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Fluctuations of the Magnetization for Ising models on Erdős-Rényi random graphs – the regimes of low temperature and external magnetic field

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Abstract. We continue our analysis of Ising models on the (directed) Erdős-Rényi random graph $G(N, p)$. We prove a quenched Central Limit Theorem for the magnetization and describe the fluctuations of the log-partition function. In the current note we consider the low temperature regime $\beta > 1$ and the case when an external magnetic field is present. In both cases, we assume that $p = p(N)$ satisfies $p^3 N \to \infty$.

1. Introduction and main results

1.1. *Description of the model.* In this paper we continue our investigation of Ising models on the Erdős-Rényi random graph. Technically speaking they are disordered ferromagnets in the sense of Fröhlich's lecture Fröhlich (1986). The model we are studying was introduced and first analyzed by Bovier and Gayrard in Bovier and Gayrard (1993). The topology of this model is given by a directed Erdős-Rényi graph $G = G(N, p)$. Its vertex set is the set $\{1, \dots, N\}$, and each of the directed edges (i, j) is realized with probability $p \in (0, 1]$ independently of all other edges. For simplicity, the case $i = j$ is admitted. The random variables $\varepsilon_{i,j}^N = \varepsilon_{i,j}, i, j \in \{1, \dots, N\}$, which indicate whether an edge (i, j) is present or not, are thus assumed to be i.i.d. with distribution

$$\mathbb{P}[\varepsilon_{i,j} = 1] = p, \qquad \mathbb{P}[\varepsilon_{i,j} = 0] = 1 - p.$$

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Our general assumption is that $p = p(N)$ satisfies $p^3 N \to \infty$ as $N \to \infty$. This is more than enough to ensure that asymptotically almost surely the graph is connected. It is likely that one could prove variants of our central results also under the weaker assumption $pN \to \infty$, but our main technique runs into problems.

On a fixed realization of this Erdős-Rényi random graph G we define the Hamiltonian or energy function of the Ising model. It is given by the function

$$H = H_N : \{-1, +1\}^N \to \mathbb{R}$$

defined as

$$H(\sigma) := -\frac{1}{2Np} \sum_{i,j=1}^{N} \varepsilon_{i,j} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \tag{1.1}$$

for $\sigma = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$. We note that H also depends on the parameter h, but for the sake of readability we suppress this dependence in the notation. Here $h > 0$ plays the role of an external magnetic field. With such an energy function H we associate a Gibbs measure on $\{-1, +1\}^N$. This is a random probability measure with respect to the randomness encoded by the $(\varepsilon_{i,j})_{i,j=1}^N$. It is given by

$$\mu_{\beta,h,N}(\sigma) := \frac{1}{Z_N(\beta, h)} \exp(-\beta H(\sigma)), \qquad \sigma \in \{-1, +1\}^N, \tag{1.2}$$

where $\beta > 0$ is called the inverse temperature. The normalizing constant is the partition function given by

$$Z_N(\beta, h) := \sum_{\sigma \in \{-1,+1\}^N} \exp(-\beta H(\sigma)).$$

The well-studied Curie-Weiss model is a special case of this setup, namely the situation where $p \equiv 1$. For a survey over many results on this model see Ellis (2006). The Curie-Weiss model is an interesting model for ferromagnetism because it exhibits a phase transition at the critical temperature $\beta_c = 1$. This phase transition can be seen by analyzing the magnetization per particle

$$m_N(\sigma) := \frac{\sum_{i=1}^N \sigma_i}{N} = \frac{|\sigma|}{N}.$$

Here we have introduced the notation

$$|\sigma| := \sum_{i=1}^{N} \sigma_i = N m_N(\sigma).$$

To avoid possible confusion, observe that $|\sigma|$ can be negative. In the Curie-Weiss model with $h = 0$ the distribution of the magnetization per particle m_N under the Gibbs measure converges to

$$\frac{1}{2}(\delta_{m^+(\beta)} + \delta_{m^-(\beta)}).$$

Here δ_x is the Dirac-measure in a point x, while $m^+(\beta)$ is the largest solution of

$$z = \tanh(\beta z), \tag{1.3}$$

and $m^-(\beta) = -m^+(\beta)$. Observe that for $\beta \leq 1$ the above equation (1.3) has only the trivial solution $m^+(\beta) = 0$. Therefore in the high temperature regime $\beta \leq 1$ the magnetization per particle m_N converges to 0 in probability. For $\beta > 1$ the largest solution of (1.3) is strictly positive. Hence in the low temperature regime $\beta > 1$ the magnetization m_N is asymptotically concentrated in two values, a positive one and a negative one.

Let us now turn to the Curie-Weiss model with $h > 0$ and $\beta > 0$. Then, it is known Ellis (2006) that the distribution of the magnetization per particle m_N under the Gibbs measure converges to $\delta_{m^+(\beta,h)}$. Here $m^+(\beta,h)$ is the largest solution of

$$z = \tanh(\beta(z + h)). \tag{1.4}$$

In Bovier and Gayrard (1993) the authors show that the same limit theorems for m_N hold in the dilute Curie-Weiss Ising model, that we defined in (1.1) and (1.2), as long as $pN \to \infty$. More concretely, if $\beta \leq 1$ and $h = 0$, then for almost all realizations of the random graph, the magnetization m_N converges to 0 in probability under the Gibbs measure, while for $\beta > 1$ and $h = 0$ it converges to $\frac{1}{2}(\delta_{m^+(\beta)} + \delta_{m^-(\beta)})$ in distribution. Moreover, for all $\beta > 0$ and $h > 0$, m_N again converges in probability to $\delta_{m^+(\beta,h)}$. This latter fact is not explicitly stated in Bovier and Gayrard (1993), however, it can be easily derived with the methods introduced there.

The starting point of our investigations in Kabluchko et al. (2019), Kabluchko et al. (2021), and Kabluchko et al. (2020), as well as the current note, is the observation that in the Curie-Weiss model one can also prove a Central Limit Theorem for the magnetization when $\beta < 1$ and $h = 0$ (see, e.g. Chatterjee and Shao (2011), Eichelsbacher and Löwe (2010), Ellis (2006), Ellis and Newman (1978)). These references show that $\sqrt{N} m_N$ converges in distribution to a normal random variable with mean 0 and variance $\frac{1}{1-\beta}$. Furthermore, as can be expected from this result, at $\beta = 1$, there is no such standard Central Limit Theorem and one has to scale in a different way. The result is that $\sqrt[4]{N} m_N$ converges in distribution to a non-normal random variable with density proportional to $\exp(-\frac{1}{12}x^4)$ with respect to the Lebesgue measure. If $\beta > 1$, one has to consider the conditional distribution of $\sqrt{N}(m_N - m^+(\beta))$ conditioned to m_N being positive. In this case $\sqrt{N}(m_N - m^+(\beta))$ conditioned on $m_N > 0$ converges in distribution to a normal distribution with expectation 0 and variance

$$\sigma^2(\beta, 0) := \sigma^2(\beta) := \frac{1 - m^+(\beta)^2}{1 - \beta(1 - m^+(\beta)^2)}. \tag{1.5}$$

Similarly, when $h > 0$ the random variable $\sqrt{N}(m_N - m^+(\beta,h))$ converges in distribution to a normal distribution with expectation 0 and variance

$$\sigma^2(\beta, h) := \frac{1 - m^+(\beta,h)^2}{1 - \beta(1 - m^+(\beta,h)^2)}. \tag{1.6}$$

These results can be found in Ellis et al. (1980).

The general question we have been investigating in previous articles was, whether such limit theorems also hold in the dilute setting introduced above. To state our results let us introduce the following *random* element of the space of finite measures on $\mathbb{R}$, denoted by $\mathcal{M}(\mathbb{R})$:

$$L_N := \frac{1}{Z_N(\beta,h)} \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \delta_{\frac{1}{\sqrt{N}}(\sum_{i=1}^N \sigma_i - Nm)}, \tag{1.7}$$

where m is either $m^+(\beta)$ (resp. $m^-(\beta)$) or $m^+(\beta,h)$ depending on the case we consider. In the notation, we suppress the dependence of L_N on β and h. Then the probability measure L_N is random, since it depends on the random variables $\varepsilon_{i,j}, i,j \in \{1,\ldots,N\}$. In Kabluchko et al. (2019) we showed that, if $p^3N^2 \to \infty$, $\beta < 1$, and $h = 0$ the random element L_N converges in probability to the normal distribution with mean 0 and variance $\frac{1}{1-\beta}$, denoted by $\mathfrak{N}_{0,1/(1-\beta)}$, which is an element in $\mathcal{M}(\mathbb{R})$. In Kabluchko et al. (2020) we extended this result to the situation of $pN \to \infty$, $\beta < 1$, and $h = 0$. Moreover, for $p^4N^3 \to \infty$, $\beta = 1$, and $h = 0$ we considered

$$L_N^1 := \frac{1}{Z_N(\beta,h)} \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \delta_{\frac{1}{N^{3/4}} \sum_{i=1}^N \sigma_i}$$

and showed that it converges in probability to $\mathfrak{M} \in \mathcal{M}(\mathbb{R})$, the probability measure with density

$$\psi(x) := \frac{\mathrm{e}^{-\frac{1}{12}x^4}}{\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{12}y^4} dy}.$$

For smaller values of p, $\beta = 1$, and $h = 0$ we showed that suitable versions of L_N^1 again have a normal distribution as limiting distribution (again in probability). Finally, in Kabluchko et al. (2021) we analyzed the fluctuations of the random partition function $Z_N(\beta, h)$.

Note that the situation we analyzed in Kabluchko et al. (2019), Kabluchko et al. (2021), and Kabluchko et al. (2020), as well as in the present note is of a different character than the results for sparse (Erdős-Rényi) graphs. Such situations were deeply studied by Dembo and Montanari in Dembo and Montanari (2010a) and Dembo and Montanari (2010b) and by van der Hofstad and coauthors in Dommers et al. (2016), Dommers et al. (2010), Dommers et al. (2014), Giardinà et al. (2016), and Giardinà et al. (2015). In particular, we will comment on related results by Giardina et al. in Giardinà et al. (2016) and Giardinà et al. (2015) at the end of the next subsection.

1.2. *Main results.* As announced, in this note we will study the fluctuations of m_N, when either $h = 0$ and $\beta > 1$, or when $h > 0$ and $\beta > 0$ is arbitrary. In both cases we require that p is such that $p^3 N \to \infty$. The results will be formulated in terms of the quantity L_N defined in (1.7) and related quantities L_N^+ and L_N^- to be defined below. Recall that L_N is a random element of $\mathcal{M}(\mathbb{R})$, the set of finite measures on $\mathbb{R}$ and that $\mathcal{M}(\mathbb{R})$. We endow $\mathcal{M}(\mathbb{R})$ with the topology of weak convergence and denote by ρ_{weak} any metric generating the weak topology and turning $\mathcal{M}(\mathbb{R})$ into a complete separable metric space.

Our main results are Central Limit Theorems for m_N. The first one is

Theorem 1.1. *Assume that $h = 0$, $\beta > 1$, and $p^3 N \to \infty$ as $N \to \infty$. Then,*

$$L_N^+ := \frac{2}{Z_N(\beta, h)} \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \delta_{\frac{1}{\sqrt{N}}(\sum_{i=1}^N \sigma_i - N m^+(\beta))} 1_{\sum_{i=1}^N \sigma_i > 0},$$

considered as a random element of the set $\mathcal{M}(\mathbb{R})$, converges in probability to the normal distribution $\mathfrak{N}_{0,\sigma^2(\beta)}$, which is considered as a deterministic element of $\mathcal{M}(\mathbb{R})$. Here the variance $\sigma^2(\beta)$ is given by (1.5). In other words, for every $\varepsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}[\rho_{weak}\big(L_N^+, \mathfrak{N}_{0,\sigma^2(\beta)}\big) > \varepsilon] = 0.$$

An analogous assertion holds true, if, in the definition of L_N^+, we replace $m^+(\beta)$ by $-m^+(\beta) = m^-(\beta)$ and restrict our attention to configurations with negative magnetization, i.e. if we consider

$$L_N^- := \frac{2}{Z_N(\beta, h)} \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \delta_{\frac{1}{\sqrt{N}}(\sum_{i=1}^N \sigma_i - N m^-(\beta))} 1_{\sum_{i=1}^N \sigma_i \le 0}.$$

Remark 1.2. Intuitively, approximately one half of the configurations are such that the magnetization per particle is close to $m^+(\beta)$, whereas for the other half it is close to $m^-(\beta)$. This is very different from the high-temperature setting considered in the previous publications Kabluchko et al. (2019) and Kabluchko et al. (2020), where the magnetization per particle was concentrated near 0. In L_N^+ we only take the configurations with positive magnetization into account, which is why a factor of 2 is necessary in its definition. Without the factor, the limit would be a measure of total mass 1/2.

Our second main theorem is

Theorem 1.3. *Assume that $h > 0, \beta > 0$, and $p^3 N \to \infty$ as $N \to \infty$. Then,*

$$L_N := \frac{1}{Z_N(\beta,h)} \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \delta_{\frac{1}{\sqrt{N}}(\sum_{i=1}^N \sigma_i - N m^+(\beta,h))},$$

considered as a random element of $\mathcal{M}(\mathbb{R})$, converges in probability to $\mathfrak{N}_{0,\sigma^2(\beta,h)}$. Here $m^+(\beta,h)$ and $\sigma^2(\beta,h)$ are given by (1.4) and (1.6). In other words, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}[\rho_{weak}(L_N, \mathfrak{N}_{0,\sigma^2(\beta,h)}) > \varepsilon] = 0.$$

In Kabluchko et al. (2021) we also analyzed the fluctuations of $Z_N(\beta,h)$ for $\beta < 1$, $h = 0$. Before we state the corresponding result for $h = 0$, $\beta > 1$ and for $h > 0, \beta > 0$ we introduce some notation. We set

$$m := \begin{cases} m^+(\beta) & \text{if } h = 0, \beta > 1 \\ m^+(\beta,h) & \text{if } h > 0, \beta > 0 \end{cases}$$

and

$$\tilde{Z}_N(\beta,h) := \sum_{\sigma \in \{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \exp\Big(-\frac{\beta}{2Np} m^2 \sum_{i,j=1}^N \varepsilon_{i,j} \Big). \tag{1.8}$$

We comment on the role of $\tilde{Z}_N(\beta,h)$ at the beginning of Section 3. In the following theorem, it is possible to replace $\mathbb{E}\tilde{Z}_N(\beta,h)$ by its asymptotic forms given in (3.19) below and (4.1), but we prefer not to state these long expressions explicitly.

Theorem 1.4. *Assume that $h = 0$, $\beta > 1$ or $h > 0$, $\beta > 0$. Further assume that $p = p(N)$ is such that $p^3 N \to \infty$ as $N \to \infty$ and, moreover, that $p(N)$ is bounded away from 1. Then,*

$$\frac{\log(Z_N(\beta,h)/\mathbb{E}\tilde{Z}_N(\beta,h)) - \frac{\beta N m^2}{2}}{\sqrt{\frac{\beta^2 m^4 (1-p)}{4p}}} \to \mathfrak{N}_{0,1}$$

in distribution.

For fixed $0 < p < 1$ that does not depend on N this may be rewritten as

$$\frac{Z_N(\beta,h)/\mathbb{E}\tilde{Z}_N(\beta,h)}{\mathrm{e}^{\frac{\beta N m^2}{2}}} \to \mathrm{e}^{\zeta}$$

in distribution. Here ζ denotes a normal random variable with expectation 0 and variance $\frac{\beta^2 m^4(1-p)}{4p}$.

We will prepare the proof of these theorems analytically in the following section. The actual proofs will follow in Sections 3–5.

Remark 1.5. In Giardinà et al. (2016) and Giardinà et al. (2015) the authors prove Central Limit Theorems for the magnetization of Ising models on *sparse* random graphs. They consider the distribution of the magnetization under three different measures: the random quenched setting, the averaged quenched measure, and the annealed measure. The first of these situations is (apart from the random graph considered) similar to our setting. The second situation considers the measure $\mu_{\beta,h,N}$ averaged over the distribution of the random edges in the graph, while the annealed measure is given by $\mu^{\mathrm{an}}_{\beta,h,N} = \frac{1}{\mathbb{E}Z_N(\beta,h)} \mathbb{E}\exp(-\beta H(\sigma))$, where the expectation is with respect to the bond distribution. The results in Giardinà et al. (2016, Theorem 1.3) and Giardinà et al. (2015, Theorems 1.3 and 1.5) indicate that for sparse graphs there is a clear distinction between the (averaged) quenched case and the annealed case. Under all three of the above measures the magnetization satisfies a CLT. However, while the variances for the normal distribution under the random quenched measure and the averaged quenched measure agree, the variance for the annealed measure in general differs from them. In our setting the annealed CLT is easy to analyze using the techniques developed in Kabluchko et al. (2019), Kabluchko et al. (2020), as well as in the next two sections. The result is that also under the annealed measure the magnetization obeys a CLT with the same variance as in the random quenched case. Given that even in the sparse situation the asymptotic variances of the distribution of the magnetizations agree under the random quenched and the averaged quenched measure and that a large expectation of the degree distribution has an averaging effect for the important observables, it appears likely that also under the averaged quenched measure m_N obeys a CLT with the same variance as in the other two situations. However, this is rather difficult to prove and thus cannot be done in the article.

Remark 1.6. While we were working on this manuscript Deb and Mukherjee published some really interesting results on the fluctuations of the magnetization of Ising models on general almost regular graphs in Deb and Mukherjee (2020). Their results partially confirm our results in Kabluchko et al. (2019) and Kabluchko et al. (2020), as well as those of Theorems 1.1 and 1.3. However, their techniques are completely different from ours. On the other hand, our third main result, Theorem 1.4, is new. Its proof relies on the results developed in order to prove Theorems 1.1 and 1.3.

2. Technical preparation

In the proof of our main theorems we will encounter some functions for which we will need an expansion up to certain orders. These functions will be studied and analyzed in this section.

More precisely, for arbitrary complex variables z and p let us define the function

$$F(p,z) := \log(1-p+p\mathrm{e}^z). \tag{2.1}$$

Note that in this section we do not assume that p is a probability. We will next compute the power series expansion of some linear combinations of $F(p,z)$ in p and z variables around the origin $(0,0)$.

Note that, for $|p|<2$ and $|z|<z_0$ with sufficiently small $z_0>0$, we can estimate that

$$|p(\mathrm{e}^z-1)|<1.$$

Thus, the function $F(p,z)$ is an analytic function of two complex variables p and z on the domain

$$\mathcal{D}=\{(p,z)\in\mathbb{C}^2\colon |p|<2, |z|<z_0\}.$$

Therefore, it has a power series expansion which converges uniformly and absolutely on compact subsets of this domain. In particular, by absolute convergence, we can re-arrange and re-group the terms arbitrarily. We will use the following first terms of the power series expansion in our computations:

$$F(p,z)=pz+\frac{p(1-p)}{2}z^2+\frac{p(2p^2-3p+1)}{6}z^3+\frac{p(-6p^3+12p^2-7p+1)}{24}z^4+\mathcal{O}(z^5). \tag{2.2}$$

The following claim has already been proven in Kabluchko et al. (2019), Lemma 1.

Lemma 2.1.

$$F(p,z)=p\sum_{k=1}^{\infty}\frac{P_k(p)}{k!}z^k,$$

where $P_k(p)$ is a power series in p with constant term $P_k(0)=1$ for all $k\in\mathbb{N}$.

Corollary 2.2. *If z and y range in a compact subset of $\mathbb{R}$ and p,γ are such that $(p,\gamma(\pm z+y))\in\mathcal{D}$, then*

(i)

$$\frac{F(p,\gamma(z+y))+F(p,\gamma(-z+y))}{2}=p\gamma y+\frac{1}{2}p(1-p)\gamma^2(z^2+y^2)+\mathcal{O}(p\gamma^3),$$

(ii)
$$\frac{F(p,\gamma(z+y)) - F(p,\gamma(-z+y))}{2} = p\gamma z + p(1-p)\gamma^2 zy + \mathcal{O}(p\gamma^3).$$

In both cases, the constant in the $\mathcal{O}$-term is uniform as long as z, y stay in compact subsets of $\mathbb{R}$ and $(p, \gamma(\pm z + y)) \in \mathcal{D}$.

Proof: From Lemma 2.1 we have

$$\frac{F(p,\gamma(z+y)) + F(p,\gamma(-z+y))}{2} = \frac{p}{2}\sum_{k=1}^{\infty} \frac{P_k(p)}{k!}\gamma^k \Big((z+y)^k + (-z+y)^k\Big)$$
$$= p\gamma y + \frac{1}{2}p(1-p)\gamma^2(z^2+y^2) + \frac{p}{2}\gamma^3 \sum_{k=3}^{\infty} \frac{P_k(p)}{k!}\gamma^{k-3}\Big((z+y)^k + (-z+y)^k\Big)$$
$$= p\gamma y + \frac{1}{2}p(1-p)\gamma^2(z^2+y^2) + \mathcal{O}(p\gamma^3).$$

Similarly, we have

$$\frac{F(p,\gamma(z+y)) - F(p,\gamma(-z+y))}{2} = \frac{p}{2}\sum_{k=1}^{\infty} \frac{P_k(p)}{k!}\gamma^k \Big((z+y)^k - (-z+y)^k\Big)$$
$$= p\gamma z + p(1-p)\gamma^2 zy + \mathcal{O}(p\gamma^3).$$

□

3. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. The proof of Theorem 1.3 is quite similar and in Section 4 we will basically point out the differences between the two proofs. Hence, throughout this section we will assume that $h = 0$ and $\beta > 1$. The main idea, for both Theorem 1.1 and Theorem 1.3, is to consider the following modification of the partition function for $g \in \mathcal{C}^b(\mathbb{R})$ (meaning that g is globally bounded and continuous) such that $g \geq 0, g \not\equiv 0$

$$Z_N^+(\beta, h, g) := \sum_{\sigma \in \{-1,+1\}^N} g\bigg(\frac{|\sigma| - Nm}{\sqrt{N}}\bigg) \mathrm{e}^{-\beta H(\sigma)}\, 1_{|\sigma|>0},$$

where we put $m := m^+(\beta)$. Then we have

$$\frac{1}{2}\int_0^{+\infty} g(x) L_N^+(dx) = \mathbb{E}_{\mu_{\beta,h,N}}\bigg[g\bigg(\frac{|\sigma| - Nm}{\sqrt{N}}\bigg) 1_{|\sigma|>0}\bigg] = \frac{Z_N^+(\beta,h,g)}{Z_N(\beta,h)}, \tag{3.1}$$

where, for a fixed disorder $(\varepsilon_{i,j})_{i,j=1}^N$, $\mathbb{E}_{\mu_{\beta,h,N}}$ denotes the expectation with respect to the Gibbs measure $\mu_{\beta,h,N}$.

Instead of $Z_N^+(\beta,h,g)$ we study the related quantity $\tilde{Z}_N^+(\beta,h,g)$ in which the summands, as we will show, behave like asymptotically independent random variables. To define $\tilde{Z}_N^+(\beta,h,g)$, for $\sigma \in \{\pm 1\}^N$ and for fixed $\beta > 0$ we first introduce

$$\gamma := \frac{\beta}{2Np}$$

as well as

$$T(\sigma) := T_{\beta,N}(\sigma) := \exp\Bigg(\gamma \sum_{i,j=1}^N \varepsilon_{i,j}\sigma_i\sigma_j - \gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}\Bigg).$$

We will suppress the indices β and N in $T_{\beta,N}(\sigma)$ in the rest of this section. We will usually use $T(\sigma)$ to compute its expectation, its variance or covariances of the form $\mathrm{Cov}(T(\sigma),T(\tau))$. From the expressions that we obtain it will be obvious that $T(\sigma)$ depends on β and N.

Then, for $g \in \mathcal{C}^b(\mathbb{R})$ such that $g \geq 0, g \not\equiv 0$, we set

$$\tilde{Z}_N^+(\beta,h,g) := \sum_{\sigma\in\{-1,+1\}^N} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) T(\sigma)\, 1_{|\sigma|>0}\,.$$

Note that in (1.8) we defined a related quantity

$$\tilde{Z}_N(\beta,h) := \sum_{\sigma\in\{-1,+1\}^N} \mathrm{e}^{-\beta H(\sigma)} \exp\left(-\gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}\right),$$

which does not exactly correspond to $\tilde{Z}_N^+(\beta,h,1)$ since in the latter quantity the summation is restricted to $|\sigma|>0$. Since

$$Z_N^+(\beta,h,g) = \tilde{Z}_N^+(\beta,h,g) \exp\left(\gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}\right) \tag{3.2}$$

we have

$$\frac{Z_N^+(\beta,h,g)}{Z_N(\beta,h)} = \frac{\tilde{Z}_N^+(\beta,h,g)}{\tilde{Z}_N(\beta,h)}. \tag{3.3}$$

Let us investigate, how $\tilde{Z}_N^+(\beta,h,g)$ behaves. We start our analysis by the following computation

Lemma 3.1. *For $h=0$, $\beta>1$, and $p^3N\to\infty$ as $N\to\infty$ we have for all $\sigma\in\{-1,+1\}^N$*

$$\mathbb{E}T(\sigma) = \exp\left(\frac{\beta}{2N}(|\sigma|^2 - m^2N^2) + \frac{(1-p)\beta^2}{8p}\left(m^4 - \frac{2m^2|\sigma|^2}{N^2} + 1\right) + o(1)\right)$$

with an $o(1)$-term that is uniform over $\sigma\in\{-1,+1\}^N$.

Proof: We compute

$$\mathbb{E}T(\sigma) = \prod_{i,j=1}^N \mathbb{E}\Big[\mathrm{e}^{\gamma\varepsilon_{i,j}\sigma_i\sigma_j - \gamma m^2\varepsilon_{i,j}}\Big] = \prod_{i,j=1}^N \Big(1-p+p\mathrm{e}^{\gamma(\sigma_i\sigma_j - m^2)}\Big).$$

If we introduce

$$f(x) = f(x;p,\gamma) = \log(1-p+p\mathrm{e}^{\gamma(x-m^2)}) = F(p,\gamma(x-m^2)),$$

where F is given by (2.1), we can continue by

$$\mathbb{E}T(\sigma) = \exp\left(\sum_{i,j=1}^N \log(1-p+p\mathrm{e}^{\gamma(\sigma_i\sigma_j-m^2)})\right) = \exp\left(\sum_{i,j=1}^N f(\sigma_i\sigma_j)\right).$$

Note that $\sigma_i\in\{\pm1\}$ for all i and hence $\sigma_i\sigma_j\in\{\pm1\}$. For the two values ±1 we can rewrite f in a linear form. More precisely, we write

$$f(x) = a_0 + a_1 x, \qquad x\in\{-1,+1\}.$$

The two coefficients a_0 and a_1 naturally are dependent on p, m and γ. They can be computed from the following two equations

$$a_0 = \frac{f(1)+f(-1)}{2} = \frac{\log(1-p+p\mathrm{e}^{\gamma(1-m^2)}) + \log(1-p+p\mathrm{e}^{\gamma(-1-m^2)})}{2},$$

$$a_1 = \frac{f(1)-f(-1)}{2} = \frac{\log(1-p+p\mathrm{e}^{\gamma(1-m^2)}) - \log(1-p+p\mathrm{e}^{\gamma(-1-m^2)})}{2}$$

to obtain

$$\mathbb{E}T(\sigma) = \exp(N^2 a_0 + a_1|\sigma|^2).$$

Using Corollary 2.2 we see immediately that

$$a_0 = -\gamma p m^2 + \frac{\gamma^2}{2} p(1-p)(m^4+1) + \mathcal{O}(p\gamma^3)$$

and

$$a_1 = p\gamma - \gamma^2 p(1-p)m^2 + \mathcal{O}(p\gamma^3).$$

Thus

$$\mathbb{E}T(\sigma) = \exp\left(\frac{\beta}{2N}(|\sigma|^2 - m^2N^2) + \frac{(1-p)\beta^2}{8p}\left(m^4 - \frac{2m^2|\sigma|^2}{N^2} + 1\right) + o(1)\right)$$

with an o-term that is uniform in $\sigma \in \{-1,+1\}^N$. This was the assertion. □

We will now compute the asymptotic expectation of $\tilde{Z}_N^+(\beta,h,g)$. To this end, we will introduce the following set of spin configurations. Set

$$S_N^1 := \left\{\sigma \in \{\pm 1\}^N : \big|\,|\sigma| - Nm\,\big| \le \sqrt{N}\kappa_N\right\}$$

with $\kappa_N = p\sqrt{N}/(p^3N)^{2/5}$. These spin configurations will be called typical in the following proof. The corresponding set of values of $|\sigma|$ is denoted by $W_{N,m}$, i.e.

$$W_{N,m} := \{|\sigma| : \sigma \in S_N^1\} = \{\lceil Nm - \sqrt{N}\kappa_N \rceil, \ldots, \lfloor Nm + \sqrt{N}\kappa_N \rfloor\}. \tag{3.4}$$

The set of the atypical spin configurations is denoted by

$$S_N^{1c} := \{\sigma : \big|\,|\sigma| - Nm\big| > \sqrt{N}\kappa_N, |\sigma| > 0\}.$$

Proposition 3.2. *For all $g \in \mathcal{C}^b(\mathbb{R}), g \ge 0, g \not\equiv 0,\ h = 0, \beta > 1,$ and p with $Np^3 \to \infty$ we have*

$$\lim_{N\to\infty} \frac{\mathbb{E}\tilde{Z}_N^+(\beta,h,g)}{\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(1-m^4) - NI(m)} 2^{N+1} \frac{1}{\sqrt{1-m^2}} \sigma(\beta)\mathbb{E}_\xi[g(\xi)]} = 1.$$

Here, the function I is given by formula (3.6) below and

$$\mathbb{E}_\xi[g(\xi)] = \frac{1}{\sqrt{2\pi\sigma^2(\beta)}} \int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\frac{x^2}{2\sigma^2(\beta)}}\,dx,$$

i.e. ξ denotes a normally distributed random variable with expectation 0 and variance $\sigma^2(\beta)$.

Proof: By decomposing $\{\pm 1\}^N$ into typical and atypical σ's, defined by S_N^1, we have

$$\mathbb{E}\tilde{Z}_N^+(\beta,h,g) = \sum_{\sigma \in S_N^1} g\left(\frac{|\sigma| - Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) + \sum_{\sigma \in S_N^{1c}} g\left(\frac{|\sigma| - Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma).$$

For the typical configurations with $|\sigma| \in W_{N,m}$ we have from Lemma 3.1

$$\begin{aligned}
&\sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) \qquad (3.5)\\
=&\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)} \sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(|\sigma|^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2|\sigma|^2}{N^2}+o(1)}\\
=&\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)} \sum_{k\in W_{N,m}} \sum_{\substack{\sigma\in\{\pm1\}^N:\\ |\sigma|=k}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(|\sigma|^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2|\sigma|^2}{N^2}+o(1)}\\
=&\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)} \sum_{k\in W_{N,m}} g\left(\frac{k-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2k^2}{N^2}+o(1)}\binom{N}{\frac{N+k}{2}}.
\end{aligned}$$

Note that from Stirling's formula

$$\log(n!) = n\log n - n + \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + \mathcal{O}(1/n),$$

we obtain for $k \in W_{N,m}$

$$\begin{aligned}
2^{-N}\binom{N}{\frac{N+k}{2}} &= \sqrt{\frac{2}{\pi N}}\frac{1}{\sqrt{(1-\frac{k}{N})(1+\frac{k}{N})}}\mathrm{e}^{-NI\left(\frac{k}{N}\right)+o(1)}\\
&= (1+o(1))\sqrt{\frac{2}{\pi N}}\frac{1}{\sqrt{(1-\frac{k^2}{N^2})}}\mathrm{e}^{-NI\left(\frac{k}{N}\right)}\\
&= (1+o(1))\sqrt{\frac{2}{\pi N(1-m^2)}}\mathrm{e}^{-NI\left(\frac{k}{N}\right)}
\end{aligned}$$

with an $o(1)$-term that is uniform for all k such that $k \in W_{N,m}$. We have used that $\kappa_N = o(\sqrt{N})$. Here,

$$I(x) := \frac{1-x}{2}\log(1-x) + \frac{1+x}{2}\log(1+x) \qquad \text{for } x \in [-1,1]. \tag{3.6}$$

Hence, we arrive at

$$\begin{aligned}
&\sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma)\\
&= (1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)}2^N\frac{1}{\sqrt{1-m^2}}\sqrt{\frac{2}{\pi N}}\\
&\times \sum_{k\in W_{N,m}} g\left(\frac{k-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2k^2}{N^2}-NI\left(\frac{k}{N}\right)}.
\end{aligned}$$

Let us write $\frac{k}{N} = m + \frac{c_k}{\sqrt{N}}$ with $|c_k| \le \kappa_N$. Then,

$$NI\left(\frac{k}{N}\right) = NI(m) + I'(m)c_k\sqrt{N} + I''(m)\frac{c_k^2}{2} + \mathcal{O}\left(\frac{\kappa_N^3}{\sqrt{N}}\right)$$

and the $\mathcal{O}$-term is uniform for all $k \in W_{N,m}$ and can be estimated above by $o(1)$ because of our choice of κ_N and since $p^3 N \to \infty$. Then

$$\begin{aligned}
&\sum_{\sigma \in S_N^1} g\Big(\frac{|\sigma| - Nm}{\sqrt{N}}\Big)\mathbb{E}T(\sigma) \\
&\quad = (1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1) - NI(m)} 2^N \frac{1}{\sqrt{1-m^2}}\sqrt{\frac{2}{\pi N}} \\
&\qquad \times \sum_{k \in W_{N,m}} g\Big(\frac{k - Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2N}\left(2N^{3/2}mc_k + c_k^2 N\right) - \frac{(1-p)\beta^2}{8p}\frac{2m^4N^2 + 4m^3N^{3/2}c_k + 2m^2c_k^2N}{N^2}} \\
&\qquad\qquad \mathrm{e}^{-I'(m)c_k\sqrt{N} - I''(m)\frac{c_k^2}{2}}. \qquad (3.7)
\end{aligned}$$

Now the linear term in c_k in the above expression is

$$c_k\bigg(\beta m\sqrt{N} - \frac{\beta^2(1-p)m^3}{2p\sqrt{N}} - \sqrt{N}I'(m)\bigg) = c_k\big(\beta m\sqrt{N} - \sqrt{N}I'(m)\big) + o(1) = o(1),$$

where the first equality follows from $\kappa_N = o(p\sqrt{N})$ while the second equality follows from $I'(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \operatorname{artanh}(x)$ and $m = \tanh(\beta m)$. Therefore, with an $o(1)$-term that is uniform for typical σ we have that

$$\begin{aligned}
&\sum_{\sigma \in S_N^1} g\Big(\frac{|\sigma| - Nm}{\sqrt{N}}\Big)\mathbb{E}T(\sigma) \\
&\quad = (1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1) - NI(m)} 2^N \frac{1}{\sqrt{1-m^2}}\sqrt{\frac{2}{\pi N}} \\
&\qquad \sum_{k \in W_{N,m}} g\Big(\frac{k - Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2}c_k^2 - \frac{(1-p)\beta^2}{8p}\frac{2m^4N^2 + 2m^2c_k^2N}{N^2}}\mathrm{e}^{-I''(m)\frac{c_k^2}{2}} \\
&\quad = (1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(-m^4+1) - NI(m)} 2^N \frac{1}{\sqrt{1-m^2}}\sqrt{\frac{2}{\pi N}} \\
&\qquad \sum_{k \in W_{N,m}} g\Big(\frac{k - Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta - I''(m)}{2}c_k^2},
\end{aligned}$$

where we used that $\kappa_N^2 = o(pN)$. Note that for the term in the last exponential we have

$$\frac{\beta - I''(m)}{2}c_k^2 = -\frac{1-\beta(1-m^2)}{2(1-m^2)}c_k^2 = -\frac{1}{2\sigma(\beta)^2}c_k^2 = -\frac{1}{2\sigma(\beta)^2}\Big(\frac{k-Nm}{\sqrt{N}}\Big)^2,$$

where we used $c_k = (k - Nm)/\sqrt{N}$. We see that

$$\sqrt{\frac{1}{2\pi N\sigma^2(\beta)}}\sum_{k \in W_{N,m}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big)\exp\bigg(-\frac{1}{2\sigma(\beta)^2}\Big(\frac{k-Nm}{\sqrt{N}}\Big)^2\bigg)$$

converges to

$$\sqrt{\frac{1}{2\pi\sigma^2(\beta)}}\int_{-\infty}^{\infty} g(x)\exp\bigg(-\frac{x^2}{2\sigma(\beta)^2}\bigg)dx = \mathbb{E}_\xi[g(\xi)].$$

Indeed, this is basically the approximation of an integral by its Riemann sum. Note that for $k \in W_{N,m}$ the variable $c_k = (k - Nm)/\sqrt{N}$ ranges in $[-\kappa_N, +\kappa_N]$ intersected with a lattice of mesh size $1/\sqrt{N}$. Over each fixed interval, the Riemann approximation argument applies. Since

$\kappa_N \to \infty$, we need an additional justification for the applicability of the Riemann approximation over intervals of growing size. Note that

$$\sqrt{\frac{1}{2\pi\sigma^2(\beta)}}\int_{-\infty}^{\infty} g(x)\exp\Big(-\frac{x^2}{2\sigma(\beta)^2}\Big)dx<\infty.$$

Hence, for all $\varepsilon>0$, there is a compact interval I_ε such that

$$\sqrt{\frac{1}{2\pi\sigma^2(\beta)}}\int_{x\notin I_\varepsilon} g(x)\exp\Big(-\frac{x^2}{2\sigma(\beta)^2}\Big)dx<\varepsilon$$

as well as

$$\sqrt{\frac{1}{2\pi N\sigma^2(\beta)}}\sum_{\substack{k\in W_{N,m}:\\ c_k\in I_\varepsilon^c}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big)\exp\Big(-\frac{1}{2\sigma(\beta)^2}\Big(\frac{k-Nm}{\sqrt{N}}\Big)^2\Big)<\varepsilon,$$

where I_ε^c denotes the complement of I_ε in $\mathbb{R}$. For the second claim, bound g by its supremum and estimate the remaining sum by the corresponding integral using monotonicity. On the fixed interval I_ε we have convergence of Riemann sums to Riemann integrals, meaning that for sufficiently large N, the difference between both is at most ε. But this means

$$\Bigg|\sqrt{\frac{1}{2\pi N\sigma^2(\beta)}}\sum_{k\in W_{N,m}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big)\exp\Big(-\frac{1}{2\sigma(\beta)^2}\Big(\frac{k-Nm}{\sqrt{N}}\Big)^2\Big) - \sqrt{\frac{1}{2\pi\sigma^2(\beta)}}\int_{-\infty}^{\infty} g(x)\exp\Big(-\frac{x^2}{2\sigma(\beta)^2}\Big)dx\Bigg|<3\varepsilon$$

for all N sufficiently large.

To prove the proposition, it suffices to show that the atypical spin configurations do not contribute to the asymptotic size of $\tilde{Z}_N^+(\beta,h,g)$. Recall that the set of atypical spin configurations is denoted by

$$S_N^{1c}:=\{\sigma:\big||\sigma|-Nm\big|>\sqrt{N}\kappa_N,|\sigma|>0\}.$$

The corresponding set of possible values for $|\sigma|$ will be called $W_{N,m}^c$. We will use the following bound on the binomial coefficient which again is a consequence of Markov's inequality:

$$\binom{N}{\frac{N+k}{2}}\le 2^N \mathrm{e}^{-NI(\frac{k}{N})},\qquad |k|\le N. \tag{3.8}$$

We can use Lemma 3.1 in combination with (3.8) to obtain

$$\begin{aligned}
&\sum_{\sigma\in S_N^{1c}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big)\mathbb{E}T(\sigma)\\
\le&\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)}||g||_\infty \sum_{k\in W_{N,m}^c} \mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2k^2}{N^2}+o(1)}\binom{N}{\frac{N+k}{2}}\\
\le&C\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)}2^N \sum_{k\in W_{N,m}^c} \mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{8p}\frac{2m^2k^2}{N^2}-NI(\frac{k}{N})}
\end{aligned}\tag{3.9}$$

for some constant $C>0$. For the next step again write $k=Nm+c_k\sqrt{N}$, this time $\kappa_N<|c_k|<C_m\sqrt{N}$ (the exact value for C_m depends on m). For the second term in the exponent we obtain the

estimate

$$
\begin{aligned}
&-\frac{(1-p)\beta^2}{8p}\frac{2m^2k^2}{N^2} = -\frac{m^4(1-p)\beta^2}{4p} - \frac{c_k(1-p)\beta^2}{2p\sqrt{N}}m^3 - \frac{c_k^2(1-p)\beta^2}{4pN}m^2 \\
&\leq -\frac{m^4(1-p)\beta^2}{4p} - \frac{c_k(1-p)\beta^2}{2p\sqrt{N}}m^3.
\end{aligned}
$$

Inserting this estimate into (3.9) leads to

$$
\begin{aligned}
&\sum_{\sigma\in S_N^{1c}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) \\
&\leq C\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)}2^N \sum_{k\in W_{N,m}^c} \mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{m^4(1-p)\beta^2}{4p}-\frac{c_k(1-p)\beta^2}{2p\sqrt{N}}m^3-NI(\frac{k}{N})}. \qquad (3.10)
\end{aligned}
$$

Next observe that from the analysis of the Curie-Weiss model (see Ellis (2006) for the large deviations regime and Eichelsbacher and Löwe (2004) for the regime of moderate deviations) we know that the function $\frac{k}{N}\mapsto \frac{\beta}{2N}k^2 - NI(\frac{k}{N})$ attains its maximum for $\frac{k}{N}$ positive at m and that

$$
\frac{\beta}{2N}k^2 - NI\left(\frac{k}{N}\right) \leq N\left(\frac{\beta}{2}m^2 - I(m)\right) - K_1 c_k^2 \qquad (3.11)
$$

for some sufficiently small constant $K_1 > 0$. Moreover, c_k^2 will be at least of order κ_N^2. Hence

$$
\begin{aligned}
&\sum_{\sigma\in S_N^{1c}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) \\
&\leq C\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(-m^4+1)}2^N\mathrm{e}^{-NI(m)} \sum_{k\in W_{N,m}^c} \mathrm{e}^{-\frac{2c_k(1-p)\beta^2}{4p\sqrt{N}}m^3-K_1c_k^2} \\
&\leq C\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(-m^4+1)}2^N\mathrm{e}^{-NI(m)} \sum_{k\in W_{N,m}^c} \mathrm{e}^{-K_2c_k^2} \qquad (3.12)
\end{aligned}
$$

for some other constant $K_2 > 0$. We used that $c_k/(p\sqrt{N}) = c_k^2/(c_k p\sqrt{N})$ with $c_k p\sqrt{N}\to\infty$ in the denominator.

Now $c_k^2 \geq \kappa_N^2 \geq N^{1/10}$ and the sum contains at most N summands, which yields

$$
\lim_{N\to\infty}\sum_{k\in W_{N,m}^c} \mathrm{e}^{-K_2\kappa_N^2} = 0. \qquad (3.13)
$$

This shows that the contribution of the spin configurations in S_N^{1c} is negligible and therefore proves the proposition. □

In the next step we will control the variance of $\tilde{Z}_N^+(\beta,h,g)$ in order to show that it is of smaller order than the squared expectation. This would imply that the quantity $\tilde{Z}_N^+(\beta,h,g)$ is self-averaging meaning that $\tilde{Z}_N^+(\beta,h,g)/\mathbb{E}\tilde{Z}_N^+(\beta,h,g)$ converges in probability to 1. Our first step in this direction is

Lemma 3.3. *For $h=0$, $\beta>1$, all $p=p(N)$ such that $p^3N\to\infty$, and all $\sigma,\tau\in\{-1,+1\}^N$ we have*

$$
\mathbb{E}(T(\sigma)T(\tau)) = \exp(N^2b_0 + b_1|\sigma|^2 + b_2|\tau|^2 + b_{12}|\sigma\tau|^2),
$$

where

$$b_0 = -2(m^2p)\gamma + (p + 2m^4p - p^2 - 2m^4p^2)\gamma^2 + \mathcal{O}(p\gamma^3),$$
$$b_1 = b_2 = p\gamma + (-2m^2p + 2m^2p^2)\gamma^2 + \mathcal{O}(p\gamma^3),$$
$$b_{12} = (p - p^2)\gamma^2 + \mathcal{O}(p\gamma^3),$$

and the $\mathcal{O}$-term is uniform over $\sigma, \tau \in \{-1, +1\}^N$. Here, we set

$$|\sigma\tau| := \sum_{i=1}^{N} \sigma_i\tau_i.$$

Proof: We have

$$\mathbb{E}(T(\sigma)T(\tau)) = \prod_{i,j=1}^{N} \left(1 - p + pe^{\gamma(\sigma_i\sigma_j + \tau_i\tau_j - 2m^2)}\right) = \exp\left(\sum_{i,j=1}^{N} f(\sigma_i\sigma_j + \tau_i\tau_j)\right),$$

where

$$f(x) = f(x; p, \gamma) = \log(1 - p + pe^{\gamma(x-2m^2)}) = F(p, \gamma(x - 2m^2)),$$

and F is given by (2.1). Note that for fixed m, $f(\sigma_i\sigma_j + \tau_i\tau_j)$ is a function of the arguments $x_1 = \sigma_i\sigma_j$ and $x_2 = \tau_i\tau_j$, where x_1 and x_2 take values in $\{\pm 1\}$. Hence, for these values of x_1 and x_2, we can write

$$f(x_1 + x_2) = b_0 + b_1x_1 + b_2x_2 + b_{12}x_1x_2,$$

where the coefficients are given by

$$b_0 = \frac{f(2) + f(-2) + 2f(0)}{4},$$
$$b_{12} = \frac{f(2) + f(-2) - 2f(0)}{4},$$
$$b_1 = b_2 = \frac{f(2) - f(-2)}{4}.$$

The representation of the coefficients is then an immediate consequence of Corollary 2.2. Using (2.2) the lemma is proved. □

From here we start to estimate the variance of $\tilde{Z}_N^+(\beta, h, g)$.

Proposition 3.4. *For $h = 0$, $\beta > 1$, all $p = p(N)$ such that $p^3N \to \infty$, and all $g \in \mathcal{C}^b(\mathbb{R}), g \geq 0$, $g \not\equiv 0$ we have that*

$$\mathbb{V}(\tilde{Z}_N^+(\beta, h, g)) = o\Big(\mathbb{E}^2[\tilde{Z}_N^+(\beta, h, g)]\Big).$$

Proof: Obviously,

$$\mathbb{V}(\tilde{Z}_N^+(\beta, h, g)) = \mathbb{E}\left[\left(\tilde{Z}_N^+(\beta, h, g)\right)^2\right] - \mathbb{E}^2[\tilde{Z}_N^+(\beta, h, g)].$$

The asymptotics of the second term on the right is already known from Proposition 3.2. Our aim is to show that the first term satisfies

$$\mathbb{E}\left[\left(\tilde{Z}_N^+(\beta, h, g)\right)^2\right] \leq (1 + o(1))\mathbb{E}^2[\tilde{Z}_N^+(\beta, h, g)].$$

Since the variance cannot become negative, this would imply the assertion. Introduce the set of typical pairs of spin configurations

$$S_N^2 := \{(\sigma, \tau) : |\sigma|, |\tau| \in W_{N,m}, \big||\sigma\tau| - Nm^2\big| \leq C'\sqrt{N}\kappa_N\},$$

where $W_{N,m}$ is defined as in (3.4), C' is a constant to be specified below, and we recall that $|\sigma\tau| := \sum_{i=1}^N \sigma_i\tau_i$. The pairs of spin configurations (σ,τ) that are not in S_N^2 will be called atypical. The set of atypical pairs (σ,τ) that satisfy $|\sigma| > 0$, $|\tau| > 0$ will be denoted by S_N^{2c}. We split $\mathbb{E}[(\tilde{Z}_N^+(\beta,h,g))^2]$ into the contribution of typical and atypical pairs of spin configurations as follows:

$$\begin{aligned}
\mathbb{E}\Big[\Big(\tilde{Z}_N^+(\beta,h,g)\Big)^2\Big] &= \sum_{(\sigma,\tau)\in\{\pm1\}^N\times\{\pm1\}^N} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))\\
&= \sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))\\
&\quad + \sum_{(\sigma,\tau)\in S_N^{2c}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau)).
\end{aligned}$$

Let us first consider the typical spin configurations. Using Lemma 3.3 we obtain

$$\begin{aligned}
&\sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))\\
&= \sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathrm{e}^{-2m^2N^2p\gamma+N^2p(1-p)(2m^4+1)\gamma^2+o(1)}\\
&\qquad\times \exp\Big(\big(p\gamma+\big(-2m^2p+2m^2p^2\big)\gamma^2\big)\big(|\sigma|^2+|\tau|^2\big)+\big(p-p^2\big)\gamma^2|\sigma\tau|^2\Big)\\
&= \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}+o(1)} \sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\\
&\qquad\times \mathrm{e}^{\frac{\beta}{2N}\big((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2)\big)-\frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2}.
\end{aligned} \tag{3.14}$$

Our first observation is that the term with $|\sigma\tau|^2$ can be asymptotically replaced by a term not depending on σ and τ. Indeed, for $(\sigma,\tau)\in S_N^2$ we have that

$$\frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2 = \frac{(1-p)\beta^2m^4}{4p}+o(1)$$

because $\kappa_N = o(pN)$. Our second observation is that the same can be done for the term involving $|\sigma|^2+|\tau|^2$ since

$$\frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2} = \frac{(1-p)\beta^2}{4p}\frac{2m^2\cdot 2N^2m^2}{N^2}+o(1) = \frac{(1-p)\beta^2m^4}{p}+o(1),$$

where we used that $\kappa_N = o(p\sqrt{N})$. Overall, we get

$$\begin{aligned}
&\sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))\\
=&\mathrm{e}^{\frac{(1-p)\beta^2(1-m^4)}{4p}+o(1)} \sum_{(\sigma,\tau)\in S_N^2} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2N}\big((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2)\big)}.
\end{aligned}$$

Now we have the upper estimate

$$\begin{aligned}&\sum_{(\sigma,\tau)\in S_N^2} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right) g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathbb{E}(T(\sigma)T(\tau))\\ &\le \mathrm{e}^{\frac{(1-p)\beta^2(1-m^4)}{4p}+o(1)} \sum_{(\sigma,\tau)\in S_N^1\times S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right) g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\\ &\quad\times \mathrm{e}^{\frac{\beta}{2N}\left((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2)\right)}\\ &\le \mathrm{e}^{\frac{(1-p)\beta^2(1-m^4)}{4p}+o(1)}\left(\sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(|\sigma|^2-m^2N^2)}\right)^2.\end{aligned}$$

To justify the inequality, observe that in the first line we sum over a smaller set of pairs (σ,τ) because S_N^2 involves an additional constraint on $|\sigma\tau|$, and recall that $g\ge 0$.

Now we proceed similarly to the proof of Proposition 3.2. Indeed, in the same way we prove that

$$\sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(|\sigma|^2-m^2N^2)} = \mathrm{e}^{-NI(m)}2^{N+1}\frac{1+o(1)}{\sqrt{1-m^2}}\sigma(\beta)\mathbb{E}_\xi[g(\xi)].$$

This, together with the statement of Proposition 3.2 shows that

$$\sum_{(\sigma,\tau)\in S_N^2} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right) g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathbb{E}(T(\sigma)T(\tau)) \le (1+o(1))\left(\mathbb{E}\tilde{Z}_N^+(\beta,h,g)\right)^2.$$

We will now show that the contribution of the atypical spins to the variance of $\tilde{Z}_N^+(\beta,h,g)$ is negligible. We need to show that

$$\sum_{(\sigma,\tau)\in S_N^{2c}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right) g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathbb{E}(T(\sigma)T(\tau)) = o\left(\left(\mathbb{E}\tilde{Z}_N^+(\beta,h,g)\right)^2\right).$$

Note that the pairs of spin configurations $(\sigma,\tau)\in S_N^{2c}$ either satisfy

$$\big||\sigma|-Nm\big| > \sqrt{N}\kappa_N \quad\text{or,}\quad \big||\tau|-Nm\big| > \sqrt{N}\kappa_N \quad\text{or,}\quad \big||\sigma\tau|-Nm^2\big| > C'\sqrt{N}\kappa_N.$$

In the case when

$$\big||\sigma|-Nm\big| > \sqrt{N}\kappa_N \quad\text{or}\quad \big||\tau|-Nm\big| > \sqrt{N}\kappa_N$$

we can proceed similarly as in the proof of Proposition 3.2. For concreteness, let us assume that we consider the situation where $||\sigma|-Nm| > \sqrt{N}\kappa_N$ and τ is arbitrary and let us denote the corresponding set of spin configurations by $S_{N,A}^{2c}$. Then, starting from (3.14), we estimate

$$\begin{aligned}&\sum_{(\sigma,\tau)\in S_{N,A}^{2c}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right) g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathbb{E}(T(\sigma)T(\tau))\\ &\le \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}+o(1)}\|g\|_\infty \sum_{(\sigma,\tau)\in S_{N,A}^{2c}} g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2))}\\ &\qquad\times \mathrm{e}^{-\frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2}\\ &\le \mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{4p}+o(1)}\|g\|_\infty \sum_{(\sigma,\tau)\in S_{N,A}^{2c}} g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2))}\\ &\qquad\times \mathrm{e}^{-\frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2}}\end{aligned}$$

because $\frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2 \le \frac{(1-p)\beta^2}{4p}$. Thus

$$\sum_{(\sigma,\tau)\in S_{N,A}^{2c}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))$$
$$\le \mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{4p}+o(1)}\|g\|_\infty$$
$$\times \sum_{\substack{(k,l):\\ |k-Nm|>\sqrt{N}\kappa_N}} g\Big(\frac{l-Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2))-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}}\binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}}$$
$$\le \mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{4p}}2^N\|g\|_\infty \sum_{k:|k-Nm|>\sqrt{N}\kappa_N}\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{4p}\frac{2m^2k^2}{N^2}-NI\left(\frac{k}{N}\right)}$$
$$\times \sum_l g\Big(\frac{l-Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2N}(l^2-m^2N^2)-\frac{(1-p)\beta^2}{4p}\frac{2m^2l^2}{N^2}}\binom{N}{\frac{N+l}{2}}$$

where the last step follows from

$$2^{-N}\binom{N}{\frac{N+k}{2}} \le \exp\Big(-NI\Big(\frac{k}{N}\Big)\Big).$$

Note that as in Proposition 3.2, especially equation (3.5) and the following equations we obtain that

$$\mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{4p}}\sum_l g\Big(\frac{l-Nm}{\sqrt{N}}\Big)\mathrm{e}^{\frac{\beta}{2N}(l^2-m^2N^2)-\frac{(1-p)\beta^2}{4p}\frac{2m^2l^2}{N^2}}\binom{N}{\frac{N+l}{2}}$$
$$=(1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{2p}-NI(m)}2^{N+1}\frac{1}{\sqrt{1-m^2}}\sigma(\beta)\mathbb{E}_\xi[g(\xi)]$$
$$=(1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2(3+m^4)}{8p}}\mathbb{E}\tilde{Z}_N^+(\beta,h,g).$$

This implies that

$$\sum_{(\sigma,\tau)\in S_{N,A}^{2c}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))$$
$$\le (1+o(1))\mathbb{E}\tilde{Z}_N^+(\beta,h,g)\mathrm{e}^{\frac{(1-p)\beta^2(3+m^4)}{8p}}2^N\|g\|_\infty\sum_{k:|k-Nm|>\sqrt{N}\kappa_N}\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{(1-p)\beta^2}{4p}\frac{2m^2k^2}{N^2}-NI\left(\frac{k}{N}\right)}$$
$$\le (1+o(1))\mathbb{E}\tilde{Z}_N^+(\beta,h,g)\mathrm{e}^{\frac{(1-p)\beta^2(3+m^4)}{8p}}2^N\|g\|_\infty\sum_{k:|k-Nm|>\sqrt{N}\kappa_N}\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-NI\left(\frac{k}{N}\right)}.$$

But following the steps in (3.11), (3.12), and (3.13) we see that

$$\mathrm{e}^{\frac{(1-p)\beta^2(3+m^4)}{8p}}2^N\sum_{k:|k-Nm|>\sqrt{N}\kappa_N}\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-NI\left(\frac{k}{N}\right)}$$
$$=\mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{8p}}\mathrm{e}^{\frac{(1-p)\beta^2(1-m^4)}{8p}}2^N\sum_{k:|k-Nm|>\sqrt{N}\kappa_N}\mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-NI\left(\frac{k}{N}\right)}$$
$$\le\mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{8p}}\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(-m^4+1)}2^N\mathrm{e}^{-NI(m)}\sum_{k\in W_{N,m}^c}\mathrm{e}^{-K_2c_k^2}$$

with the set $W^c_{N,m}$ defined as in the proof of Proposition 3.2. But

$$\mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{8p}} \sum_{k\in W^c_{N,m}} \mathrm{e}^{-K_2 c_k^2} \le C\mathrm{e}^{\frac{(1-p)\beta^2(2+2m^4)}{8p}} \mathrm{e}^{-K_2\kappa_N^2} \to 0$$

because $\kappa_N^2 p \to \infty$. Together with Proposition 3.2 this shows that

$$\sum_{(\sigma,\tau)\in S^{2c}_{N,A}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau)) = o\Big(\Big(\mathbb{E}\tilde{Z}^+_N(\beta,h,g)^2\Big)\Big).$$

Hence the contribution of the pairs of spin configurations from $S^{2c}_{N,A}$ is asymptotically negligible.

The contributions of pairs of spin configurations from $S^{2c}_{N,B}$, the set where

$$\big|\,|\tau| - Nm\big| > \sqrt{N}\kappa_N$$

and σ is arbitrary, is bounded in the same way.

It remains to estimate the contribution of the pairs of spin configurations of the set

$$S^{2c}_{N,C} := \Big\{(\sigma,\tau)\in S^{2c}_N : \big|\,|\sigma| - Nm\big| \le \sqrt{N}\kappa_N,$$
$$\big|\,|\tau| - Nm\big| \le \sqrt{N}\kappa_N, \big|\,|\sigma\tau| - Nm^2\big| > C'\sqrt{N}\kappa_N\Big\}.$$

Let us denote by $R^{2c}_{N,C}$ the set of possible values (k,l,n) the vector $(|\sigma|,|\tau|,|\sigma\tau|)$ can take, when $(\sigma,\tau)\in S^{2c}_{N,C}$, formally

$$R^{2c}_{N,C} := \{(k,l,n) : \exists(\sigma,\tau)\in S^{2c}_{N,C} \text{ with } (k,l,n) = (|\sigma|,|\tau|,|\sigma\tau|)\}.$$

Moreover, denote by $V_N(k,l,n)$ the set of pairs

$$(\sigma,\tau)\in\{\pm1\}^N\times\{\pm1\}^N \quad \text{for which } |\sigma| = k, |\tau| = l, \text{and } |\sigma\tau| = n$$

and set $\nu_N(k,l,n) := \#V_N(k,l,n)$. Note in particular that by the definition of $|\sigma|, |\tau|$ and $|\sigma\tau|$ we have

$$-(N+k)\le l+n\le N+k \quad \text{and} \quad -(N-k)\le l-n\le N-k. \tag{3.15}$$

In order to treat the corresponding contribution we need to compute the distribution of $|\sigma\tau|$ in greater detail. We begin by using Lemma 3.3 again:

$$\begin{aligned}
&\sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))\\
&= \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}+o(1)} \sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\\
&\quad\times \mathrm{e}^{\frac{\beta}{2N}((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2)) - \frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2} + \frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2}\\
&= \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}+o(1)} \sum_{(k,l,n)\in R^{2c}_{N,C}} \sum_{(\sigma,\tau)\in V_N(k,l,n)} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\\
&\quad\times \mathrm{e}^{\frac{\beta}{2N}((|\sigma|^2-m^2N^2)+(|\tau|^2-m^2N^2)) - \frac{(1-p)\beta^2}{4p}\frac{2m^2(|\sigma|^2+|\tau|^2)}{N^2} + \frac{(1-p)\beta^2}{4N^2p}|\sigma\tau|^2}\\
&= \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}+o(1)} \sum_{(k,l,n)\in R^{2c}_{N,C}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big) g\Big(\frac{l-Nm}{\sqrt{N}}\Big)\\
&\quad\times \mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2)) - \frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2} + \frac{(1-p)\beta^2}{4N^2p}n^2} \nu_N(k,l,n).
\end{aligned}$$

As observed in Kabluchko et al. (2020) $\nu_N(k,l,n)$ divided by 2^{2N} is a probability mass function, which can be written in terms of a conditional probability:

$$\begin{aligned}2^{-2N}\nu_N(k,l,n) &= \mathbb{P}_{\text{unif}}(|\sigma|=k,|\tau|=l,|\sigma\tau|=n)\\ &= \mathbb{P}_{\text{unif}}(|\sigma\tau|=n \,\Big|\, |\sigma|=k,|\tau|=l)\mathbb{P}(|\sigma|=k)\mathbb{P}(|\tau|=l)\\ &= 2^{-N}\binom{N}{\frac{N+k}{2}}2^{-N}\binom{N}{\frac{N+l}{2}}\mathbb{P}(|\sigma\tau|=n \,\Big|\, |\sigma|=k,|\tau|=l).\end{aligned}$$

Here, $\mathbb{P}_{\text{unif}}$ denotes the probability distribution under which (σ,τ) is uniformly distributed on $\{\pm1\}^N\times\{\pm1\}^N$. Using the hypergeometric distribution, we can express the conditional probability $\mathbb{P}(|\sigma\tau|=n \,\big|\, |\sigma|=k,|\tau|=l)$ as the following fraction:

$$\mathbb{P}(|\sigma\tau|=n \,\Big|\, |\sigma|=k,|\tau|=l) = \frac{\binom{\frac{N+k}{2}}{\frac{N+k+l+n}{4}}\binom{\frac{N-k}{2}}{\frac{N+l-k-n}{4}}}{\binom{N}{\frac{N+l}{2}}}. \tag{3.16}$$

We will use

$$cN^{-1/2}2^N\mathrm{e}^{-NI(\frac{k}{N})-\lambda_N(k)} \le \binom{N}{\frac{N+k}{2}} \le CN^{-1/2}2^N\mathrm{e}^{-NI(\frac{k}{N})-\lambda_N(k)}, \qquad |k|\le N$$

for some constants $c,C>0$ which was shown in Kabluchko et al. (2020), Eqn. (4.11), as a consequence of Stirling's formula. Here

$$\lambda_N(k) := \frac{1}{2}\log\left(\frac{(N+1)^2-k^2}{N^2}\right).$$

With this formula we can treat the binomial coefficients in (3.16) (where we bound the log-correction in the exponent of the denominator by 0) to obtain

$$\begin{aligned}\mathbb{P}(|\sigma\tau|=n \,\Big|\, |\sigma|=k,|\tau|=l) &\le C\sqrt{\frac{N}{(N+k)(N-k)}}\\ &\quad\times \mathrm{e}^{-N\left(\frac{N+k}{2N}I(\frac{l+n}{N+k})+\frac{N-k}{2N}I(\frac{l-n}{N-k})-I(\frac{l}{N})\right)-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})}\end{aligned}$$

for some positive constant C. Note that $k = N$ or $l = N$ is excluded by the definition of $R^{2c}_{N,C}$. Moreover, on $R^{2c}_{N,C}$ we have that $\sqrt{\frac{N}{(N+k)(N-k)}} \le C/\sqrt{N}$ (again for another $C > 0$). Thus

$$\begin{aligned}
&\sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big) \mathbb{E}(T(\sigma)T(\tau)) \\
&\le C \mathrm{e}^{\frac{(1-p)\beta^2(1+2m^4)}{4p}} \sum_{(k,l,n)\in R^{2c}_{N,C}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big) g\Big(\frac{l-Nm}{\sqrt{N}}\Big) \\
&\quad \times \mathrm{e}^{\frac{\beta}{2N}\left((k^2-m^2N^2)+(l^2-m^2N^2)\right)-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}n^2} \binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}} \\
&\quad \times \frac{1}{\sqrt{N}} \mathrm{e}^{-N\left(\frac{N+k}{2N}I(\frac{l+n}{N+k})+\frac{N-k}{2N}I(\frac{l-n}{N-k})-I(\frac{l}{N})\right)-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})} \\
&= C \mathrm{e}^{\frac{(1-p)\beta^2(1+3m^4)}{4p}} \sum_{(k,l,n)\in R^{2c}_{N,C}} g\Big(\frac{k-Nm}{\sqrt{N}}\Big) g\Big(\frac{l-Nm}{\sqrt{N}}\Big) \\
&\quad \times \mathrm{e}^{\frac{\beta}{2N}\left((k^2-m^2N^2)+(l^2-m^2N^2)\right)-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}(n^2-m^4N^2)} \binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}} \\
&\quad \times \frac{1}{\sqrt{N}} \mathrm{e}^{-N\left(\frac{N+k}{2N}I(\frac{l+n}{N+k})+\frac{N-k}{2N}I(\frac{l-n}{N-k})-I(\frac{l}{N})\right)-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})}.
\end{aligned}$$

Now, borrowing an idea from Kabluchko et al. (2020) we observe that

$$\frac{l}{N} = \frac{N-k}{2N}\frac{l-n}{N-k} + \frac{N+k}{2N}\frac{l+n}{N+k},$$

i.e. $\frac{l}{N}$ is a convex combination of $\frac{l+n}{N+k}$ and $\frac{l-n}{N-k}$ with weights $\frac{N+k}{2N}$ and $\frac{N-k}{2N}$, respectively. On the other hand, I is a convex function, and, even more, considering its Taylor expansion

$$NI(l/N) = \frac{l^2}{2N} + \sum_{j\ge 2} d_{2j}\frac{l^{2j}}{N^{2j-1}}$$

with positive coefficients d_{2j} we see that it is a positive linear combination of convex functions. Using that $d_2 = \frac{1}{2}$ we obtain

$$\begin{aligned}
&-N\left(\frac{N+k}{2N} I\Big(\frac{l+n}{N+k}\Big) + \frac{N-k}{2N} I\Big(\frac{l-n}{N-k}\Big) - I\Big(\frac{l}{N}\Big)\right) \\
&= -N\left(\frac{N+k}{2N}\sum_{j=1}^{\infty} d_{2j}\Big(\frac{l+n}{N+k}\Big)^{2j} + \frac{N-k}{2N}\sum_{j=1}^{\infty} d_{2j}\Big(\frac{l-n}{N-k}\Big)^{2j} - \sum_{j=1}^{\infty} d_{2j}\Big(\frac{l}{N}\Big)^{2j}\right) \\
&\le -\frac{N}{2}\left(\frac{N+k}{2N}\Big(\frac{l+n}{N+k}\Big)^2 + \frac{N-k}{2N}\Big(\frac{l-n}{N-k}\Big)^2 - \Big(\frac{l}{N}\Big)^2\right) \\
&= -\frac{1}{2}\frac{(Nn-lk)^2}{N(N^2-k^2)} = -\frac{1}{2}\frac{(n-\frac{lk}{N})^2}{(N-\frac{k^2}{N})},
\end{aligned}$$

where for the inequality we used that for each $j \ge 2$

$$\frac{N+k}{2N} d_{2j}\Big(\frac{l+n}{N+k}\Big)^{2j} + \frac{N-k}{2N} d_{2j}\Big(\frac{l-n}{N-k}\Big)^{2j} - d_{2j}\Big(\frac{l}{N}\Big)^{2j} \ge 0.$$

Thus

$$\begin{aligned}
&\mathrm{e}^{-N\left(\frac{N+k}{2N}I(\frac{l+n}{N+k})+\frac{N-k}{2N}I(\frac{l-n}{N-k})-I(\frac{l}{N})\right)-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})} \\
&\leq \exp\left(-\frac{(n-\frac{kl}{N})^2}{2(N-\frac{k^2}{N})}-\lambda_{\frac{N+k}{2}}\left(\frac{n+l}{2}\right)-\lambda_{\frac{N-k}{2}}\left(\frac{l-n}{2}\right)\right) \\
&\leq C\exp\left(-\frac{(n-\frac{kl}{N})^2}{2(N-\frac{k^2}{N})}\right).
\end{aligned}$$

We used that $\mathrm{e}^{-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})} \leq C$ as by the definition of λ and the lower bounds in (3.15)

$$\mathrm{e}^{-\lambda_{\frac{N+k}{2}}(\frac{n+l}{2})} = \left(1+\frac{1}{2}\frac{n+l}{N+k}\right)^{-\frac{1}{2}} \leq C$$

and

$$\mathrm{e}^{-\lambda_{\frac{N-k}{2}}(\frac{l-n}{2})} = \left(1+\frac{1}{2}\frac{l-n}{N-k}\right)^{-\frac{1}{2}} \leq C.$$

Therefore

$$\begin{aligned}
&\sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right)\mathbb{E}(T(\sigma)T(\tau)) \\
&\leq C\mathrm{e}^{\frac{(1-p)\beta^2(1+3m^4)}{4p}} \sum_{(k,l,n)\in R^{2c}_{N,C}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)g\left(\frac{|\tau|-Nm}{\sqrt{N}}\right) \\
&\quad\times \mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2))-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}(n^2-m^4N^2)}\binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}} \\
&\quad\times \frac{1}{\sqrt{N}}\exp\left(-\frac{(n-\frac{kl}{N})^2}{2(N-\frac{k^2}{N})}\right).
\end{aligned} \tag{3.17}$$

Note that on $R^{2c}_{N,C}$ we have that $\frac{kl}{N}$ differs from Nm^2 by at most $C\sqrt{N}\kappa_N$ for some constant C. On the other hand, n differs from Nm^2 by at least $C'\sqrt{N}\kappa_N$ by our definition of $S^{2c}_{N,C}$. Choosing $C' := 2C$ we have that $|n-\frac{kl}{N}| \geq C\sqrt{N}\kappa_N$ and hence

$$\frac{1}{\sqrt{N}}\exp\left(-\frac{(n-\frac{kl}{N})^2}{2(N-\frac{k^2}{N})}\right) \leq \exp\left(-C^2N\kappa_N^2/(2N)\right) = \exp\left(-C^2\kappa_N^2/2\right).$$

On the other hand,

$$\frac{(1-p)\beta^2}{4N^2p}(n^2-m^4N^2) \leq K/p$$

for some constant p. Hence the sum over n on the right hand side of (3.17) (which contains at most N summands) can be bounded by

$$N\exp\left(-C^2\kappa_N^2/2+\frac{K}{p}\right) = o(1).$$

The latter is true, since $\frac{1}{p} = o(\kappa_N^2)$, because $p\kappa_N^2 \to \infty$ as $N \to \infty$ and

$$N\exp\left(-C^2\kappa_N^2/2\right) = o(1).$$

Thus we have that

$$\sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau))$$
$$\le o(1)\sum_{k,l\in W_{N,m}} \mathrm{e}^{\frac{(1-p)\beta^2(1+3m^4)}{4p}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)$$
$$\times\, \mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2))-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}+\frac{(1-p)\beta^2}{4N^2p}(n^2-m^4N^2)}\binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}}.$$

Following the lines of the proof of Proposition 3.2, we see that for $k,l\in W_{N,m}$ we have

$$\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}=\frac{(1-p)\beta^2m^4}{p}+o(1).$$

Hence

$$\sum_{k,l\in W_{N,m}} \mathrm{e}^{\frac{(1-p)\beta^2(1+3m^4)}{4p}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)$$
$$\times\, \mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2))-\frac{(1-p)\beta^2}{4p}\frac{2m^2(k^2+l^2)}{N^2}}\binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}}$$
$$=\sum_{k,l\in W_{N,m}} \mathrm{e}^{\frac{(1-p)\beta^2(1-m^4)}{4p}} g\Big(\frac{|\sigma|-Nm}{\sqrt{N}}\Big) g\Big(\frac{|\tau|-Nm}{\sqrt{N}}\Big)$$
$$\times\, \mathrm{e}^{\frac{\beta}{2N}((k^2-m^2N^2)+(l^2-m^2N^2))}\binom{N}{\frac{N+k}{2}}\binom{N}{\frac{N+l}{2}}$$

and – as in Proposition 3.2 – the sum on the right-hand side is bounded above by $[\mathbb{E}\tilde{Z}_N^+(\beta,h,g)]^2$. Thus

$$\sum_{(\sigma,\tau)\in S^{2c}_{N,C}} g\Big(\frac{|\sigma|}{\sqrt{N}}\Big) g\Big(\frac{|\tau|}{\sqrt{N}}\Big)\mathbb{E}(T(\sigma)T(\tau)) = o\Big(\Big(\mathbb{E}\tilde{Z}_N^+(\beta,h,g)\Big)^2\Big),$$

which shows that also the contribution from $S^{2c}_{N,C}$ is negligible. This finishes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Proposition 3.4 shows that, when $p^3N\to\infty$, we have that $\mathbb{V}(\tilde{Z}_N^+(\beta,h,g)) = o((\mathbb{E}\tilde{Z}_N^+(\beta,h,g))^2)$ for all non-negative $g\in\mathcal{C}_b(\mathbb{R})$, $g\not\equiv 0$. This immediately implies

$$\frac{\tilde{Z}_N^+(\beta,h,g)}{\mathbb{E}\tilde{Z}_N^+(\beta,h,g)}\to 1$$

in L^2, for all non-negative $g\in\mathcal{C}_b(\mathbb{R})$, $g\not\equiv 0$. By Chebyshev's inequality, this implies that

$$\frac{\tilde{Z}_N^+(\beta,h,g)}{\mathbb{E}\tilde{Z}_N^+(\beta,h,g)}\to 1$$

in probability. Recall from Proposition 3.2 that

$$\lim_{N\to\infty}\frac{\mathbb{E}\tilde{Z}_N^+(\beta,h,g)}{\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(1-m^4)-NI(m)}2^{N+1}\frac{1}{\sqrt{1-m^2}}\sigma(\beta)\mathbb{E}_\xi[g(\xi)]}=1.$$

Arguing in the same way but without the restriction to $|\sigma|>0$ we get

$$\frac{\tilde{Z}_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)}\to 1 \tag{3.18}$$

in probability and

$$\lim_{N\to\infty} \frac{\mathbb{E}\tilde{Z}_N(\beta,h)}{e^{\frac{(1-p)\beta^2}{8p}(1-m^4)-NI(m)}2^{N+1}\frac{1}{\sqrt{1-m^2}}\sigma(\beta)} = 2. \tag{3.19}$$

Note that the limit equals 2 because the summation can be split into two sums over configurations with $|\sigma| > 0$ and $|\sigma| \le 0$ having the same asymptotic behavior.

Recall that for the convergence of the random probability measure L_N^+ defined in (1.7) we need to consider its integral against all non-negative $g \in \mathcal{C}_b(\mathbb{R})$. Moreover recall that, according to (3.1) and (3.3) we have

$$\int_0^{+\infty} g(x) L_N^+(dx) = 2\mathbb{E}_{\mu_{\beta,h,N}}\left[g\left(\frac{\sum_{i=1}^N \sigma_i - Nm}{\sqrt{N}}\right)\right] = \frac{\tilde{Z}_N^+(\beta,h,g)}{\frac{1}{2}\tilde{Z}_N(\beta,h)}.$$

The above claims and Slutsky's lemma yield

$$\lim_{N\to\infty}\int_0^{+\infty} g(x) L_N^+(dx) = \lim_{N\to\infty} \frac{\mathbb{E}\tilde{Z}_N^+(\beta,h,g)}{\frac{1}{2}\mathbb{E}\tilde{Z}_N(\beta,h)} = \mathbb{E}_\xi[g(\xi)] \tag{3.20}$$

in probability, where ξ denotes a normally distributed random variable with expectation 0 and variance $\sigma^2(\beta)$. Hence, we have shown that the measure L_N^+, considered as a random element of the space of finite measures, converges in probability to a normal distribution with mean 0 and variance $\sigma^2(\beta)$. This is the assertion of Theorem 1.1. □

4. Proof of Theorem 1.3

We go through the proof of Theorem 1.1 and indicate where we need to adjust the arguments. At the beginning of Section 3 we already defined $T(\sigma) = T_{\beta,N}(\sigma)$ for $h = 0$. Now, in the definition of T and in the rest of the proof we use

$$m = m^+(\beta, h)$$

and set

$$T(\sigma) := T_{\beta,h,N}(\sigma) := \exp\left(\gamma \sum_{i,j=1}^N \varepsilon_{i,j}\sigma_i\sigma_j - \gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j} + \beta h \sum_{i=1}^N \sigma_i\right)$$

and

$$\tilde{Z}_N(\beta,h,g) := \sum_{\sigma\in\{-1,+1\}^N} g\left(\frac{|\sigma| - Nm}{\sqrt{N}}\right) T(\sigma).$$

Again, we will suppress the indices β, h and N in $T_{\beta,h,N}(\sigma)$ in the rest of this section.

In the expansion of $\mathbb{E}T(\sigma)$, we get an additional deterministic term $\exp(\beta h|\sigma|)$, i.e. we get immediately

Lemma 4.1. *Assume $h > 0, \beta > 0$. Then for all $\sigma \in \{-1,+1\}^N$ we have*

$$\mathbb{E}T(\sigma) = \exp\left(\frac{\beta}{2N}(|\sigma|^2 - m^2N^2) + \frac{(1-p)\beta^2}{8p^2}\left(m^4 - \frac{2m^2|\sigma|^2}{N^2} + 1\right)\right)\exp(h\beta|\sigma| + o(1))$$

with an $o(1)$-term that is uniform over $\sigma \in \{-1,+1\}^N$.

The analogue of Proposition 3.2 now reads as follows.

Proposition 4.2. *For all $g \in \mathcal{C}^b(\mathbb{R}), g \ge 0, g \not\equiv 0,$ $h > 0, \beta > 0,$ and p with $Np^3 \to \infty$ we have*

$$\lim_{N\to\infty} \frac{\mathbb{E}\tilde{Z}_N(\beta,h,g)}{e^{\frac{(1-p)\beta^2}{8p}(-m^4+1)-NI(m)+\beta hNm}2^{N+1}\frac{1}{\sqrt{1-m^2}}\sigma(\beta,h)\mathbb{E}_\xi[g(\xi)]} = 1, \tag{4.1}$$

where ξ denotes a normally distributed random variable with expectation 0 and variance $\sigma^2(\beta,h)$.

Proof: We note that, in contrast to the case $h = 0$, there is an additional term $e^{\beta hNm}$ in the denominator in the assertion of the proposition. We proceed as in the proof of Proposition 3.2. The main difference is that $\mathbb{E}T(\sigma)$ has an supplementary factor $e^{\beta h|\sigma|}$, which results in an extra factor $e^{\beta hk}$ in (3.5). The remaining notation is the same as in the proof of Proposition 3.2. Again we use $\frac{k}{N} = m + \frac{c_k}{\sqrt{N}}$ with $|c_k| \leq \kappa_N$, i.e.

$$e^{\beta hk} = \mathrm{e}^{\beta hNm} e^{\beta h\sqrt{N}c_k}.$$

Instead of (3.7) we have

$$\begin{aligned}
&\sum_{\sigma\in S_N^1} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) \\
&\quad = (1+o(1))\mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)-NI(m)+\beta hNm} 2^N \frac{1}{\sqrt{1-m^2}}\sqrt{\frac{2}{\pi N}} \\
&\qquad \times \sum_{k\in W_{N,m}} g\left(\frac{k-Nm}{\sqrt{N}}\right)\mathrm{e}^{\frac{\beta}{2N}(2N^{3/2}mc_k+c_k^2N)-\frac{(1-p)\beta^2}{8p}\frac{2m^4N^2+4m^3N^{3/2}c_k+2m^2c_k^2N}{N^2}+\beta h\sqrt{N}c_k} \\
&\qquad\quad \times \mathrm{e}^{-NI'(m)\frac{c_k}{\sqrt{N}}-I''(m)\frac{c_k^2}{2}}.
\end{aligned}$$

The linear term in c_k in the exponent is

$$\begin{aligned}
&c_k\left(\beta m\sqrt{N} - \frac{\beta^2(1-p)m^3}{2p\sqrt{N}} - \sqrt{N}I'(m) + \beta h\sqrt{N}\right) \\
&= c_k\left(\beta m\sqrt{N} - \sqrt{N}I'(m) + \beta h\sqrt{N}\right) + o(1) \\
&= o(1),
\end{aligned}$$

where the first equality follows from the definition of κ_N and $Np^3 \to \infty$ and the second equality follows from $m^+(\beta,h) = \tanh(\beta(m^+(\beta,h)+h))$, which is the defining relation for $m^+(\beta,h)$. The rest of the proof for typical spin configurations can remain unchanged, except for the additional factor $e^{\beta hNm}$, which now appears in the denominator in the assertion of the proposition. For atypical spin configurations we get again an extra factor $e^{\beta hk}$ in the expansion of $\mathbb{E}T(\sigma)$, i.e. instead of (3.10) we arrive at

$$\begin{aligned}
&\sum_{\sigma\in S_N^{1c}} g\left(\frac{|\sigma|-Nm}{\sqrt{N}}\right)\mathbb{E}T(\sigma) \\
&\leq C||g||_\infty \mathrm{e}^{\frac{(1-p)\beta^2}{8p}(m^4+1)} 2^N \sum_{\substack{k: \\ k^2\in W_{N,m,0}^c}} \mathrm{e}^{\frac{\beta}{2N}(k^2-m^2N^2)-\frac{m^4(1-p)\beta^2}{4p}-\frac{2c_k(1-p)\beta^2}{4p\sqrt{N}}m^3-NI(\frac{k}{N})} \mathrm{e}^{\beta hk}.
\end{aligned}$$

The function $\frac{k}{N} \mapsto \frac{\beta}{2N}k^2 - NI(\frac{k}{N}) + \beta hk$ attains its maximum in $m^+(\beta,h)$ and hence

$$\frac{\beta}{2N}k^2 - NI(\frac{k}{N}) + \beta hk \leq N\left(\frac{\beta}{2}m^2 - I(m) + \beta hm\right) - k_1c_k^2,$$

for some $k_1 > 0$. The rest of the proof is completely analogous to the case $h = 0$. □

In the expansion of $\mathbb{E}(T(\sigma)T(\tau))$ we get two extra terms, which leads immediately to the following analogue of Lemma 3.3.

Lemma 4.3. *For $h > 0$, $\beta > 0$, all $p = p(N)$ such that $p^3N \to \infty$ and all $\sigma,\tau \in \{-1,+1\}^N$ we have*

$$\mathbb{E}(T(\sigma)T(\tau)) = \exp(N^2b_0 + b_1|\sigma|^2 + b_2|\tau|^2 + b_{12}|\sigma\tau|^2 + h\beta(|\sigma|+|\tau|))$$

with coefficients b_0, b_1, b_2, b_{12} given in Lemma 3.3.

The proof of

$$\mathbb{V}(\tilde{Z}_N(\beta,h,g)) = o\Big(\mathbb{E}^2[\tilde{Z}_N(\beta,h,g)]\Big),$$

i.e. of the result corresponding to Proposition 3.4, as well as the rest of the proof is completely analogous to the case $h=0$, where in (3.20) ξ is now a normally distributed random variable with expectation 0 and variance $\sigma^2(\beta,h)$. In particular, it follows that

$$\frac{\tilde{Z}_N(\beta,h,g)}{\mathbb{E}\tilde{Z}_N(\beta,h,g)} \to 1, \qquad \frac{\tilde{Z}_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} \to 1, \tag{4.2}$$

in probability.

5. Proof of Theorem 1.4

We give the proof for the two cases $h=0$, $\beta>1$ and $h>0$, $\beta>0$ simultaneously. The main ingredient for the fluctuations of $Z_N(\beta,h)$ is the relation (see (3.2))

$$Z_N(\beta,h) = \tilde{Z}_N(\beta,h)\exp\left(\gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}\right)$$

together with

$$\frac{\tilde{Z}_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} \to 1 \quad \text{in probability;}$$

see (3.18) for the case $h=0$, $\beta>1$, and (4.2) for the case $h>0$, $\beta>0$. The term $\exp\left(\gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}\right)$ can easily be treated by the Central Limit Theorem.

Proof of Theorem 1.4: We have

$$\log\frac{Z_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} = \log\frac{\tilde{Z}_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} + \gamma m^2 \sum_{i,j=1}^N \varepsilon_{i,j}.$$

Note that $\gamma\sum_{i,j=1}^N \varepsilon_{i,j}$ is a sum of independent random variables with

$$\mathbb{E}\left(\gamma\sum_{i,j=1}^N \varepsilon_{i,j}\right) = \gamma p N^2 = \frac{\beta}{2}N$$

and

$$\mathbb{V}\left(\gamma\sum_{i,j=1}^N \varepsilon_{i,j}\right) = \gamma^2 N^2 p(1-p) = \frac{\beta^2}{4p}(1-p).$$

Hence, by the Central Limit Theorem (which applies since $N^2p(1-p)\to\infty$) we have for $N\to\infty$

$$\frac{\gamma m^2\sum_{i,j=1}^N \varepsilon_{i,j} - m^2\frac{\beta}{2}N}{\sqrt{m^4\frac{\beta^2}{4p}(1-p)}} \to \mathfrak{N}_{0,1}$$

in distribution. It follows that

$$\frac{\log\frac{Z_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} - m^2\frac{\beta}{2}N}{\sqrt{m^4\frac{\beta^2}{4p}(1-p)}} = \frac{\log\frac{\tilde{Z}_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)}}{\sqrt{m^4\frac{\beta^2}{4p}(1-p)}} + \frac{\gamma m^2\sum_{i,j=1}^N \varepsilon_{i,j} - m^2\frac{\beta}{2}N}{\sqrt{m^4\frac{\beta^2}{4p}(1-p)}}.$$

The numerator of the first summand converges to 0 in probability, while the denominator is bounded away from 0 because p is bounded away from 1. It follows that the first term converges to 0 in probability and hence

$$\frac{\log \frac{Z_N(\beta,h)}{\mathbb{E}\tilde{Z}_N(\beta,h)} - m^2 \frac{\beta}{2} N}{\sqrt{m^4 \frac{\beta^2}{4p}(1-p)}} \to \mathfrak{N}_{0,1}$$

in distribution. This completes the proof. □

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