). The function δ replaces the function γ used in previous studies on coming down from infinity for Ξ-coalescents.

The function ϕ is formally related to δ in the same way as the function ψ is related to γ. However, in contrast to ψ, ϕ does not seem to be the characteristic exponent of a Lévy process anymore. A probabilistic interpretation of the function ϕ remains unclear (may be there is none). Thus, ϕ is not in the focus of the article and only introduced since it is sometimes a bit simpler to do calculus with exponentials e^{-qx} rather than with powers $(1-x)^q$. The analysis of the function ϕ may also serve as a basis for future work.

Lemma 4.1. *For all $q\in(0,\infty)$, $\delta(q)\le\phi(q)<\infty$.*

Proof: The inequality $\delta(q)\le\phi(q)$ is clear, since $(1-x)^q\le e^{-qx}$ for all $q\in(0,\infty)$ and all $x\in[0,1]$. In order to show that $\delta(q)<\infty$ and $\phi(q)<\infty$ we can assume (without loss of generality) that $a:=\Xi(\{0\})=0$. By (1.2), for $n\in\mathbb{N}$, $\delta(n)/n=\int_\Delta -\log \mathrm{E}(Y/n)\,\nu(dx)$, where we write Y instead of $Y(n,x)$ for convenience. Since the map $x\mapsto -\log x$ is convex, Jensen's inequality yields

$$\begin{aligned}\frac{\delta(n)}{n} \quad &\le \quad \int_\Delta \mathrm{E}\Big(-\log\frac{Y}{n}\Big)\,\nu(dx) \quad = \quad \int_\Delta\sum_{k=1}^{n-1}\Big(-\log\frac{k}{n}\Big)P(Y=k)\,\nu(dx)\\ &= \quad \sum_{k=1}^{n-1}\Big(-\log\frac{k}{n}\Big)\int_\Delta P(Y=k)\,\nu(dx) \quad = \quad \sum_{k=1}^{n-1}\Big(-\log\frac{k}{n}\Big)g_{nk} \;<\; \infty.\end{aligned}$$

The finiteness of $\delta(q)$ for arbitrary $q\in(0,\infty)$ follows from the fact that, since $a=0$, $\delta(q)$ (even $\delta(q)/q$) is non-decreasing in $q\in(0,\infty)$ by Lemma 3.1 (see also Corollary 4.2).

We now verify that $\phi(q)<\infty$. For $x\in\Delta$ let $(\tilde X_{-1}(n,x),\tilde X_0(n,x),\tilde X_1(n,x),\ldots)$ be random variables having a multinomial distribution with parameters $n\in\mathbb{N}$ and $p=(p_{-1},p_0,p_1,p_2,\ldots)$ with $p_{-1}:=|x|-\sum_{i=1}^\infty(1-e^{-x_i})=\sum_{i=1}^\infty(x_i-1+e^{-x_i})$, $p_0:=1-|x|$ and $p_i:=1-e^{-x_i}$ for $i\in\mathbb{N}$. The random variable $\tilde Y:=\tilde Y(n,x):=\tilde X_0(n,x)+\sum_{i=1}^\infty 1_{\{\tilde X_i(n,x)\ge1\}}$ has expectation $\mathrm{E}(\tilde Y)\;=\;np_0+\sum_{i=1}^\infty(1-(1-p_i)^n)\;=\;n(1-|x|)+\sum_{i=1}^\infty(1-e^{-nx_i})$, so we can rewrite $\phi(n)$ in the form $\phi(n)/n=\int_\Delta -\log(1-\frac{1}{n}\sum_{i=1}^\infty(e^{-nx_i}-1+nx_i))\,\nu(dx)=\int_\Delta -\log\mathrm{E}(\tilde Y/n)\,\nu(dx)$. We therefore obtain $(\phi(n)-\delta(n))/n=\int_\Delta(\log\mathrm{E}(Y/n)-\log\mathrm{E}(\tilde Y/n))\,\nu(dx)$. Applying the formula $\log b-\log a\le(b-a)/a$, $0<a\le b<\infty$, with $a:=\mathrm{E}(\tilde Y/n)$ and $b:=\mathrm{E}(Y/n)$ yields

$$\frac{\phi(n)-\delta(n)}{n} \;\le\; \int_\Delta\frac{\mathrm{E}(Y-\tilde Y)}{\mathrm{E}(\tilde Y)}\,\nu(dx).$$

Note that $p_{-1}=\sum_{i=1}^\infty(e^{-x_i}-1+x_i)\le\sum_{i=1}^\infty x_i^2/2\le\sum_{i=1}^\infty x_i/2\le 1/2$, and, therefore, $\mathrm{E}(\tilde Y)=\int_{\{\tilde Y\ge1\}}\tilde Y\,dP\ge 1-P(\tilde Y=0)=1-P(\tilde X_{-1}(n,x)=n)=1-p_{-1}^n\ge1-(1/2)^n$. Thus,

$$\frac{\phi(n)-\delta(n)}{n} \;\le\; \frac{1}{1-(1/2)^n}\int_\Delta \mathrm{E}(Y-\tilde Y)\,\nu(dx) \;=\; \frac{1}{1-(1/2)^n}(\psi(n)-\gamma(n)) \;<\; \infty$$

and, hence, $\phi(n)<\infty$ for all $n\in\mathbb{N}$. The finiteness of $\phi(q)$ for arbitrary $q\in(0,\infty)$ follows from the fact that $\phi(q)$ (even $\phi(q)/q$) is non-decreasing in $q\in(0,\infty)$. $\square$

The following corollary concerns monotonicity properties of the four maps $q\mapsto f(q)/q$, $f\in\{\gamma,\delta,\psi,\phi\}$. Note that the monotonicity of the map $q\mapsto\delta(q)/q$ on $[1,\infty)$ is crucial for the proof of Theorem 2.4.

Corollary 4.2. *The maps $q \mapsto \gamma(q)/q$ and $q \mapsto \delta(q)/q$, with $\gamma(q)$ and $\delta(q)$ defined in (1.4) and (2.1), are both non-decreasing on $[1,\infty)$. The maps $q \mapsto \psi(q)/q$ and $q \mapsto \phi(q)/q$, with $\psi(q)$ and $\phi(q)$ defined in (1.6) and (2.2), are both non-decreasing on $(0,\infty)$.*

Proof: This is an immediate consequence of Lemma 3.1, since $\gamma(q) = a\binom{q}{2} + q\int_\Delta g(q,x)\,\nu(dx)$, $\delta(q) = a\binom{q}{2} + q\int_\Delta(-\log(1-g(q,x)))\,\nu(dx)$, $\psi(q) = aq^2/2 + q\int_\Delta h(q,x)\,\nu(dx)$ and $\phi(q) = aq^2/2 + q\int_\Delta(-\log(1-h(q,x)))\,\nu(dx)$. □

Remark 4.3. Suppose that $a=0$. Then $\gamma(q)/q = \int_\Delta g(q,x)\,\nu(dx)$ and $\psi(q)/q = \int_\Delta h(q,x)\,\nu(dx)$. By Lemma 3.1, $g(q,x) \nearrow |x|$ and $h(q,x) \nearrow |x|$ as $q\to\infty$. By monotone convergence it follows that

$$\lim_{q\to\infty}\frac{\gamma(q)}{q} \;=\; \lim_{q\to\infty}\frac{\psi(q)}{q} \;=\; \int_\Delta |x|\,\nu(dx) \;\in\; (0,\infty]. \tag{4.1}$$

Similarly, since we have $\delta(q)/q = \int_\Delta(-\log(1-g(q,x)))\,\nu(dx)$ as well as $\phi(q)/q = \int_\Delta(-\log(1-h(q,x)))\,\nu(dx)$, it follows from Lemma 3.1 that $-\log(1-g(q,x)) \nearrow -\log(1-|x|)$ and $-\log(1-h(q,x)) \nearrow -\log(1-|x|)$ as $q\to\infty$. Monotone convergence yields

$$\lim_{q\to\infty}\frac{\delta(q)}{q} \;=\; \lim_{q\to\infty}\frac{\phi(q)}{q} \;=\; \int_\Delta -\log(1-|x|)\,\nu(dx) \;\in\; (0,\infty]. \tag{4.2}$$

Note that there exist Ξ-coalescents (even Λ-coalescents) such that (4.1) is finite but (4.2) is infinite. For example, if Λ assigns for each $m\in\mathbb{N}$ mass m^{-2} to $x_m := 1-e^{-m}$, then $\int x^{-1}\Lambda(dx) = \sum_{m=1}^\infty (1-e^{-m})^{-1}m^{-2} \le (1-e^{-1})^{-1}\sum_{m=1}^\infty m^{-2} < \infty$ but $\int -\log(1-x)x^{-2}\Lambda(dx) = \sum_{m=1}^\infty m(1-e^{-m})^{-2}m^{-2} \ge \sum_{m=1}^\infty m^{-1} = \infty$.

Lemma 4.4. *The maps $q\mapsto\delta(q)/q$ and $q\mapsto\phi(q)/q$ are both differentiable with derivatives*

$$\frac{d}{dq}\frac{\delta(q)}{q} \;=\; \frac{a}{2} + \int_\Delta \frac{g'(q,x)}{1-g(q,x)}\,\nu(dx), \qquad q\in(0,\infty)$$

and

$$\frac{d}{dq}\frac{\phi(q)}{q} \;=\; \frac{a}{2} + \int_\Delta \frac{h'(q,x)}{1-h(q,x)}\,\nu(dx), \qquad q\in(0,\infty),$$

with $g(q,x)$ and $h(q,x)$ defined in Lemma 3.1.

Proof: Without loss of generality assume that $a=0$. Let us verify the result for δ. It suffices to verify that it is allowed to differentiate

$$\frac{\delta(q)}{q} \;=\; \int_\Delta(-\log(1-g(q,x)))\,\nu(dx)$$

below the integral. The result then follows immediately since

$$\frac{d}{dq}(-\log(1-g(q,x))) \;=\; \frac{g'(q,x)}{1-g(q,x)}.$$

By the well known differentiation lemma, taking the derivative below the integral is allowed if, for arbitrary but fixed $0<a\le b<\infty$, there exists a ν-integrable function $d:\Delta\to[0,\infty)$ such that

$$\frac{g'(q,x)}{1-g(q,x)} \;\le\; d(x) \tag{4.3}$$

for all $x \in \Delta$ and all $q \in [a,b]$. Note that the dominating function d is allowed to depend on a and b. Let us now verify that we can choose $d(x) := (x,x)\max(1/a^2, b)$. Note that $\int_\Delta d(x)\,\nu(dx) = \max(1/a^2, b)\Xi(\Delta) < \infty$. By Lemma 3.1,

$$g'(q,x) \;=\; \frac{1}{q^2}\sum_{i=1}^\infty \big(1-(1-x_i)^q + (1-x_i)^q\log((1-x_i)^q)\big).$$

For $t \in [0,1]$ and $q \geq 1$ we have

$$\begin{aligned}1-(1-t)^q+(1-t)^q\log((1-t)^q) &= 1-(1-t)^q(1-q\log(1-t))\\ &\leq 1-(1+qt)(1+qt) \;=\; q^2t^2.\end{aligned}$$

Moreover, the expression on the left hand side is non-decreasing in $q \in (0,\infty)$, so for $q \in (0,1]$ we obtain the bound

$$1-(1-t)^q+(1-t)^q\log((1-t)^q) \;\leq\; 1-(1-x)+(1-x)\log(1-x) \;\leq\; x^2.$$

Thus, $1-(1-t)^q+(1-t)^q\log((1-t)^q) \leq t^2\max(1,q^2)$, $q \in [0,\infty)$, $t \in [0,1]$. Applying this inequality with $t = x_i$ it follows that $g'(q,x) \leq (x,x)\max(1/q^2, 1)$. For $q \in (0,1]$ we have $g(q,x) \leq 0$ and hence $1/(1-g(q,x)) \leq 1$. Assume now that $q \geq 1$. Then, $1-(1-x_i)^q \geq x_i$, and, therefore,

$$q(1-g(q,x)) \;=\; q-q|x|+\sum_{i=1}^\infty(1-(1-x_i)^q) \;\geq\; q(1-|x|)+\sum_{i=1}^\infty x_i \;\geq\; 1,$$

or, equivalently, $1/(1-g(q,x)) \leq q$. Thus, $1/(1-g(q,x)) \leq \max(1,q)$ for all $q \in (0,\infty)$ and all $x \in \Delta$. Thus,

$$\frac{g'(q,x)}{1-g(q,x)} \;\leq\; (x,x)\max(1/q^2,1)\max(1,q) \;=\; (x,x)\max(1/q^2,q)$$

for all $q \in (0,\infty)$ and all $x \in \Delta$, so (4.3) holds with $d(x) := (x,x)\max(1/a^2, b)$.

The proof for ϕ works similar. We now have to find a dominating function for $h'(q,x)/(1-h(q,x))$. It is readily checked that $1-e^{-t}-te^{-t} \leq t^2/2$ for all $t \in [0,\infty)$. Applying this inequality with $t := qx_i$ it follows that

$$\begin{aligned}h'(q,x) &= \frac{1}{q^2}\sum_{i=1}^\infty(1-e^{-qx_i}-qx_ie^{-qx_i})\\ &\leq \frac{1}{q^2}\sum_{i=1}^\infty\frac{(qx_i)^2}{2} \;=\; \frac{(x,x)}{2}, \qquad q \in (0,\infty), x \in \Delta.\end{aligned}$$

Moreover, for $q \in [1,\infty)$,

$$\begin{aligned}q(1-h(q,x)) &= q(1-|x|)+\sum_{i=1}^\infty(1-e^{-qx_i}) \;\geq\; 1-|x|+\sum_{i=1}^\infty(1-e^{-x_i})\\ &\geq 1-|x|+\sum_{i=1}^\infty\Big(x_i-\frac{x_i^2}{2}\Big) \;=\; 1-\frac{(x,x)}{2} \;\geq\; \frac{1}{2},\end{aligned}$$

or, equivalently, $1/(1-h(q,x)) \leq 2q$. For $q \in (0,1]$, $1/(1-h(q,x)) \leq 1/(1-h(1,x)) \leq 2$. Thus, $1/(1-h(q,x)) \leq \max(2,2q)$ for all $q \in (0,\infty)$ and, consequently

$$\frac{h'(q,x)}{1-h(q,x)} \;\leq\; \frac{(x,x)}{2}\max(2,2q) \;=\; (x,x)\max(1,q), \qquad x \in \Delta, q \in (0,\infty).$$

For arbitrary but fixed $0 < a \le b < \infty$, it follows that $h'(q,x)/(1-h(q,x)) \le (x,x)\max(1,b)$ for all $x \in \Delta$ and all $q \in [a,b]$. So we can work with the dominating function $x \mapsto (x,x)\max(1,b)$ and all other arguments work the same as in the first part of the proof concerning the function δ. □

In general it seems to be not straightforward to verify that $\phi(q) \sim \delta(q)$ as $q \to \infty$. Let Δ^* denote the set of all $x \in \Delta$ satisfying $|x| = 1$. We verify the following result.

Proposition 4.5. *If $\Xi(\Delta^*) = 0$ and if $\int_\Delta 1/(1-|x|)\,\Xi(dx) < \infty$, then $\phi(q) \sim \delta(q)$ as $q \to \infty$.*

Remark 4.6. Example 6.2 in Section 6 shows that there exist critical coalescents satisfying $\int_\Delta 1/(1-|x|)\,\Xi(dx) = \infty$ and critical coalescents satisfying $\int_\Delta 1/(1-|x|)\,\Xi(dx) < \infty$.

Proof: Define $C := \int_\Delta 1/(1-|x|)\,\Xi(dx)$. Let us first verify that $\phi(q)-\delta(q) \le (C/2)q$ for all $q \in (0,\infty)$. We have

$$\frac{\phi(q)-\delta(q)}{q} \;=\; \frac{a}{2} + \int_\Delta \big(\log(1-g(q,x)) - \log(1-h(q,x))\big)\,\nu(dx),$$

where the functions g and h are defined in Lemma 3.1. Note that $0 \le g(q,x) \le h(q,x) < |x|$ for all $q \in (0,\infty)$ and all $x \in \Delta\setminus\{0\}$. Expanding the logarithms yields

$$\log(1-g(q,x)) - \log(1-h(q,x)) \;=\; \sum_{k=1}^\infty \frac{(h(q,x))^k - (g(q,x))^k}{k}.$$

Applying the inequality $b^k - a^k \le kb^{k-1}(b-a)$, $0 \le a \le b$, with $a := g(q,x)$ and $b := h(q,x)$ yields

$$\begin{aligned}\log(1-g(q,x)) - \log(1-h(q,x)) \;&\le\; (h(q,x)-g(q,x))\sum_{k=1}^\infty (h(q,x))^{k-1}\\ &=\; \frac{h(q,x)-g(q,x)}{1-h(q,x)} \;\le\; \frac{(x,x)}{2(1-|x|)},\end{aligned}$$

since $1-h(q,x) \ge 1-|x|$ and since

$$\begin{aligned}h(q,x)-g(q,x) \;&=\; \sum_{i=1}^\infty \frac{e^{-qx_i}-(1-x_i)^q}{q}\\ &\le\; \sum_{i=1}^\infty (e^{-x_i}-(1-x_i)) \;\le\; \frac{1}{2}\sum_{i=1}^\infty x_i^2 \;=\; \frac{(x,x)}{2}.\end{aligned}$$

Thus,

$$\frac{\phi(q)-\delta(q)}{q} \;\le\; \frac{a}{2} + \int_\Delta \frac{(x,x)}{2(1-|x|)}\,\nu(dx) \;=\; \frac{1}{2}\int_\Delta \frac{1}{1-|x|}\Xi(dx) \;=\; \frac{C}{2}.$$

Let us now verify that $\phi(q) \sim \delta(q)$ as $q \to \infty$. We proceed similar as in the proof of Lemma 3.3. If $a = 0$ and $\int_\Delta -\log(1-|x|)\,\nu(dx) < \infty$, then, by (4.2), $\lim_{q\to\infty}\delta(q)/q = \lim_{q\to\infty}\phi(q)/q = \int_\Delta -\log(1-|x|)\,\nu(dx) < \infty$ and, therefore, $\lim_{q\to\infty}\phi(q)/\delta(q) = 1$. Suppose now that $a > 0$ or $\int_\Delta -\log(1-|x|)\,\nu(dx) = \infty$. If $a = 0$, then obviously, $\delta(q)/q \ge a(q-1)/2 \to \infty$. Otherwise, we have $a = 0$ and, by (4.2), $\delta(q)/q \to \int_\Delta -\log(1-|x|)\,\nu(dx) = \infty$. Therefore, $q/\delta(q) \to 0$ as $q \to \infty$, so for each $\varepsilon > 0$ there exists $q_0 = q_0(\varepsilon) > 0$ such that $q/\delta(q) < 2\varepsilon/C$ for all $q > q_0$. For all $q > q_0$ it follows that

$$\delta(q) \;\le\; \phi(q) \;=\; \phi(q) - \delta(q) + \delta(q) \;\le\; \frac{C}{2}q + \delta(q) \;<\; \varepsilon\delta(q) + \delta(q),$$

or, equivalently, $1 \le \phi(q)/\delta(q) < 1 + \varepsilon$ for all $q > q_0$. Thus, $\phi(q) \sim \delta(q)$ as $q \to \infty$. □

5. Proofs

Proof: (of Proposition 2.1) The first statement is clear, since for all $k \in \mathbb{N}$ and all $\varepsilon \in (0,1)$

$$\int_\Delta |x|^k\,\nu(dx) \;\ge\; \int_{\Delta\setminus\Delta^\varepsilon} |x|^k\,\nu(dx) \;\ge\; (1-\varepsilon)^k \nu(\Delta\setminus\Delta^\varepsilon). \tag{5.1}$$

In order to verify the second assertion fix $k \in \mathbb{N}$ and suppose that $\Xi(\Delta\setminus\Delta_k) = 0$. Fix $\varepsilon \in (0,1)$. Applying the inequality $(x_1 + \cdots + x_k)^2 \le k\sum_{i=1}^k x_i^2$ to $x = (x_1, x_2, \ldots) \in \Delta_k$ satisfying $|x| > 1 - \varepsilon$, it follows that $(x,x) = \sum_{i=1}^k x_i^2 \ge (x_1 + \cdots + x_k)^2/k = |x|^2/k \ge (1-\varepsilon)^2/k$. Thus,

$$\nu(\Delta\setminus\Delta^\varepsilon) \;=\; \int_{\{|x|>1-\varepsilon\}} \frac{\Xi(dx)}{(x,x)} \;\le\; \frac{k}{(1-\varepsilon)^2}\Xi(\Delta) \;<\; \infty,$$

which contradicts our assumption that the coalescent is critical. □

Proof: (of Theorem 2.3) Since $\log(n+1)/\log n \to 1$, we can work with $\log(n+1)$ in the following. We want the sequence $(a_n)_{n\in\mathbb{N}} := (\log(n+1))_{n\in\mathbb{N}}$ to be the solution of a recursion of the type

$$a_n \;=\; b_n + \sum_{k=1}^{n-1} p_{nk} a_k, \qquad n \in \{2,3,\ldots\}, \tag{5.2}$$

with $p_{nk} := g_{nk}/g_n$ for $n \ge 2$ and $1 \le k < n$. Therefore we define $b_n := \sum_{k=1}^{n-1} p_{nk}(a_n - a_k)$. Plugging in the representation (1.1) of the infinitesimal rates $g_{nk} = \int_\Delta P(Y(n,x) = k)\,\nu(dx)$ we get

$$g_n b_n \;=\; \sum_{k=1}^{n-1}(a_n - a_k)g_{nk} \;=\; \int_\Delta \mathrm{E}\left(-\log\frac{Y(n,x)+1}{n+1}\right)\nu(dx).$$

We are going to show that $\lim_{n\to\infty} g_n b_n = \int_\Delta -\log(1-|x|)\,\nu(dx)$. Then applying Marynych (2010, Theorem 5.1) proves Theorem 2.3, since $(a'_n)_{n\in\mathbb{N}} := (\mathrm{E}(T_n))_{n\in\mathbb{N}}$ satisfies the recursion

$$a'_n \;=\; \frac{1}{g_n} + \sum_{k=1}^{n-1} p_{nk} a'_k, \qquad n \in \{2,3,\ldots\}. \tag{5.3}$$

First, Jensen's inequality, the concavity of the function log and $X_0(n,x) \le Y(n,x)$ yield

$$\begin{aligned} \mathrm{E}\left(-\log\frac{Y(n,x)+1}{n+1}\right) \;&\le\; \mathrm{E}\left(\log\frac{n+1}{X_0(n,x)+1}\right) \;\le\; \log \mathrm{E}\left(\frac{n+1}{X_0(n,x)+1}\right) \\ &=\; \log\frac{1-|x|^{n+1}}{1-|x|} \le -\log(1-|x|) \end{aligned}$$

for all $x \in \Delta$ with $|x| < 1$. Note that the last equality can be calculated fairly easy, since $X_0(n,x)$ is binomially distributed with parameters n and $1-|x|$. Furthermore, note that the inequality above actually holds for all $x \in \Delta$. Therefore we get

$$\limsup_{n\to\infty} \int_\Delta \mathrm{E}\left(-\log\frac{Y(n,x)+1}{n+1}\right)\nu(dx) \;\le\; \int_\Delta -\log(1-|x|)\,\nu(dx).$$

On the other hand, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Delta \mathrm{E}\left(-\log\frac{Y(n,x)+1}{n+1}\right)\nu(dx)\\
\ge\;& \liminf_{n\to\infty} \int_\Delta -\log \mathrm{E}\left(\frac{Y(n,x)+1}{n+1}\right)\nu(dx)\\
\ge\;& \int_\Delta \liminf_{n\to\infty}\left(-\log \mathrm{E}\left(\frac{Y(n,x)+1}{n+1}\right)\right)\nu(dx),
\end{aligned}$$

where the first inequality holds again by Jensen's inequality and the second inequality follows from Fatou's lemma. Since $\lim_{n\to\infty}\mathrm{E}((Y(n,x)+1)/(n+1)) = 1-|x|$ by Lemma 3.1 and by (1.2), we conclude

$$\liminf_{n\to\infty} \int_\Delta \mathrm{E}\left(-\log\frac{Y(n,x)+1}{n+1}\right)\nu(dx) \;\ge\; \int_\Delta -\log(1-|x|)\,\nu(dx),$$

resulting in $\lim_{n\to\infty} g_n b_n = \int_\Delta -\log(1-|x|)\,\nu(dx)$. □

The following proof of Theorem 2.4 has much in common with the previous proof of Theorem 2.3. The main difference is that the 'global' sequence $(a_n)_{n\in\mathbb{N}} = (\log(n+1))_{n\in\mathbb{N}}$ is carefully replaced by a more involved sequence, which depends on the measure Ξ of the coalescent.

Proof: (of Theorem 2.4) Define the auxiliary map $h : \{2,3,\ldots\} \to (0,\infty)$ via $h(n) := n/\delta(n)$ for all integers $n \ge 2$. Note that h cannot be defined for $n = 1$, since $\delta(1) = 0$. Define the sequence $(a_n)_{n\in\mathbb{N}}$ via

$$a_n \;:=\; \int_{(1,n]} \frac{h(\lceil q\rceil)}{q}\,\lambda(dq) \;=\; \int_{(0,\log n]} h(\lceil e^t\rceil)\,\lambda(dt), \qquad n\in\mathbb{N},$$

where $\lceil x\rceil := \inf\{z\,|\,z\in\mathbb{Z}, z\ge x\}$, $x\in\mathbb{R}$, denotes the (left-continuous) upper Gauss bracket. Defining again $b_n := \sum_{k=1}^{n-1} p_{nk}(a_n - a_k)$ and $p_{nk} := g_{nk}/g_n$ for $n \ge 2$, the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies the recursion (5.2). Note furthermore that, for all $n\ge 2$,

$$\begin{aligned}
n(a_n - a_{n-1}) \;&=\; n\int_{(\log(n-1),\log n]} h(\lceil e^t\rceil)\,\lambda(dt)\\
&=\; nh(n)(\log n - \log(n-1)) \;\ge\; h(n).
\end{aligned}$$

By Corollary 4.2, the auxiliary map h is non-increasing. Thus, for all $n\in\{2,3,\ldots\}$ and all $x\in\Delta$, writing Y instead of $Y(n,x)$ for convenience, we obtain

$$\begin{aligned}
a_n - a_Y \;&=\; \int_{(\log Y,\log n]} h(\lceil e^t\rceil)\,\lambda(dt)\\
&\ge\; h(n)\int_{(\log Y,\log n]} 1\,\lambda(dt) \;=\; h(n)\Big(-\log\Big(\frac{Y}{n}\Big)\Big).
\end{aligned}$$

Plugging in the representation (1.1), as in the previous proof, it follows for all $n \geq 2$ that

$$\begin{aligned} g_n b_n &= (a_n - a_{n-1}) a \binom{n}{2} + \int_\Delta \mathrm{E}(a_n - a_Y)\, \nu(dx) \\ &\geq h(n) a \frac{n-1}{2} + \int_\Delta \mathrm{E}\bigg(h(n) \bigg(-\log\bigg(\frac{Y}{n}\bigg)\bigg)\bigg)\, \nu(dx) \\ &= h(n) \bigg(a\frac{n-1}{2} + \int_\Delta \mathrm{E}\bigg(-\log\bigg(\frac{Y}{n}\bigg)\bigg)\, \nu(dx)\bigg) \\ &\geq h(n) \bigg(a\frac{n-1}{2} - \int_\Delta \log \mathrm{E}\bigg(\frac{Y}{n}\bigg)\, \nu(dx)\bigg) \;=\; h(n)\frac{\delta(n)}{n} \;=\; 1, \end{aligned}$$

where the last inequality follows by Jensen's inequality, since the map $x \mapsto -\log x$ is convex. Thus, $1/g_n \leq b_n$ for all $n \geq 2$. Comparing the recursion (5.2), which the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies, with the recursion (5.3) of the sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ it therefore follows inductively on n that $\mathrm{E}(T_n) \leq a_n$ for all $n \in \mathbb{N}$. Note that $E(T_1) = 0$ and that $a_1 = 0$. Thus, for all $n \in \mathbb{N}$

$$\begin{aligned} \mathrm{E}(T_n) \;\leq\; a_n &= \int_{(1,n]} \frac{h(\lceil q \rceil)}{q}\, \lambda(dq) \;=\; \sum_{k=2}^n \int_{k-1}^k \frac{h(k)}{q}\, dq \\ &= \sum_{k=2}^n \frac{1}{\delta(k)} \int_{k-1}^k \frac{k}{q}\, dq \;\leq\; \sum_{k=2}^n \frac{1}{\delta(k)} \frac{k}{k-1} \;\leq\; 2 \sum_{k=2}^n \frac{1}{\delta(k)}. \end{aligned}$$

In particular, if $\sum_{k=2}^\infty 1/\delta(k) < \infty$, or, equivalently, if $\int_2^\infty dq/\delta(q) < \infty$, then the sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ is bounded, which implies that $\mathrm{E}(T_\infty) < \infty$, so the coalescent comes down from infinity. □

Remark 5.1. In this remark it is explained that our proof differs significantly from related proofs (see Limic, 2010 and Foucart, 2011, 2012) in the literature. Fix $n \in \mathbb{N}$ and let G_n denote the generator of the block counting process $N^{(n)} = (N_t^{(n)})_{t\geq 0}$, i.e. $G_n f(i) = \sum_{j=1}^{i-1} (f(j) - f(i)) g_{ij}$ for all $f : \{1, \ldots, n\} \to \mathbb{R}$ and all $i \in \{1, \ldots, n\}$. For the particular function $f(i) := (\sup_{1\leq j\leq n} a_j) - a_i$, $i \in \{1, \ldots, n\}$, with the sequence $(a_i)_{i\in\mathbb{N}}$ as defined in the previous proof, we have just verified that $G_n f(i) = \sum_{j=1}^{i-1} (a_i - a_j) g_{ij} = g_i b_i \geq 1$ for all $i \in \{1, \ldots, n\}$. The process $(X_t^{(n)})_{t\geq 0}$, defined via $X_t^{(n)} := f(N_t^{(n)}) - \int_0^t (G_n f)(N_s^{(n)})\, ds$ for all $t \geq 0$, is a martingale. Applying for arbitrary but fixed $k \in \mathbb{N}$ the optional stopping theorem to the bounded stopping time $T_n \wedge k$ yields

$$\begin{aligned} f(n) &= \mathrm{E}(X_0^{(n)}) \;=\; \mathrm{E}(X_{T_n\wedge k}^{(n)}) \\ &= \mathrm{E}(f(N_{T_n\wedge k}^{(n)})) - \mathrm{E}\bigg(\int_0^{T_n\wedge k} (G_n f)(N_s^{(n)})\, ds \bigg) \\ &\leq \mathrm{E}(f(N_{T_n\wedge k}^{(n)})) - \mathrm{E}(T_n \wedge k), \end{aligned}$$

since $G_n f \geq 1$. For $k \to \infty$ it follows by monotone convergence and dominated convergence that $f(n) \leq \mathrm{E}(f(N_{T_n}^{(n)})) - \mathrm{E}(T_n) = f(1) - \mathrm{E}(T_n)$, or, equivalently, $\mathrm{E}(T_n) \leq f(1) - f(n) = a_n$. This alternative method to verify that $\mathrm{E}(T_n) \leq a_n$ for all $n \in \mathbb{N}$, is essentially a slight modification of related proofs of Limic (2010) and Foucart (2011, 2012). Instead of using a martingale argument and the optional stopping theorem, our proof is based on an elementary comparison analysis of the recursion (1.3). Our proof also differs from that of Schweinsberg (see Schweinsberg, 2000b, Lemma 6 and Schweinsberg, 2000a, Proposition 32). He uses the monotonicity of the function γ, whereas we have managed to exploit the monotonicity of the map $q \mapsto q/\delta(q)$ leading to an improved criterion involving δ instead of γ. Moreover we make direct use of the recursion (1.3) whereas in Schweinsberg's proof of Lemma 6 in Schweinsberg (2000b) the recursive structure is used rather implicitly and hidden in calculations involving conditional expectations.

Remark 5.2. In the proof of Theorem 2.4 only two inequalities are used. The first inequality is based on the fact that the auxiliary function h is non-increasing. The examples studied in Section 6 indicate that the function h is smoothly decreasing (usually regular varying and often even slowly varying at infinity). This leads to the intuition that, for large n, this first inequality should be quite sharp. The second estimation is based on Jensen's inequality. It is known (see, for example, Möhle, 2010) that $Y(n,x)/n \to 1-|x|$ in probability and in L^2 as $n \to \infty$. Thus, for large n the random variable $Y(n,x)/n$ behaves nearly like a constant. The Jensen inequality used in the last proof should hence be as well pretty sharp, at least for large n. These arguments lead to the intuition that both inequalities should be 'nearly' an equality for sufficiently large n. This indicates that Proposition 2.5 could hold without the restriction that the coalescent is non-critical (see Problem 2.6).

Proof: (of Proposition 2.8) Follow the proof of Theorem 2.4, but with $h(n) = n/\delta(n)$ replaced by $\tilde{h}(n) := n/\tilde{\delta}(n)$. It turns out that the proof works the same except for the following two minor changes. (i) Instead of using Corollary 4.2 it is now assumed that the function $\tilde{h}$ is non-increasing. (ii) Jensen's inequality is not needed anymore. □

6. Examples

Since the problem of coming down from infinity is solved for all non-critical coalescents, we focus on examples which are critical.

Example 6.1. Let $(p_m)_{m\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 < p_m \le 1/2$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^\infty p_m < \infty$. Suppose that Ξ assigns for each $m \in \mathbb{N}$ mass p_m to the point $x^{(m)} \in \Delta$ whose first $\lfloor (1-p_m)/p_m \rfloor$ coordinates are all equal to p_m and all other coordinates are equal to 0. Schweinsberg's example Schweinsberg (2000a, p. 42, Example 34) corresponds to $p_m := 2^{-m}$, $m \in \mathbb{N}$. Note that $\Xi(\Delta) = \sum_{m=1}^\infty p_m < \infty$. Moreover, $|x^{(m)}| = \lfloor (1-p_m)/p_m \rfloor p_m \le 1 - p_m < 1$ for all $m \in \mathbb{N}$ and, hence, $\Xi(\Delta_f) = 0$ and $\Xi(\Delta^*) = 0$. Recall the definition of Δ^* before Proposition 4.5. The assumption $p_m \le 1/2$ for all $m \in \mathbb{N}$ ensures that $\Xi(\{0\}) = 0$, so excludes a Kingman part.

In order to verify that the Ξ-coalescent is critical fix $\varepsilon \in (0,1)$. Since $|x^{(m)}| = \lfloor (1-p_m)/p_m \rfloor p_m \ge ((1-p_m)/p_m - 1)p_m = 1 - 2p_m \to 1$ as $m \to \infty$, there exists a constant $m_0 = m_0(\varepsilon) \in \mathbb{N}$ (which may depend on $(p_m)_{m\in\mathbb{N}}$) such that $|x^{(m)}| > 1-\varepsilon$ for all $m > m_0$. Thus,

$$\nu(\Delta \setminus \Delta^\varepsilon) \ge \sum_{m>m_0} \nu(\{x^{(m)}\}) = \sum_{m>m_0} \frac{p_m}{\lfloor (1-p_m)/p_m \rfloor p_m^2} \ge \sum_{m>m_0} \frac{1}{1-p_m} = \infty,$$

so the coalescent is critical. By (5.1), $\int |x|^k\,\nu(dx)=\infty$ for all $k\in\mathbb{N}$. The coalescent is in particular not regular. Note that $1-|x^{(m)}|\le 2p_m$ for all $m\in\mathbb{N}$ and, hence, $\int_\Delta 1/(1-|x|)\,\Xi(dx)=\sum_{m=1}^\infty p_m/(1-|x^{(m)}|)\ge\sum_{m=1}^\infty 1/2=\infty$. Clearly,

$$\begin{aligned}\psi(q) &= \int_\Delta\sum_{i=1}^\infty(e^{-qx_i}-1+qx_i)\nu(dx)\\ &= \sum_{m=1}^\infty\sum_{i=1}^{\lfloor(1-p_m)/p_m\rfloor}(e^{-qp_m}-1+qp_m)\frac{p_m}{\lfloor(1-p_m)/p_m\rfloor p_m^2}\\ &= \sum_{m=1}^\infty\frac{e^{-qp_m}-1+qp_m}{p_m},\qquad q\ge 0.\end{aligned}\tag{6.1}$$

Assume now that p_m is non-increasing in $m\in\mathbb{N}$. Let $f:[0,\infty)\to(0,\infty)$ be a non-increasing function such that $f(m)=p_m$ for all $m\in\mathbb{N}$. Since the map $x\mapsto(e^{-qf(x)}-1+qf(x))/f(x)$ is non-increasing in $x\in[0,\infty)$ it follows that

$$\int_{[1,\infty)}\frac{e^{-qf(x)}-1+qf(x)}{f(x)}\,\lambda(dx)\ \le\ \psi(q)\ \le\ \int_{[0,\infty)}\frac{e^{-qf(x)}-1+qf(x)}{f(x)}\,\lambda(dx).$$

Since for any constant $a\in[0,\infty)$ we have $\int_{[0,a]}(e^{-qf(x)}-1+qf(x))/f(x)\,dx\le\int_{[0,a]}q\,\lambda(dx)=aq$ and since, by (4.1), $\lim_{q\to\infty}\psi(q)/q=\int_\Delta|x|\,\nu(dx)=\infty$, it follows that

$$\psi(q)\ \sim\ \int_{(a,\infty)}\frac{e^{-qf(x)}-1+qf(x)}{f(x)}\,\lambda(dx),\qquad q\to\infty,\tag{6.2}$$

no matter how $a\in[0,\infty)$ is chosen. In the following it is assumed that p_m is even strictly decreasing in $m\in\mathbb{N}$. Moreover, we assume that the interpolating function f can be chosen such that it is strictly decreasing on $[0,\infty)$ and differentiable on $(0,\infty)$. If $v:=f^{-1}:(0,f(0)]\to[0,\infty)$ denotes the inverse of f, then the substitution $u=qf(x)$ leads to

$$\psi(q)\ \sim\ \int_{(0,qf(a))}\frac{e^{-u}-1+u}{u}\Big(-v'\Big(\frac{u}{q}\Big)\Big)\,\lambda(du),\qquad q\to\infty,\tag{6.3}$$

no matter how $a\in[0,\infty)$ is chosen. Eq. (6.3) will be used in the following concrete examples to determine the asymptotics of $\psi(q)$ as $q\to\infty$. Note that, by Lemma 3.3, $\gamma(q)\sim\psi(q)$ as $q\to\infty$. The analysis of $\phi(q)$ and $\delta(q)$ as $q\to\infty$ is more involved. In order to get rid of the disturbing logarithm in (2.2) it turns out to be useful to consider the derivative of $\phi(q)/q$. By Lemma 4.4 it is allowed to differentiate $\phi(q)/q=-\int_\Delta\log(1-h(q,x))\,\nu(dx)$ below the integral, so we have

$$\frac{d}{dq}\frac{\phi(q)}{q}\ =\ \int_\Delta\frac{h'(q,x)}{1-h(q,x)}\,\nu(dx)\ =\ \sum_{m=1}^\infty\frac{h'(q,x^{(m)})}{1-h(q,x^{(m)})}\,\nu(\{x^{(m)}\}).$$

Plugging in

$$\begin{aligned}h(q,x^{(m)}) &= \Big\lfloor\frac{1-p_m}{p_m}\Big\rfloor\frac{e^{-qp_m}-1+qp_m}{q},\\ h'(q,x^{(m)}) &= \Big\lfloor\frac{1-p_m}{p_m}\Big\rfloor\frac{1-e^{-qp_m}-qp_me^{-qp_m}}{q^2}\end{aligned}$$

and

$$\nu(\{x^{(m)}\}) \;=\; \frac{\Xi(\{x^{(m)}\})}{(x^{(m)},x^{(m)})} \;=\; \frac{p_m}{\lfloor\frac{1-p_m}{p_m}\rfloor p_m^2} \;=\; \frac{1}{\lfloor\frac{1-p_m}{p_m}\rfloor p_m}$$

it follows that

$$\frac{d}{dq}\frac{\phi(q)}{q} \;=\; \sum_{m=1}^\infty \frac{\frac{1-e^{-qp_m}-qp_me^{-qp_m}}{q^2p_m}}{1-\lfloor\frac{1-p_m}{p_m}\rfloor\frac{e^{-qp_m}-1+qp_m}{q}}.$$

Since the Gauss bracket in the denominator is bounded above and below via $(1-2p_m)/p_m = (1-p_m)/p_m - 1 \le \lfloor(1-p_m)/p_m\rfloor \le (1-p_m)/p_m$, we obtain the bounds

$$b_2(q) \;\le\; \frac{d}{dq}\frac{\phi(q)}{q} \;\le\; b_1(q), \tag{6.4}$$

where, for $c\in\{1,2\}$,

$$b_c(q) \;:=\; \sum_{m=1}^\infty \frac{\frac{1-e^{-qp_m}-qp_me^{-qp_m}}{q^2p_m}}{1-(1-cp_m)\frac{e^{-qp_m}-1+qp_m}{qp_m}}, \qquad q\ge 0.$$

Note that $b_c(q)$ is even defined for real parameter $c\in[0,2]$, however, we only need the two particular bounds $b_1(q)$ and $b_2(q)$. In general, the bound $b_c(q)$ is not so simple to analyze further. For many choices of the sequence $(p_m)_{m\in\mathbb{N}}$ (see the following concrete examples), the asymptotics of $b_c(q)$ as $q\to\infty$ does not depend on the constant c which yields the asymptotics of $(d/dq)(\phi(q)/q)$ and, by an application of de l'Hospital's rule, the asymptotics of $\phi(q)$ as $q\to\infty$. We now study three insightful examples of sequences $(p_m)_{m\in\mathbb{N}}$. A summarizing table is provided after Example 6.1 c).

Example 6.1 a) Fix $\alpha\in(1,\infty)$ and assume that $p_m := (m+1)^{-\alpha}$ for all $m\in\mathbb{N}$. Note that $\Xi(\Delta)=\sum_{m=1}^\infty (m+1)^{-\alpha} = \zeta(\alpha)-1<\infty$. In order to verify that

$$\psi(q) \;\sim\; c_\alpha\, q^{1+\frac{1}{\alpha}}, \qquad q\to\infty, \tag{6.5}$$

with $c_\alpha := \alpha^{-1}\int_{(0,\infty)}(e^{-u}-1+u)/u^{2+1/\alpha}\,\lambda(du)\in(0,\infty)$, define $f(x):=(x+1)^{-\alpha}$ for all $x\in[0,\infty)$. Note that the inverse $v:=f^{-1}:(0,1]\to\mathbb{R}$ of f is given by $v(t)=t^{-1/\alpha}-1$, $t\in(0,1]$, and that $v'(t)=-\alpha^{-1}t^{-1/\alpha-1}$, $t\in(0,1)$. By (6.3) (applied with $a:=0$) it follows that

$$\begin{aligned}\psi(q) \;&\sim\; \int_{(0,q)}\frac{e^{-u}-1+u}{u}\frac{1}{\alpha}\Big(\frac{u}{q}\Big)^{-\frac{1}{\alpha}-1}\lambda(du) \;=\; \frac{q^{1+\frac{1}{\alpha}}}{\alpha}\int_{(0,q)}\frac{e^{-u}-1+u}{u^{2+\frac{1}{\alpha}}}\lambda(du)\\ &\sim\; \frac{q^{1+\frac{1}{\alpha}}}{\alpha}\int_{(0,\infty)}\frac{e^{-u}-1+u}{u^{2+\frac{1}{\alpha}}}\lambda(du) \;=\; c_\alpha q^{1+\frac{1}{\alpha}},\end{aligned}$$

and (6.5) is established. In particular, $\sum_{n=2}^\infty 1/\psi(n)<\infty$. Since, by Lemma 3.3, $\gamma(q)\sim\psi(q)$ as $q\to\infty$, it follows that $\sum_{n=2}^\infty 1/\gamma(n)<\infty$, which implies (Theorem 1.3) that the coalescent comes down from infinity. Clearly, from $\gamma(q)\le\delta(q)$ and $\psi(q)\le\phi(q)$ it follows that $\sum_{n=2}^\infty 1/\delta(n)<\infty$ and that $\sum_{n=2}^\infty 1/\phi(n)<\infty$. Thus, alternatively, Theorem 2.4 yields as well that the coalescent comes down from infinity.

Note that, for this example, an analysis of the function γ (or ψ) is sufficient in order to conclude that the coalescent comes down from infinity. The more complicated functions ϕ and δ are not needed to determine that the coalescent comes down from infinity. For completeness we now derive as well the asymptotics of $\phi(q)$ and $\delta(q)$ as $q\to\infty$. By (6.4), $b_2(q)\le (d/dq)(\phi(q)/q)\le b_1(q)$, where

$$\begin{aligned} b_c(q) \;&:=\; \sum_{m=1}^{\infty}\frac{\frac{1-e^{-qp_m}-qp_me^{-qp_m}}{q^2p_m}}{1-(1-cp_m)\frac{e^{-qp_m}-1+qp_m}{qp_m}}\\ &\sim\; \int_{(0,\infty)}\frac{\frac{1-e^{-qf(x)}-qf(x)e^{-qf(x)}}{q^2f(x)}}{1-(1-cf(x))\frac{e^{-qf(x)}-1+qf(x)}{qf(x)}}\,\lambda(dx)\\ &=\; \int_{(0,q)}\frac{\frac{1-e^{-u}-ue^{-u}}{qu}}{1-(1-c\frac{u}{q})\frac{e^{-u}-1+u}{u}}\Big(-v'\Big(\frac{u}{q}\Big)\Big)\,\lambda(du)\\ &=\; \frac{q^{\frac{1}{\alpha}-1}}{\alpha}\int_{(0,q)}\frac{\frac{1-e^{-u}-ue^{-u}}{u^{2+\frac{1}{\alpha}}}}{1-(1-c\frac{u}{q})\frac{e^{-u}-1+u}{u}}\,\lambda(du)\;\sim\;\frac{q^{\frac{1}{\alpha}-1}}{\alpha}d_\alpha,\qquad q\to\infty, \end{aligned}$$

with $d_\alpha := \int_{(0,\infty)}(1-e^{-u}-ue^{-u})/(u^{1+1/\alpha}(1-e^{-u}))\,\lambda(du)\in(0,\infty)$. Thus,

$$\frac{d}{dq}\frac{\phi(q)}{q}\;\sim\;\frac{d_\alpha}{\alpha}q^{\frac{1}{\alpha}-1},\qquad q\to\infty.$$

An application of de l'Hospital's rule yields

$$\phi(q)\;\sim\;d_\alpha q^{1+\frac{1}{\alpha}},\qquad q\to\infty. \tag{6.6}$$

The calculations for $\delta(q)$ are essentially the same. The only difference is that many terms of the form $(1-u/q)^q$ instead of e^{-u} occur. Since $(1-u/q)^q\to e^{-u}$ as $q\to\infty$, one obtains the same asymptotics $\delta(q)\sim d_\alpha q^{1+1/\alpha}$ as $q\to\infty$. It is remarkable that, for this example, the four quantities $\psi(q)$, $\gamma(q)$, $\phi(q)$, and $\delta(q)$ are all of the same order $q^{1+1/\alpha}$ as $q\to\infty$. All four functions ψ, γ, ϕ, and δ are regularly varying at infinity of index $1+1/\alpha$.

In the following Example 6.1 b), Schweinsberg's criterion (1.5) does not work, whereas Theorem 2.4 is applicable.

Example 6.1 b) Fix $p\in(0,1/2]$ and suppose that $p_m:=p^m$ for all $m\in\mathbb{N}$. For $p=1/2$ this is Schweinsberg's example Schweinsberg (2000a, p. 42, Example 34), where the measure Ξ assigns for each $m\in\mathbb{N}$ mass 2^{-m} to the point in Δ whose first 2^m-1 coordinates are all equal to 2^{-m} and all other coordinates are equal to 0. Note that $\Xi(\Delta)=\sum_{m=1}^\infty p^m=p/(1-p)<\infty$. Let us verify that

$$\psi(q)\;\sim\;\kappa_p\,q\log q,\qquad q\to\infty, \tag{6.7}$$

where $\kappa_p := -1/\log p$. The function $x\mapsto p^x$, $x\in[0,\infty)$, has inverse $v(t):=(\log t)/(\log p)$, $t\in(0,1]$, with derivative $v'(t)=1/(t\log p)$, $t\in(0,1)$. By (6.3) (applied with $a:=0$) it follows that

$$\begin{aligned} \psi(q)\;&\sim\;\int_{(0,q)}\frac{e^{-u}-1+u}{u}\Big(-\frac{q}{u\log p}\Big)\,\lambda(du)\;=\;\kappa_p q\int_{(0,q)}\frac{e^{-u}-1+u}{u^2}\,\lambda(du)\\ &\sim\;\kappa_p q\int_1^q\frac{e^{-u}-1+u}{u^2}\,du\;\sim\;\kappa_p q\int_1^q\frac{1}{u}\,du\;=\;\kappa_p q\log q,\quad q\to\infty, \end{aligned}$$

and (6.7) is established. By Lemma 3.3, $\gamma(q)\sim\psi(q)\sim\kappa_p q\log q$ as $q\to\infty$. In particular, $\sum_{n=2}^\infty 1/\gamma(n)=\infty$, a result known for $p=1/2$ already by Schweinsberg Schweinsberg (2000a). Note that both maps $q\mapsto\psi(q)/q$ and $q\mapsto\gamma(q)/q$ are slowly varying at infinity. Let us now turn to the functions δ and ϕ. In order to verify that

$$\delta(q) \;\sim\; \frac{\kappa_p}{4} q(\log q)^2, \qquad q \to \infty, \tag{6.8}$$

it is sufficient (by de l'Hospital's rule) to verify that

$$\frac{d}{dq}\frac{\delta(q)}{q} \;\sim\; \frac{\kappa_p}{2}\frac{\log q}{q}, \qquad q \to \infty. \tag{6.9}$$

By (6.4), $b_2(q) \le (d/dq)(\delta(q)/q) \le b_1(q)$, where, for $c \in \{1,2\}$,

$$b_c(q) \;:=\; \sum_{m=1}^\infty \frac{1-(1-p^m)^q + (1-p^m)^q \log((1-p^m)^q)}{1-(1-cp^m)^{\frac{(1-p^m)^q - 1 + qp^m}{qp^m}}} \;\sim\; \frac{\kappa_p}{q}\int_0^q f_q(u)\,du,$$

with

$$f_q(u) \;:=\; \frac{\frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u^2}}{1-(1-c\frac{u}{q})^{\frac{(1-\frac{u}{q})^q-1+u}{u}}}, \qquad q \in (0,\infty), u \in (0,q).$$

It is readily checked that $0 \le f_q(u) \le q/u^3$ for $q \in (0,\infty)$ and $u \in (0,q)$. Thus, $\int_{\sqrt{q}}^q f_q(u)\,du \le \frac{1}{2}$, so it suffices to verify that $\int_0^{\sqrt{q}} f_q(u)\,du \sim \frac{1}{2}\log q$ as $q \to \infty$. We have

$$\begin{aligned} f_q(u) \;&\le\; \frac{\frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u^2}}{1-\frac{(1-\frac{u}{q})^q-1+u}{u}} \;=\; \frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u(1-(1-\frac{u}{q})^q)} \\ &=\; \frac{1}{u} + \frac{(1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u(1-(1-\frac{u}{q})^q)} \;\le\; \frac{1}{u}, \end{aligned}$$

which yields the upper bound $\int_0^{\sqrt{q}} f_q(u)\,du \sim \int_1^{\sqrt{q}} f_q(u)\,du \le \int_1^{\sqrt{q}} \frac{1}{u}\,du = \frac{1}{2}\log q$. In order to derive a lower bound note that for $q \in (0,\infty)$ and $u \in (0,\sqrt{q}]$

$$\begin{aligned} f_q(u) \;&=\; \frac{\frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u^2}}{\frac{1-(1-\frac{u}{q})^q}{u} + c\frac{(1-\frac{u}{q})^q-1+u}{q}} \;\ge\; \frac{\frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u^2}}{\frac{1-(1-\frac{u}{q})^q}{u} + \frac{c}{\sqrt{q}}} \\ &\ge\; \frac{1-(1-\frac{u}{q})^q + (1-\frac{u}{q})^q\log((1-\frac{u}{q})^q)}{u(1-(1-\frac{u}{q})^q + c\frac{u}{\sqrt{q}})} \\ &=\; \frac{1}{u} - \frac{\frac{-\log((1-\frac{u}{q})^q)}{u}(1-\frac{u}{q})^q + \frac{c}{\sqrt{q}}}{1-(1-\frac{u}{q})^q + c\frac{u}{\sqrt{q}}} \;\ge\; \frac{1}{u} - \frac{(1-\frac{u}{q})^{q-1} + \frac{c}{\sqrt{q}}}{1-(1-\frac{u}{q})^q + c\frac{u}{\sqrt{q}}}, \end{aligned}$$

since $(1-\frac{u}{q})(-\log(1-\frac{u}{q})) \le \frac{u}{q}$. Plugging in this lower bound and integration yields

$$\begin{aligned} \int_0^{\sqrt{q}} f_q(u)\,du \;&\ge\; \int_0^{\sqrt{q}} \left(\frac{1}{u} - \frac{(1-\frac{u}{q})^{q-1} + \frac{c}{\sqrt{q}}}{1-(1-\frac{u}{q})^q + c\frac{u}{\sqrt{q}}}\right) du \\ &=\; \left[\log u - \log\left(1-\left(1-\frac{u}{q}\right)^q + c\frac{u}{\sqrt{q}}\right)\right]_0^{\sqrt{q}} \\ &=\; \log\sqrt{q} - \log\left(1+c-\left(1-\frac{1}{\sqrt{q}}\right)^q\right) + \log(1+c/\sqrt{q}) \\ &\sim\; \frac{1}{2}\log q. \end{aligned}$$

Thus, (6.9) and, therefore, (6.8) holds. Similarly it follows that

$$\phi(q) \;\sim\; \frac{\kappa_p}{4} q(\log q)^2, \qquad q \to \infty. \tag{6.10}$$

In particular, both maps $q \mapsto \delta(q)/q$ and $q \mapsto \phi(q)/q$ are slowly varying at infinity. Moreover, $\sum_{n=2}^{\infty} 1/\delta(n) < \infty$. By Theorem 2.4, the coalescent comes down from infinity, For $p = 1/2$, Schweinsberg (2000a) verified that the coalescent comes down from infinity by showing with different direct methods that the sequence $(\mathrm{E}(T_n))_{n\in\mathbb{N}}$ of the mean absorption times is bounded.

In the following Example 6.1 c) the sequence $(p_m)_{m\in\mathbb{N}}$ tends extremely quickly to 0 as $m \to \infty$.

Example 6.1 c) Fix $p > 0$ sufficiently small such that $p^e \le 1/2$ and assume that $p_m := p^{e^m}$ for all $m \in \mathbb{N}$. In order to verify that

$$\psi(q) \;\sim\; q \log\log q, \qquad q \to \infty, \tag{6.11}$$

note that the map $x \mapsto f(x) := p^{e^x}$, $x \in [0, \infty)$, has inverse $v(t) = \log(\log t/\log p) = \log(-\log t) - \log(-\log p)$, $t \in (0, p]$ and that $v'(t) = 1/(t \log t)$, $t \in (0, p)$. By (6.3) (with $a := 0$) it follows that

$$\psi(q) \;\sim\; \int_{(0,pq)} \frac{e^{-u} - 1 + u}{u}\Big(-\frac{1}{\frac{u}{q}\log\frac{u}{q}}\Big)\,\lambda(du) \;=\; q\int_{(0,pq)} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,\lambda(du).$$

Since, for all $q \ge 2$ and all $u \in (0, 1]$, $-\log(u/q) = \log q - \log u \ge \log q \ge \log 2$, and, therefore,

$$\int_{(0,1]} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,\lambda(du) \;\le\; \frac{1}{\log 2}\int_{(0,1]} \frac{e^{-u}-1+u}{u^2}\,\lambda(du) \;<\; \infty$$

for all $q \ge 2$, it suffices to verify that

$$\int_1^{pq} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,du \;\sim\; \log\log q, \qquad q \to \infty.$$

Since $e^{-u} - 1 \le 0$, we obtain the upper bound

$$\int_1^{pq} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,du \;\le\; \int_1^{pq} \frac{1}{u(-\log\frac{u}{q})}\,du \;=\; [-\log(-\log\tfrac{u}{q})]_1^{pq} \;\sim\; \log\log q$$

and the lower estimation

$$\begin{aligned}
\int_1^{pq} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,du \;&\ge\; \int_{\sqrt{q}}^{pq} \frac{e^{-u}-1+u}{u^2(-\log\frac{u}{q})}\,du \;\sim\; \int_{\sqrt{q}}^{pq} \frac{1}{u(-\log\frac{u}{q})}\,du\\
&=\; [-\log(-\log\tfrac{u}{q})]_{\sqrt{q}}^{pq}\\
&=\; \log\log\sqrt{q} - \log(-\log p) \;\sim\; \log\log q,
\end{aligned}$$

and (6.11) is established. In the following the notation $a(q) = \Theta(b(q))$ as $q \to \infty$ means that there exists $q_0 \in (0, \infty)$ and constants $c, d \in (0, \infty)$ such that $cb(q) \le a(q) \le db(q)$ for all $q > q_0$.

In order to prove that $\phi(q) = \Theta(q \log q)$ as $q \to \infty$ it is sufficient to verify that $(d/dq)(\phi(q)/q) = \Theta(1/q)$ as $q \to \infty$. By (6.4), $b_2(q) \le (d/dq)(\phi(q)/q) \le b_1(q)$, where, for $c\in\{1,2\}$,

$$\begin{aligned} b_c(q) &:= \sum_{m=1}^\infty \frac{\frac{1-e^{-qp_m}-qp_me^{-qp_m}}{q^2p_m}}{1-(1-cp_m)\frac{e^{-qp_m}-1+qp_m}{qp_m}}\\ &\sim \int_{(0,\infty)} \frac{\frac{1-e^{-qf(x)}-qf(x)e^{-qf(x)}}{q^2f(x)}}{1-(1-cf(x))\frac{e^{-qf(x)}-1+qf(x)}{qf(x)}}\,\lambda(dx).\end{aligned}$$

Substituting $t=f(x)$ and noting that $dx/dt=1/(t\log t)$ it follows that

$$\begin{aligned} b_c(q) &\sim -\int_{(0,p)} \frac{\frac{1-e^{-qt}-qte^{-qt}}{q^2t}}{1-(1-ct)\frac{e^{-qt}-1+qt}{qt}}\frac{1}{t\log t}\,\lambda(dt)\\ &= \frac{1}{q}\int_{(0,pq)} \frac{\frac{1-e^{-u}-ue^{-u}}{u^2}}{1-(1-c\frac{u}{q})\frac{e^{-u}-1+u}{u}}\frac{1}{-\log\frac{u}{q}}\,\lambda(du)\\ &=: \frac{1}{q}\int_{(0,pq)} f_q(u)\frac{1}{-\log\frac{u}{q}}\,\lambda(du).\end{aligned}$$

In the following it is verified that the last integral is bounded above and below. For all $q\ge 1/p^2$, the last integral is obviously larger than

$$\int_1^{\sqrt{q}} f_q(u)\frac{1}{-\log\frac{u}{q}}\,du \;\ge\; \frac{1}{\log q}\int_1^{\sqrt{q}} f_q(u)\,du \;\sim\; \frac{1}{2}, \qquad q\to\infty,$$

where the last asymptotics was already verified in Example 6.1 b). In order to obtain an upper bound we decompose the integral into three integrals $\int_0^1 ...\,du$, $\int_1^{\sqrt{q}} ...\,du$, and $\int_{\sqrt{q}}^{pq} ...\,du$. To handle the first part it suffices to use the crude bound $f_q(u)\le 1/2$ and we obtain

$$\int_0^1 f_q(u)\frac{1}{-\log\frac{u}{q}}\,du \;\le\; \frac{1}{\log q}\int_0^1 f_q(u)\,du \;\le\; \frac{1}{2\log q} \;\to\; 0, \qquad q\to\infty.$$

Using the bound $f_q(u)\le 1/u$ we obtain for the second integral part the upper bound

$$\int_1^{\sqrt{q}} f_q(u)\frac{1}{-\log\frac{u}{q}}\,du \;\le\; \frac{1}{\log\sqrt{q}}\int_1^{\sqrt{q}} f_q(u)\,du \;\le\; \frac{1}{\log\sqrt{q}}\int_1^{\sqrt{q}}\frac{1}{u}\,du \;=\; 1.$$

For the third and last part we use the bound $f_q(u)\le q/u^3$ and obtain, with $\kappa_p:=1/(-\log p)$, the upper bound

$$\int_{\sqrt{q}}^{pq} f_q(u)\frac{1}{-\log\frac{u}{q}}\,du \;\le\; \kappa_p\int_{\sqrt{q}}^{pq} f_q(u)\,du \;\le\; \kappa_p\int_{\sqrt{q}}^{pq}\frac{q}{u^3}\,du \;=\; \kappa_p\Big[-\frac{q}{2u^2}\Big]_{\sqrt{q}}^{pq} \;\le\; \frac{\kappa_p}{2}.$$

Thus, we have shown that $b_c(q)=\Theta(1/q)$ as $q\to\infty$, and, hence $(d/dq)(\phi(q)/q)=\Theta(1/q)$ as $q\to\infty$. This implies that $\phi(q)=\Theta(q\log q)$ as $q\to\infty$. Similarly it can be verified that $\delta(q)=\Theta(q\log q)$ as $q\to\infty$. In particular, $\int_2^\infty dq/\delta(q)=\infty$ and $\int_2^\infty dq/\phi(q)=\infty$, so condition (2.3) is not satisfied.

Intuitively, $\mathrm{E}(T_n)$ should behave as $\sum_{k=2}^n 1/\delta(k)$ which (up to a constant) should behave as $\int_2^n 1/(q\log q)\,dq \sim \log\log n$ as $n\to\infty$. We therefore conjecture that $\mathrm{E}(T_n)\sim c\log\log n$ as $n\to\infty$ for some constant $c>0$, which would imply that the coalescent stays infinite.

The following table summarizes the results of the examples considered so far.

	Example 6.1 a)	Example 6.1 b)	Example 6.1 c)
parameter	$\alpha \in (1, \infty)$	$p \in (0, \frac{1}{2}]$	$p > 0$ with $p^e \leq \frac{1}{2}$
p_m	$(m+1)^{-\alpha}$	p^m	p^{e^m}
$\gamma(q)$, $\psi(q)$	$\sim c_\alpha q^{1+\frac{1}{\alpha}}$	$\sim \kappa_p q \log q$	$\sim q \log \log q$
$\delta(q)$, $\phi(q)$	$\sim d_\alpha q^{1+\frac{1}{\alpha}}$	$\sim \kappa_p q (\log q)^2$	$\Theta(q \log q)$
coming down	yes	yes	Conjecture: no

Table 1: Summary of Examples 6.1 a), b) and c)

Constants in the table:

$$c_\alpha := \frac{1}{\alpha} \int_0^\infty \frac{e^{-u} - 1 + u}{u^{2+1/\alpha}}\, du,\; d_\alpha := \int_0^\infty \frac{1 - e^{-u} - ue^{-u}}{u^{1+1/\alpha}(1 - e^{-u})}\, du,\; \kappa_p := \frac{1}{-\log p}.$$

All the coalescents in the following more general Example 6.2 are again critical, however the more general construction allows for coalescents which, in contrast to those studied in Example 6.1, satisfy $\int_\Delta 1/(1-|x|)\, \Xi(dx) < \infty$.

Example 6.2. As in Example 6.1, let $(p_m)_{m\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 < p_m \leq 1/2$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^\infty p_m < \infty$. Furthermore, let $f : \mathbb{N} \to [1/2, 1)$ be a function such that $\lim_{m\to\infty} f(m) = 1$. Assume that Ξ assigns for each $m \in \mathbb{N}$ mass p_m to the point $x^{(m)} \in \Delta$ whose first $\lfloor f(m)/p_m \rfloor$ coordinates are all equal to p_m and all other coordinates are equal to 0. Example 6.1 corresponds to $f(m) := 1 - p_m$. Schweinsberg's example Schweinsberg (2000a, p. 42, Example 34) corresponds to $p_m := 2^{-m}$ and $f(m) := 1 - 2^{-m}$, $m \in \mathbb{N}$. Limic Limic (2010, p. 231) considers the situation $p_m := (1/2)^m$, $m \in \mathbb{N}$, with general f. Note that $\Xi(\Delta) = \sum_{m=1}^\infty p_m < \infty$. Moreover, $|x^{(m)}| = \lfloor f(m)/p_m \rfloor p_m \leq f(m) < 1$ for all $m \in \mathbb{N}$ and, hence, $\Xi(\Delta_f) = 0$ and $\Xi(\Delta^*) = 0$. The assumptions $p_m \leq 1/2$ and $f(m) \geq 1/2$ for all $m \in \mathbb{N}$ ensure that $f(m)/p_m \geq 1$, i.e. that $\Xi(\{0\}) = 0$, so excludes a Kingman part. In order to verify that the coalescents is critical fix $\varepsilon \in (0,1)$. Since $|x^{(m)}| = \lfloor f(m)/p_m \rfloor p_m \geq (f(m)/p_m - 1)p_m = f(m) - p_m \to 1$ as $m \to \infty$, there exists a constant $m_0 = m_0(\varepsilon)$ (which may depend on $(p_m)_{m\in\mathbb{N}}$ and f) such that $|x^{(m)}| > 1 - \varepsilon$ for all $m > m_0$. Thus,

$$\nu(\Delta \setminus \Delta^\varepsilon) \;\geq\; \sum_{m>m_0} \nu(\{x^{(m)}\}) \;=\; \sum_{m>m_0} \frac{p_m}{\lfloor f(m)/p_m \rfloor p_m^2} \;\geq\; \sum_{m>m_0} \frac{1}{f(m)} \;=\; \infty,$$

so the coalescent is critical and, in particular, not regular. Note that the integral

$$\int_\Delta \frac{1}{1-|x|}\, \Xi(dx) \;=\; \sum_{m=1}^\infty \frac{p_m}{1 - |x^{(m)}|} \;=\; \sum_{m=1}^\infty \frac{p_m}{1 - \lfloor f(m)/p_m \rfloor p_m} \tag{6.12}$$

can be finite or infinite depending on the choice of the sequence $(p_m)_{m\in\mathbb{N}}$ and the function f. In Example 6.1, where $f(m) := 1 - p_m$, the integral (6.12) is infinite, as already shown in Example 6.1. In contrast, if $f(m)$ grows sufficiently slowly to 1 as $m \to \infty$, for example $f(m) := 1 - \sqrt{p_m}$, then the integral (6.12) is bounded above by $\sum_{m=1}^\infty p_m/(1 - f(m)) = \sum_{m=1}^\infty \sqrt{p_m}$, which is finite whenever the series $\sum_{m=1}^\infty \sqrt{p_m}$ converges. Because of the complexity of this example, we do not analyze it further here.

Let us finally provide an example which, in contrast to the previous examples, satisfies $\nu(\Delta^*) = \infty$.

Example 6.3. Let $(p_m)_{m\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 < p_m < 1$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^\infty (1-p_m) < \infty$, for example $p_m := 1 - 1/(m(m+1))$ or $p_m := 1-(m+1)^{-\alpha}$ for some constant $\alpha \in (1,\infty)$. Assume that Ξ assigns for each $m \in \mathbb{N}$ mass $1-p_m$ to the point $x^{(m)} := (1-p_m)(1,p_m,p_m^2,\ldots) \in \Delta$. Note that $\Xi(\Delta) = \sum_{m=1}^\infty (1-p_m) < \infty$ by assumption, so the measure Ξ is finite. Clearly, $\Xi(\Delta_f) = 0$. Each point $x^{(m)}$, $m \in \mathbb{N}$, satisfies $|x^{(m)}| = (1-p_m)\sum_{i=1}^\infty p_m^{i-1} = 1$, so the measure Ξ (and hence also ν) is concentrated on Δ^*. Moreover, for all $m \in \mathbb{N}$,

$$(x^{(m)}, x^{(m)}) \;=\; (1-p_m)^2 \sum_{i=1}^\infty p_m^{2(i-1)} \;=\; \frac{(1-p_m)^2}{1-p_m^2} \;=\; \frac{1-p_m}{1+p_m}.$$

Thus,

$$\nu(\Delta^*) \;=\; \sum_{m=1}^\infty \nu(\{x^{(m)}\}) \;=\; \sum_{m=1}^\infty \frac{\Xi(\{x^{(m)}\})}{(x^{(m)},x^{(m)})} \;=\; \sum_{m=1}^\infty (1+p_m) \;=\; \infty,$$

in contrast to the previously studied examples. In particular, $\nu(\Delta\setminus\Delta^\varepsilon) = \nu(\Delta^*) = \infty$ for all $\varepsilon \in (0,1)$, so the coalescent is critical. Note that

$$\begin{aligned}
\psi(q) &= \int_\Delta \sum_{i=1}^\infty (e^{-qx_i} - 1 + qx_i)\,\nu(dx)\\
&= \sum_{m=1}^\infty \nu(\{x^{(m)}\}) \sum_{i=1}^\infty \big(e^{-q(1-p_m)p_m^{i-1}} - 1 + q(1-p_m)p_m^{i-1}\big)\\
&= \sum_{m=1}^\infty (1+p_m) \sum_{i=0}^\infty \big(e^{-q(1-p_m)p_m^{i}} - 1 + q(1-p_m)p_m^{i}\big)\\
&= \sum_{m=1}^\infty (1+p_m) \sum_{i=0}^\infty \sum_{k=2}^\infty \frac{(-q(1-p_m)p_m^i)^k}{k!}\\
&= \sum_{m=1}^\infty (1+p_m) \sum_{k=2}^\infty \frac{(-q)^k}{k!}(1-p_m)^k \sum_{i=0}^\infty p_m^{ik}\\
&= \sum_{m=1}^\infty (1+p_m) \sum_{k=2}^\infty \frac{(-q)^k}{k!}\,\frac{(1-p_m)^k}{1-p_m^k}, \qquad q \ge 0.
\end{aligned}$$

It seems to be not straightforward to determine the asymptotics of $\psi(q)$ as $q \to \infty$. Concerning the coming down from infinity problem, coalescents satisfying $\nu(\Delta^*) = \infty$, as constructed in this Example 6.3, belong probably to the more complicated ones.

7. Appendix

For $x = (x_1, x_2, \ldots) \in \Delta$ let $X := (X_0, X_1, \ldots) := (X_0(m,x), X_1(m,x), \ldots)$ have an infinite multinomial distribution with parameters $m \in \mathbb{N}$ and $(x_0, x_1, x_2, \ldots)$, where $x_0 := 1 - |x|$. Recall that $P(\bigcap_{i=0}^\infty \{X_i = m_i\}) = m! \prod_{i=0}^\infty x_i^{m_i}/m_i!$ for all $m_0, m_1, \ldots \in \mathbb{N}_0$ with $\sum_{i=0}^\infty m_i = m$. In this appendix we provide formulas for the distribution of the random variable $Y := Y(m,x) := X_0(m,x) + \sum_{i=1}^\infty 1_{\{X_i(m,x)\ge 1\}}$. The results extent those in the appendix of Berestycki et al. (2010) to the infinite multinomial case. In the following δ_{mk} denotes the Kronecker symbol.

Lemma 7.1. *The random variable* $Y := Y(m,x)$, $m \in \mathbb{N}$, $x = (x_1, x_2, \ldots) \in \Delta$, *takes the value* $k \in \{1, \ldots, m\}$ *with probability*

$$\begin{aligned} & P(Y = k) \\ = \;& \delta_{mk} x_0^m + \sum_{l=1}^{k} \binom{m}{k-l} x_0^{k-l} \sum_{\substack{i_1,\ldots,i_l \in \mathbb{N} \\ i_1 < \cdots < i_l}} \sum_{\substack{n_1,\ldots,n_l \in \mathbb{N} \\ n_1+\cdots+n_l = m-k+l}} \frac{(m-k+l)!}{n_1! \cdots n_l!} x_{i_1}^{n_1} \cdots x_{i_l}^{n_l} \quad (7.1) \\ = \;& \delta_{mk} x_0^m + \sum_{l=1}^{k} \binom{m}{k-l} x_0^{k-l} \sum_{\substack{i_1,\ldots,i_l \in \mathbb{N} \\ i_1 < \cdots < i_l}} \sum_{j=1}^{l} (-1)^{l-j} \sum_{\substack{A \subseteq \{1,\ldots,l\} \\ |A| = j}} \Big(\sum_{a \in A} x_{i_a} \Big)^{m-k+l}. \quad (7.2) \end{aligned}$$

Remark 7.2. 1. The expression on the right hand side in (7.2) is well defined even for all real parameters $m \in [0, \infty)$ and all integers k satisfying $0 \le k \le m+1$, which formally allows to extend the definition of $P(Y(m,x) = k)$ in a natural way to real parameter $m \in [0, \infty)$.

2. If $x_1 = \cdots = x_L =: p$ for some constant $L \in \mathbb{N}$ and $x_i = 0$ for all $i > L$, then the distribution of $Y(m,x)$ simplifies to

$$\begin{aligned} & P(Y(m,x) = k) \\ = \;& \delta_{mk} x_0^m + \sum_{l=1}^{k} \binom{m}{k-l} x_0^{k-l} \sum_{\substack{i_1,\ldots,i_l \in \{1,\ldots,L\} \\ i_1 < \cdots < i_l}} p^{m-k+l} \sum_{\substack{n_1,\ldots,n_l \in \mathbb{N} \\ n_1+\cdots+n_l = m-k+l}} \frac{(m-k+l)!}{n_1! \cdots n_l!} \\ = \;& \delta_{mk} x_0^m + \sum_{l=1}^{k} \binom{m}{k-l} x_0^{k-l} \binom{L}{l} p^{m-k+l}\, l!\, S(m-k+l, l), \qquad k \in \{1, \ldots, m\}, \end{aligned}$$

where $x_0 = 1 - Lp$ and $S(.,.)$ denote the Stirling numbers of the second kind. For $L = 1$ (Λ-coalescent) the distribution of $Y(m,x)$ simplifies considerably to

$$P(Y(m,x) = k) \; = \; \delta_{mk} x_0^m + \binom{m}{k-1} x_0^{k-1} x_1^{m-(k-1)}, \qquad k \in \{1, \ldots, m\}.$$

Proof: (of Lemma 7.1) Fix $m \in \mathbb{N}$ and $x \in \Delta$. For all $k \in \{1, \ldots, m\}$,

$$\begin{aligned} & P(Y = k) \\ = \;& \sum_{l=0}^{k} P\Big(X_0 = k - l, \sum_{i=1}^{\infty} 1_{\{X_i \ge 1\}} = l\Big) \\ = \;& P(X_0 = k, X_i = 0 \ \forall\, i \in \mathbb{N}) + \\ & \sum_{l=1}^{k} \sum_{\substack{i_1,\ldots,i_l \in \mathbb{N} \\ i_1 < \cdots < i_l}} P(X_0 = k - l, X_{i_1} \ge 1, \ldots, X_{i_l} \ge 1, X_i = 0 \ \forall\, i \in \mathbb{N} \setminus \{i_1, \ldots, i_l\}) \\ = \;& \delta_{mk} x_0^m + \sum_{l=1}^{k} \binom{m}{k-l} x_0^{k-l} \sum_{\substack{i_1,\ldots,i_l \in \mathbb{N} \\ i_1 < \cdots < i_l}} \sum_{\substack{n_1,\ldots,n_l \in \mathbb{N} \\ n_1+\cdots+n_l = m-k+l}} \frac{(m-k+l)!}{n_1! \cdots n_l!} x_{i_1}^{n_1} \cdots x_{i_l}^{n_l}, \end{aligned}$$

which is (7.1). Applying the sieve formula

$$\sum_{\substack{n_1,\ldots,n_l \in \mathbb{N} \\ n_1+\cdots+n_l = n}} \frac{n!}{n_1! \cdots n_l!} p_1^{n_1} \cdots p_l^{n_l} \; = \; \sum_{j=1}^{l} (-1)^{l-j} \sum_{\substack{A \subseteq \{1,\ldots,l\} \\ |A| = j}} \Big(\sum_{a \in A} p_a \Big)^n, \qquad n, l \in \mathbb{N},$$

with $p_1 := x_{i_1}, \ldots, p_l := x_{i_l}$ and $n := m - k + l$ to the last sum in (7.1) yields (7.2). □

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